ANNALS OF PHYSICS: 54, 207–239 (1969)

Electrodynamics of Direct Interparticle Action. I. The Quantum Mechanical Response of the Universe

F. HOYLE AND J. V. NARLIKAR

Institute of Theoretical Astronomy, University of Cambridge, Cambridge, England

The present paper is the first of a series that seeks to obtain results in agreement with experience from a completely time-symmetric electromagnetic theory—i.e. which does not permit an *ad hoc* restriction to retarded solutions of time-symmetric equations. It is remarkable that the development of a wholly time-symmetric theory must be along lines entirely different from the usual electrodynamics. While a first quantisation of the particles can readily be carried out, there can be no separate quantisation of the field, since the field is wholly determined by the particles. This raises the question of how practical results that have hitherto been thought to arise from field quantisation can be obtained. The most immediate problem of this kind concerns the spontaneous transitions of atoms. Much of the present paper is directed toward showing that this problem can indeed be solved without the need for field quantisation. Although this question might appear simple compared to other issues in quantum electrodynamics—e.g. vacuum polarisation—it is not trivial in its implication, for the establishment of one such case provides a critical precedent.

The path integral method of first quantisation is used to demonstrate that provided the Universe is a perfect absorber along the future light cone the usual formulae for level shifts and for spontaneous transitions can be obtained in a steady-state model of the Universe, but not in open Friedmann models.

I. INTRODUCTION

Maxwell's equations admit advanced solutions as well as retarded solutions, and indeed of an arbitrary combination of advanced and retarded solutions. The usual theory of electrodynamics cannot therefore be taken to represent the world of experience unless either (a) we make the *ad hoc* postulate of restricting ourselves to retarded solutions, or (b) the universe happened to be set up in the first place in such a way that only retarded solutions were excited. The normal procedure is to follow (a), but then the equations of physics correspond to experience only after a metaphysical prescription has been applied to them. On the other hand if we follow (b) we accept the position that even the simplest aspects of experience are not wholly determined by the equations of physics. We have to think of the locally determined laws of physics plus cosmology as being necessary for an understanding of experience.

595/54/2-1

This prompts the question of whether we are at present making the most perceptive association of cosmology and physics. Any largescale environmental component that influences experience must be effectively constant over the small space-time regions in which terrestrial experiments are conducted. Hence we cannot investigate such an environmental component through the usual method of studying changes and variations in the laboratory. And since there must be a tendency to overestimate the importance of those features that we can study it seems only too likely that we may be exaggerating local effects at the expense of large-scale environmental effects. We may well be overloading the locally determined laws with features that do not really belong to them.

As an example of what has just been said, we shall attempt in these papers to show that the practical results of quantum electrodynamics can be obtained by a "first" quantisation of the matter but without a "second" quantisation of the field. There will be no independent degrees of freedom of the field—independent of the matter—no quantised vacuum oscillators and no quanta.

Accepting for the moment that this can be done, we then have to ask the question: could our usual idea of a quantised electromagnetic field, perhaps indeed of all quantised fields, be an illusion? Could it be that hypotheses concerning fields have been invented in order to push into the laws of physics aspects of our experience that are really environmental? We ourselves now believe the answers to these questions are probably affirmative, since we feel it would otherwise be most surprising to find the same practical results coming out of the present theory as come from the usual theory. The agreement over practical results does not spring from some deep intrinsic equivalence of the two theories, since the correct results can only be obtained in the following theory subject to an appropriate cosmological structure for the whole universe, whereas the practical results of the usual quantum electrodynamics do not make any such demand. Rather do we interpret the agreement as a tribute to the ingenuity of theoretical physicists who by an appropriate choice of postulates have been able to get the right results out of a wrong theory. The sceptic will of course wish to see how the details turn out.

It was already suggested by Gauss that Newton's concept of instantaneous action at a distance should be modified to action transmitted at the speed of light. This idea was developed in the present century by Schwarzschild (*1*), Tetrode (*2*) and Fokker (*3*). In the form given by Fokker the laws of classical electrodynamics are derived from an action principle, the action being given by

$$S = -\sum_{a} \int m_a \, da - \sum_{a<b}\sum e_a e_b \iint \delta(s_{AB}^2) \, \eta_{ik} \, da^i \, db^k. \tag{1}$$

Here a, b,... label the particles; e_a, m_a being the charge and mass of particle a.

a^i are the Minkowski coordinates and da is the element of proper time on the world line of a. η_{ik} is the space-time metric and s_{AB}^2 is the square of the four-dimensional distance between A and B, typical points on the world lines of a and b. The 4-potential generated by charge a at a general point B is defined by

$$A_i^{(a)}(B) = e_a \int \delta(s_{AB}^2)\, \eta_{ik}\, da^k. \tag{2}$$

By virtue of this definition Maxwell's equations and the Lorentz gauge condition are identically satisfied. The equations of particle motion are obtained by the variation of the world lines, and are the usual Lorentz-force equations, but without self-action.

The status of Maxwell's equations is now quite different, however, from the usual situation in classical electrodynamics. Although the equations again have both advanced and retarded solutions there is no arbitrariness about which solution must be chosen, because the action principle based on (1) is now the starting point of the theory, not Maxwell's equations. Indeed Maxwell's equations apply only in relation to the definition (2), and this definition requires the field produced by the charge a to be, not the usually observed retarded field $F_{\text{ret}}^{(a)}$, but

$$F^{(a)} = \tfrac{1}{2}[F_{\text{ret}}^{(a)} + F_{\text{adv}}^{(a)}], \tag{3}$$

where $F_{\text{adv}}^{(a)}$ is the time-reversed form of $F_{\text{ret}}^{(a)}$. Here $F^{(a)}$ stands for any particular field component.

The presence of advanced fields was considered a drawback of the theory and prevented further progress until it was reconsidered by Wheeler and Feynman (*4*). These authors pointed out that (3) represents the field generated by a single particle. In the actual universe there is a large number of particles and interference takes place. If the universe is a perfect absorber of all retarded disturbances produced by the motion of charge a, then a self-consistent cycle of argument can be set up in which the net field acting on a can be shown to be

$$\sum_{b \neq a} F_{\text{ret}}^{(b)} + \tfrac{1}{2}[F_{\text{ret}}^{(a)} - F_{\text{adv}}^{(a)}]. \tag{4}$$

The first term here denotes the usual observed retarded contribution from all other particles in the universe. This term is the "external" field acting on a due to the other particles, and it is calculated in the usual way—it has nothing to do with the instantaneous motion of a itself. The second term, on the other hand, does arise from the instantaneous motion of a. Unlike the usual theory, this second term is the result of the response of the universe—it does not arise from self-action.

The self-consistent cycle of argument is set up in the following way. It is assumed that the field acting on any particle b other than a, due to the finite (bounded) disturbance of a is just the usually calculated retarded field of a. This includes the usually calculated absorptive and dispersive effects of all particles other than a and b. The response field of b is then calculated from (2) and the sum of such response fields for all particles $b(\neq a)$ is shown to be given by (4). Then the sum of (3) and (4) gives $F^{(a)}_{\mathrm{ret}}$ plus the external field $\sum_{b\neq a} F^{(b)}_{\mathrm{ret}}$. That is to say, the effect of the motion of a is $F^{(a)}_{\mathrm{ret}}$, which returns to the assumption made at the beginning of the cycle.

In the absence of an external field the equation of motion of particle a involves only $\frac{1}{2}[F^{(a)}_{\mathrm{ret}} - F^{(a)}_{\mathrm{adv}}]$, and the nonrelativistic form of the equation can be shown to be

$$\ddot{\mathbf{r}}^{(a)} = \frac{2e_a{}^2}{3m_a c^3}\,\dddot{\mathbf{r}}^{(a)} \tag{5}$$

The same equation appears in normal electrodynamics where the divergent solution $\mathbf{r}^{(a)} \propto \exp(3m_a c^3 t/2e_a{}^2)$ must be excluded by another *ad hoc* postulate. It is of interest to ask how this solution comes to be excluded in the present theory. The answer is that (4), which leads to (5), cannot be deduced unless the disturbance of a is bounded. Since this is not the case for the divergent solution (at finite t we can make $|\,\mathbf{r}^{(a)}\,|$ as large as we please) the consistency of the cycle of argument fails. Hence the divergent solution is excluded by the theory itself and no *ad hoc* postulate is required.

In this respect, and in the clear-cut requirement (3) for half the sum of the advanced and retarded solutions of Maxwell's equations, the present theory is more satisfactory than the usual theory in its classical form. Nevertheless, an ambiguity still arises if more than one consistent cycle of argument can be found. If it is possible to start by assuming an advanced field, and then carry through a cycle similar to that described above for the retarded case, then no real logical improvement has been achieved. We are faced by uncertainty as to which cycle of argument should be chosen and an *ad hoc* prescription is still needed.

This was the situation in the absorber theory of Wheeler and Feynman. It was pointed out by Hogarth (*5*) that the situation is changed, however, if the static Euclidean universe considered by Wheeler and Feynman is replaced by an expanding universe of a suitable kind. The work of Hogarth was extended by Hoyle and Narlikar (*6*), and as a result of these investigations it was established that, while in some cosmologies ambiguity still persisted, in other cosmologies only a single cycle of argument was self-consistent. In the Einstein–de Sitter cosmology, and other ever-expanding models of the Friedmann type, the cycle with advanced solutions is consistent, but the cycle with retarded solutions is not consistent. The situation is reversed in the steady-state cosmology—the cycle with retarded solutions is uniquely consistent. The steady-state theory seems to be the only cosmology in which the self-consistent cycle is both unique and correct. It was because this cosmology overcomes what we regard as a fundamental logical difficulty in electrodynamics that we were reluctant to abandon the steady-state theory when some four years ago astronomical evidence seemed to point quite strongly against it. More recently, however, the evidence against the theory has weakened (*7*). This decline of what had seemed a firm case, taken with the results reported in these papers, has gone far towards convincing us that the steady-state theory contains features belonging to the correct cosmology, features which are not contained in the so-called big-bang cosmologies.

The classical theory outlined above has not hitherto been extended to quantum theory, although the path integral formulation of quantum mechanics developed by Feynman (*8*) provides a technique that is naturally directed to this end. It gives a first quantisation for the particles. This turns out to be sufficient to show that a second quantisation of the field is not necessary in order to obtain the transition probabilities of nonrelativistic radiation theory. Because there is no field quantisation the external term, $\sum_{b \neq a} F^{(b)}_{\text{ret}}$ in (4), behaves classically. It produces transitions from a particle state m to a state n irrespective of whether the energy E_m of state m is greater or less than the energy E_n of state n. The response term, $\frac{1}{2}[F^{(a)}_{\text{ret}} - F^{(a)}_{\text{adv}}]$, on the other hand, must be discussed quantum mechanically. It turns out to produce transitions $m \to n$ only in the case $E_m > E_n$.

The resulting formulae are the same in structure as those of the usual theory. The response term gives the spontaneous transition formula, while the "external field" gives the absorption and stimulated emission formula. The spontaneous emission formula is identical with that of the usual theory, but the absorption and stimulated emission formula contains a quantity determined by the intensity of the applied field, in place of the average number $\bar{n}_{\mathbf{k}}$ of quanta in oscillators of frequency close to $\mathbf{k}$. It is a trivial matter, however, to write the absorption-stimulated emission formula in the usual way, by introducing a suitable definition of $\bar{n}_{\mathbf{k}}$. It then turns out that $\bar{n}_{\mathbf{k}}$ is related to the intensity of the applied field in exactly the same way as in the usual theory.

Only first-order radiation processes are considered in the present paper. There is little point in working through second and higher order processes in terms of a nonrelativistic particle theory, because the interesting processes in higher order are those involving positrons. In paper II we shall be concerned with extending the present methods to the relativistic case including spin, and we will then discuss higher order processes involving positrons. In II we also discuss convergence and the interpretation of renormalisation procedures in terms of the direct action theory. However we shall see in the present theory how the usual formula for the level shift can be derived. In this connection it is worth emphasis that the present treatment of the response field of the universe is relativistic—it is the particles that are nonrelativistic. The situation is analogous to working in the usual theory with a relativistic field but with the Schrödinger equation instead of the Dirac equation.

In Section II we obtain the stimulated emission and absorption formula. There is nothing new in this section but it serves to remind the reader of the aspects of the path integral method that will be used later. In Section III we derive the spontaneous emission formula, while in Section IV we show how the level shift formula may be obtained. In Section V we return to cosmology and to the likely physical nature of the absorber. The new parts of the paper are contained in these last three sections.

II. ABSORPTION AND STIMULATED EMISSION

The probability of transition $P(m \to n)$ in the time interval $0 \leqslant t \leqslant T$ from the state m to an orthogonal state n in a specified (unquantised) external field can be worked out by the methods described by Feynman and Hibbs ((9), page 340). We start with the formula

$$P(m \to n) = \iiiint \phi_n^*(\mathbf{a}_f)\, \phi_n(\mathbf{a}_f')\, J\phi_m(\mathbf{a}_i)\, \phi_m^*(\mathbf{a}_i')\, d^3\mathbf{a}_f\, d^3\mathbf{a}_f'\, d^3\mathbf{a}_i\, d^3\mathbf{a}_i', \tag{6}$$

where ϕ_m, ϕ_n are the wave functions for the states, and J is the double path integral

$$J = \iint \exp\left[\frac{i}{\hbar}\{S[\mathbf{a}(t)] - S[\mathbf{a}'(t)]\}\right] \mathscr{D}^3\mathbf{a}(t)\, \mathscr{D}^3\mathbf{a}'(t). \tag{7}$$

Here the path $\mathbf{a}(t)$ "begins" at $\mathbf{r} = \mathbf{a}_i$, $t = 0$ and "ends" at $\mathbf{r} = \mathbf{a}_f$, $t = T$, while the path $\mathbf{a}'(t)$ begins at $\mathbf{r} = \mathbf{a}_i'$, $t = 0$ and ends at $\mathbf{r} = \mathbf{a}_f'$, $t = T$. The action $S[\mathbf{a}(t)]$ is a functional of the path $\mathbf{a}(t)$ and is defined by

$$S[\mathbf{a}(t)] = \int_0^T L(\mathbf{a}, t)\, dt, \tag{8}$$

and similarly for $S[\mathbf{a}'(t)]$.

We are concerned with an electronic transition and with a situation in which

$$L = \tfrac{1}{2}m\dot{\mathbf{a}}^2 + eV(\mathbf{a}, t) - e\dot{\mathbf{a}} \cdot \mathbf{A}(\mathbf{a}, t), \tag{9}$$

where e, m are the electronic charge and mass, the velocity of light is taken as unity, the nonrelativistic kinetic energy is used, and $V(\mathbf{a}, t)$, $\mathbf{A}(\mathbf{a}, t)$ are the potentials of the specified field. In a simple one-electron atomic problem the external field is made up of the electrostatic field within the atom together with the field incident on the atom. Provided the Coulomb gauge is used for the latter, which is taken wholly transverse, V describes the atomic field and $\mathbf{A}$ the incident field. With this division of V and $\mathbf{A}$, write

$$S[\mathbf{a}(t)] = S_0[\mathbf{a}(t)] - e \int_0^T \dot{\mathbf{a}} \cdot \mathbf{A}\, dt, \tag{10}$$

so that $S_0[\mathbf{a}(t)]$ is the action for the atomic field alone.

Next we regard $\mathbf{A}$ as small enough for

$$\exp\left[-\frac{ie}{\hbar}\int_0^T \dot{\mathbf{a}} \cdot \mathbf{A}\, dt\right] = 1 - \frac{ie}{\hbar}\int_0^T \dot{\mathbf{a}} \cdot \mathbf{A}\, dt + \cdots \tag{11}$$

to be rapidly convergent. The dominant term in (6) then involves the product of the first-order term in (11) with the first-order term in the complex conjugate of (11). After a reduction that is straightforward, except perhaps for the calculation of the transition element of the velocity ((9), page 184) one obtains

$$P(m \to n) = \frac{e^2}{\hbar^2}\left(\frac{E_n - E_m}{\hbar}\right)^2 \left| \int_0^T \exp \frac{i(E_n - E_m)\, t}{\hbar} \int \phi_n^*(\mathbf{a})\, \mathbf{a} \cdot \mathbf{A}\phi_m(\mathbf{a})\, d^3\mathbf{a}\, dt \right|^2. \tag{12}$$

To express this result in a more familiar form expand the incident field in a Fourier series. For this step consider the atom as situated inside a cube with side of unit length, the latter being chosen to be very large compared with any wavelength of importance in the transition problem. Write

$$\mathbf{A}(\mathbf{a}, t) = \sqrt{4\pi} \sum_{\mathbf{k}} \{\mathbf{c}_{\mathbf{k}} \exp[i(\mathbf{k} \cdot \mathbf{a} - kt)] + \mathbf{c}_{\mathbf{k}}^* \exp[-i(\mathbf{k} \cdot \mathbf{a} - kt)]\}, \tag{13}$$

noting that $\mathbf{k} = 2\pi(n_1, n_2, n_3)$ where n_1, n_2, n_3 are integers, and that $\mathbf{k} \cdot \mathbf{c}_{\mathbf{k}} = 0$ because of the Coulomb gauge. Inserting (13) in (12) leads to a double sum, $\sum_{\mathbf{k}'} \sum_{\mathbf{k}}$, say. *It is usual to specify the incident field to be such that all terms with* $\mathbf{k}' \neq \mathbf{k}$ *average to zero.* Only a single summation, $\sum_{\mathbf{k}}$, then remains.

Next, we note that cross-products of the form

$$\int_0^T \exp\left[\frac{i}{\hbar}(E_n - E_m + \hbar k)\, t\right] dt \int_0^T \exp\left[-\frac{i}{\hbar}(E_n - E_m - \hbar k)\, t\right] dt \tag{14}$$

never give appreciable contributions. The nature of the important terms can be seen from

$$\left| \int_0^T \exp\left[\frac{i}{\hbar}(E_n - E_m \pm \hbar k)\, t\right] dt \right|^2 \cong 2\pi T \delta\left(k \pm \frac{E_n - E_m}{\hbar}\right). \tag{15}$$

When $E_n > E_m$ the important terms involve the minus sign in the delta-function, and (12) leads to

$$\frac{4\pi e^2}{\hbar^2}\left(\frac{E_n - E_m}{\hbar}\right)^2 \sum_{\mathbf{k}} \left| \mathbf{c}_{\mathbf{k}} \cdot \int \phi_n^*(\mathbf{a})\, \mathbf{a} e^{i\mathbf{k}\cdot\mathbf{a}} \phi_m(\mathbf{a})\, d^3\mathbf{a} \right|^2$$
$$\times \left| \int_0^T \exp\left[\frac{i}{\hbar}(E_n - E_m - \hbar k)\, t\right] dt \right|^2, \tag{16}$$

where we have deferred using (15) until after the sum has been converted to an integral. This is the case of absorption.

When $E_m > E_n$ we have the case of stimulated emission, and the corresponding result is

$$\frac{4\pi e^2}{\hbar^2}\left(\frac{E_m - E_n}{\hbar}\right)^2 \sum_{\mathbf{k}} \left| \mathbf{c}_{\mathbf{k}}^* \cdot \int \phi_n^*(\mathbf{a})\, \mathbf{a} e^{-i\mathbf{k}\cdot\mathbf{a}} \phi_m(\mathbf{a})\, d^3\mathbf{a} \right|^2$$
$$\times \left| \int_0^T \exp\left[\frac{i}{\hbar}(E_n - E_m + \hbar k)\, t\right] dt \right|^2. \tag{17}$$

It is not hard to see that (17) is the same as (16). Although $\mathbf{c}_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{a}}$ has replaced $\mathbf{c}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{a}}$ the roles of the states m, n have been switched; ϕ_m is the state of lower energy in (16) whereas ϕ_n is the wavefunction of the lower state in (17). This confirms the remark of Section I—that the transition probability for stimulated emission in an external field is equal to that for absorption.

It is usual to express the transition probability in terms of the intensity of the applied field. In order to do this it is necessary either to approximate (16) or to average the transition probability with respect to the orientation of the atom. We adopt the latter procedure. Define

$$\mathbf{a}_{mn}(\mathbf{k}) = \int \phi_n^*(\mathbf{a})\, \mathbf{a} e^{i\mathbf{k}\cdot\mathbf{a}} \phi_m(\mathbf{a})\, d^3\mathbf{a}, \tag{18}$$

and let $\boldsymbol{\alpha}_{\mathbf{k}}^{(1)}$, $\boldsymbol{\alpha}_{\mathbf{k}}^{(2)}$ be unit vectors which together with $\mathbf{k}/k$ form an orthogonal triad. Because of $\mathbf{c}_{\mathbf{k}} \cdot \mathbf{k} = 0$

$$\mathbf{c}_{\mathbf{k}} = \sum_{j=1,2} [\mathbf{c}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}^{(j)}]\, \boldsymbol{\alpha}_{\mathbf{k}}^{(j)}, \tag{19}$$

and

$$| \mathbf{c}_{\mathbf{k}} \cdot \mathbf{a}_{mn}(\mathbf{k})|^2 = \left| \sum_{j=1,2} (\mathbf{c}_{\mathbf{k}} \cdot \boldsymbol{\alpha}_{\mathbf{k}}^{(j)}) \int \phi_n^*(\mathbf{a})\, \boldsymbol{\alpha}_{\mathbf{k}}^{(j)} \cdot \mathbf{a} e^{i\mathbf{k}\cdot\mathbf{a}} \phi_m(\mathbf{a})\, d^3\mathbf{a} \right|^2. \tag{20}$$

We have to average (20) with respect to orientation.

It is not hard to show that the sum of

$$\int \phi_n^*(\mathbf{a})\, \boldsymbol{\alpha}_{\mathbf{k}}^{(1)} \cdot \mathbf{a} e^{i\mathbf{k}\cdot\mathbf{a}} \phi_m(\mathbf{a})\, d^3\mathbf{a} \cdot \int \phi_n(\mathbf{a})\, \boldsymbol{\alpha}_{\mathbf{k}}^{(2)} \cdot \mathbf{a} e^{-i\mathbf{k}\cdot\mathbf{a}} \phi_m^*(\mathbf{a})\, d^3\mathbf{a}$$

and its complex conjugate averages to zero, and that (20) is proportional to $|\mathbf{c}_{\mathbf{k}}|^2$ and can be written in the form $\frac{1}{3}|\mathbf{c}_{\mathbf{k}}|^2 |\mathbf{a}_{mn}(k)|^2$, where $|\mathbf{a}_{mn}(k)|^2$ depends on the magnitude but not the direction of $\mathbf{k}$, and is equal to $|\mathbf{a}_{mn}|^2$ when the factor $\exp(i\mathbf{k}\cdot\mathbf{a})$ in (20) is approximated by unity. Hence the average value of (16) is

$$\frac{4\pi e^2}{3}\left(\frac{E_n - E_m}{\hbar}\right)^2 \sum_{\mathbf{k}} |\mathbf{c}_{\mathbf{k}}|^2 \cdot |\mathbf{a}_{mn}(k)|^2 \cdot \left| \int_0^T \exp\left[\frac{i}{\hbar}(E_m - E_n + \hbar k)\, t\right] dt \right|^2. \tag{21}$$

There are $d^3\mathbf{k}/(2\pi)^3$ terms of this series in the element $d^3\mathbf{k}$ of $\mathbf{k}$-space. Defining $\overline{|\mathbf{c}_{\mathbf{k}}|^2}$ by

$$\overline{|\mathbf{c}_{\mathbf{k}}|^2}\, \frac{d^3\mathbf{k}}{(2\pi)^3} = \sum_{d^3\mathbf{k}} |\mathbf{c}_{\mathbf{k}}|^2, \tag{22}$$

we now write (21) in the integral form

$$\frac{e^2}{6\pi^2\hbar^2}\left(\frac{E_n - E_m}{\hbar}\right)^2 \int \overline{|\mathbf{c}_{\mathbf{k}}|^2} \cdot |\mathbf{a}_{mn}(k)|^2 \left| \int_0^T \exp\left[\frac{i}{\hbar}(E_n - E_m - \hbar k)\, t\right] dt \right|^2 \cdot d^3\mathbf{k} \tag{23}$$

Because (15) is approximate it is possible to choose the unit of length large enough for there to be many terms of the Fourier series in $d^3\mathbf{k}$ even though $d^3\mathbf{k}$ is small enough for the final factor in the integrand in (23) not to vary appreciably in $d^3\mathbf{k}$. This is implied in (23).

To define the intensity $I(\mathbf{k})$ per unit solid angle we note first that

$$\frac{1}{4\pi}\mathbf{E} \times \mathbf{H} = 2\sum_{\mathbf{k}} |\mathbf{c}_{\mathbf{k}}|^2 k\mathbf{k} = \frac{1}{4\pi^3}\int \overline{|\mathbf{c}_{\mathbf{k}}|^2}\, k\mathbf{k}\, d^3\mathbf{k} \tag{24}$$

and that the contribution of solid angle $d\Omega$ to (24) is

$$\frac{d\Omega}{4\pi^3} \cdot \frac{\mathbf{k}}{k} \int \overline{|\mathbf{c}_{\mathbf{k}}|^2}\, k^4\, dk, \tag{25}$$

where $\mathbf{k}$ lies in $d\Omega$. The intensity $I(\mathbf{k})$ is now defined by equating (25) to

$$d\Omega \cdot \frac{\mathbf{k}}{k} \int I(\mathbf{k})\, dk.$$

For this definition to hold irrespective of $\overline{|\mathbf{c}_\mathbf{k}|^2}$ we must have

$$I(\mathbf{k}) = \frac{1}{4\pi^3} k^4 \overline{|\mathbf{c}_\mathbf{k}|^2}. \tag{26}$$

Eliminating $\overline{|\mathbf{c}_\mathbf{k}|^2}$ between (23) and (26), and using (15), one easily obtains

$$\frac{4\pi^2 e^2}{3\hbar^2} \int d\Omega \int_0^\infty I(\mathbf{k}) \mid \mathbf{a}_{mn}(k)|^2 \, \delta\left(k - \frac{|E_n - E_m|}{\hbar}\right) dk \tag{27}$$

for the transition probability per unit time. The result (27) applies both to absorption and stimulated emission, and is the usual relation between the intensity and the transition probability. We can introduce separate intensities $I^{(j)}(\mathbf{k})$ for the "polarisation" directions $\boldsymbol{\alpha}_k^{(j)}$ by defining

$$I^{(j)}(\mathbf{k}) = \frac{k^4}{4\pi^3} \overline{|\mathbf{c}_\mathbf{k} \cdot \boldsymbol{\alpha}_\mathbf{k}^{(j)}|^2}; \qquad j = 1, 2. \tag{28}$$

Evidently

$$I(\mathbf{k}) = \sum_{j=1,2} I^{(j)}(\mathbf{k}). \tag{29}$$

The unit of length can now be changed to anything we please—for example to 1 cm. Treating mass as a length, $\hbar$ behaves as the square of a length for such a change and $I(\mathbf{k})$ behaves as an inverse length. The original unit volume must still be used, however, in the Fourier expansion of $\mathbf{A}$.

It is to be noted that the reduction to (27) implicitly assumes that $\overline{|\mathbf{c}_\mathbf{k}|^2}$ does not vary appreciably over the range of k for which

$$\left| \int_0^T \exp\left[\frac{i}{\hbar}(|E_m - E_n| - \hbar k)\, t\right] dt \right|^2$$

is appreciably different from zero. This is a further condition on the incident field.

We end the present section by introducing the concept of opacity. Suppose there are $n(k)\, dk$ atoms per unit volume in state m, where $E_m < E_n$ and $(E_n - E_m)/\hbar$ lies between k and $k + dk$. Define a mean value of $|\mathbf{a}_{mn}(k)|^2$ by

$$V \overline{|\mathbf{a}_{mn}(k)|^2} \cdot n(k)\, dk = \sum_V |\mathbf{a}_{mn}(k)|^2, \tag{30}$$

the summation being taken through the small volume V for all atoms with $(E_n - E_m)/\hbar$ between k and $k + dk$. Both $\overline{|\mathbf{a}_{mn}(k)|^2}$ and $n(k)$ can be functions of of position as well as of k.

Suppose radiation of frequency k travels along a three-dimensional spatial path Γ connecting two points P_1 and P_2. Then the opacity difference between these points is $\int_\Gamma d\tau$, where $d\tau$ is the difference for an element of the path. To obtain $d\tau$, let s be three-dimensional length along Γ and let $\mathbf{u}(s)$ be the unit tangent vector at s. The opacity differential $d\tau(k)/ds$ for frequency k is defined in terms of absorption by the equation

$$I(k\mathbf{u})\, d\Omega\, dk\, \frac{d\tau(k)}{ds} = \frac{4\pi^2 e^2}{3\hbar^2}\, d\Omega\, I(k\mathbf{u})\, \overline{|\mathbf{a}_{mn}(k)|^2}\, \hbar k n(k)\, dk, \tag{31}$$

and the opacity difference is

$$\int_{P_1}^{P_2} d\tau(k) = \frac{4\pi^2 e^2}{3\hbar} \cdot k \int_{P_1}^{P_2} \overline{|\mathbf{a}_{mn}(k)|^2}\, n(k)\, ds. \tag{32}$$

Radiation travelling from P_1 and P_2 is reduced in intensity by the factor $\exp[-\int_{P_1}^{P_2} d\tau]$, it being supposed that enough atoms are involved for the absorption probability to be averaged. It may be noted that the factor $\hbar k$ appears on the right hand side of (31) because of $E_n - E_m \simeq \hbar k$, not because of field quantisation.

III. SPONTANEOUS EMISSION

We now turn to the problem of the spontaneous transitions of system a. In Fig. 1 the interaction of a with b is shown. The section $0 \leqslant t \leqslant T$ of the path $\mathbf{a}(t)$ interacts with two sections, Δ_+ and Δ_-, of path $\mathbf{b}(t)$. These are the intercepts on $\mathbf{b}(t)$ between the light cones drawn from $\mathbf{a} = \mathbf{a}(0)$, $t = 0$ and $\mathbf{a} = \mathbf{a}(T)$, $t = T$.

The interaction between Δ_- and the segment of $\mathbf{a}(t)$ is calculated from $-e\int_0^T \mathbf{A}_{\text{ret}}^{(b)}(\mathbf{a}) \cdot \dot{\mathbf{a}}\, dt$, where $A_{\text{ret}}^{(b)}(\mathbf{a})$ is the retarded field of particle b taken at $\mathbf{a}(t)$. The summation of such interactions for all particles b gives the contribution $-e\int_0^T \dot{\mathbf{a}} \cdot \mathbf{A}\, dt$ to $S[\mathbf{a}(t)]$ in (10). The contributions of Δ_- for all b constitutes the external field acting on a, the field investigated in the previous section. We must look therefore to the interaction of Δ_+ with the segment of $\mathbf{a}(t)$ in order to obtain the spontaneous emission. This is given by $-e\int_{\Delta_+} A_{\text{ret}}^{(a)}(\mathbf{b}) \cdot \dot{\mathbf{b}}\, dt$ where $A_{\text{ret}}^{(a)}(\mathbf{b})$ is the retarded field of path $\mathbf{a}(t)$ taken at $\mathbf{b}$.

It may be asked why retarded fields are used to calculate the interactions, both for Δ_- and Δ_+. The answer to this question follows from the discussion of the self-consistent cycle of argument, given in Section I. The net field produced by a at $\mathbf{b}$ is $A_{\text{ret}}^{(a)}(\mathbf{b})$, and $\mathbf{A}_{(b)}^{\text{ret}}(\mathbf{a})$ for b at $\mathbf{a}$. The effects of all particles other than a and b are included in this statement. Moreover, we are dealing with a cosmology in which no other self-consistent cycle of argument is possible.

The points A and B of Fig. 1 are connected by a null line, so that in Minkowski space

$$t^{(a)} = t^{(b)} - r, \tag{33}$$

where $r = |\mathbf{r}|$ is the distance corresponding to the three dimensional vector $\mathbf{r} = A \to B$. The distance r is to be thought of as cosmological, very large compared to atomic dimensions and even large compared to distances over which flat space

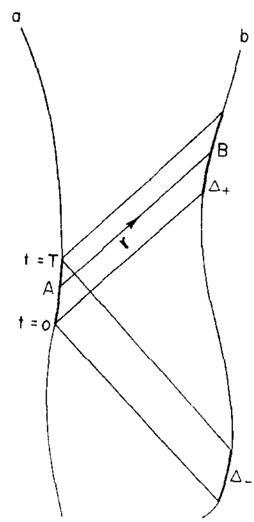

FIG. 1. The section $0 \leqslant t \leqslant T$ of the world line a is in interaction with the sections Δ_- and Δ_+ of the world line b.

considerations are valid. Nevertheless we shall first work out the spontaneous transition formula for a Minkowski space, since this avoids complicating the quantum mechanical considerations with further issues of cosmology—the calculation is easily generalised at the end to include the cosmology.

It is convenient to measure $\mathbf{a}(t)$ relative to an origin in the neighborhood of particle a and to measure $\mathbf{b}(t)$ relative to an origin in the neighbourhood of particle b. Let the two origins be connected $\mathbf{a} \to \mathbf{b}$ by the vector $\mathbf{R}$. Then

$$\mathbf{r} = \mathbf{R} + \mathbf{b} - \mathbf{a} \tag{34}$$

and to sufficient accuracy

$$r = R + \frac{1}{R} \cdot [\mathbf{R} \cdot (\mathbf{b} - \mathbf{a})]. \tag{35}$$

The spontaneous transition probability is given by an expression formally similar to (6),

$$P(m \to n) = \iiiint \phi_n^*(\mathbf{a}_f)\, \phi_n(\mathbf{a}'_f)\, K\phi_m(\mathbf{a}_i)\, \phi_m^*(\mathbf{a}'_i)\, d^3\mathbf{a}_f\, d^3\mathbf{a}'_f\, d^3\mathbf{a}_i\, d^3\mathbf{a}'_i\,, \quad (36)$$

where

$$K = \iint \exp\left[\frac{i}{\hbar}\{S_0[\mathbf{a}(t)] - S_0[\mathbf{a}'(t)]\}\right] F[\mathbf{a}(t), \mathbf{a}'(t)]\, \mathscr{D}^3\mathbf{a}(t)\, \mathscr{D}^3\mathbf{a}'(t), \quad (37)$$

$$F[\mathbf{a}(t), \mathbf{a}'(t)] = \prod_{b \neq a} F^{(b)}[\mathbf{a}(t), \mathbf{a}'(t)], \quad (38)$$

$$F^{(b)}[\mathbf{a}(t), \mathbf{a}'(t)] = \sum_f \iiiint \psi_f^*(\mathbf{b}_f)\, \psi_f(\mathbf{b}'_f)\, J^{(b)}\psi_i(\mathbf{b}_i)\, \psi_i^*(\mathbf{b}'_i)\, d^3\mathbf{b}_f\, d^3\mathbf{b}'_f\, d^3\mathbf{b}_i\, d^3\mathbf{b}'_i\,, \quad (39)$$

$$J^{(b)} = \iint \exp\left\{\frac{i}{\hbar}\{S_0[\mathbf{b}(t)] - S_0[\mathbf{b}'(t)] + S_{\text{int}}[\mathbf{a}(t), \mathbf{b}(t)] - S_{\text{int}}[\mathbf{a}'(t), \mathbf{b}'(t)]\}\right\}$$

$$\cdot\, \mathscr{D}^3\mathbf{b}(t)\, \mathscr{D}^3\mathbf{b}'(t), \quad (40)$$

$$S_{\text{int}}[\mathbf{a}(t), \mathbf{b}(t)] = -e\int_{\Delta_+} \mathbf{A}_{\text{ret}}^{(a)}(\mathbf{b}) \cdot \dot{\mathbf{b}}\, dt. \quad (41)$$

The particle **b** is taken to be in the state ψ_i at the beginning of the segment Δ_+. The summation with respect to f is a summation over a complete set of orthogonal final states for particle b, the set being chosen to include ψ_i. The above expressions follow the notation of Feynman and Hibbs ((9), page 343).

The product (38) assumes that the particles $b \neq a$ contribute independently. We shall return to this point at the end of the paper.

$F^{(b)}$ can be evaluated by the method already considered in the previous section. Before coming to the details we note that because of the cooling effect of the expansion of the universe the state ψ_i can be taken as the ground state for most particles b, so that the problem is essentially one of absorption. When $f = i$ in (39), the main contribution in the expansion

$$\exp\left[\frac{i}{\hbar}\{S_{\text{int}}[\mathbf{a}(t), \mathbf{b}(t)] - S_{\text{int}}[\mathbf{a}'(t), \mathbf{b}'(t)]\}\right] = 1 + \cdots \quad (42)$$

comes from the unity term. Thus

$$F^{(b)}[\mathbf{a}(t), \mathbf{a}'(t)] = 1 + \sum_{f \neq i} \iiiint \psi_f^*(\mathbf{b}_f)\, \psi_f(\mathbf{b}'_f)\, J^{(b)}\psi_i(\mathbf{b}_i)\, \psi_i^*(\mathbf{b}'_i)\, d^3\mathbf{b}_i\, d^3\mathbf{b}'_i\, d^3\mathbf{b}_f\, d^3\mathbf{b}'_f$$

$$+ \text{(second-order term)}. \quad (43)$$

With ψ_i the ground state, the summation $\sum_{f \neq i}$ is wholly concerned with absorption.

We shall not at the present stage work out the second-order term arising from $f = i$, since this term is not required to obtain $P(m \to n)$. In the next section we shall find, however, that it is this second-order term that leads to the level shift correction.

Using the same analysis as that which led to (12), the contribution to $F^{(b)}$ from state ψ_f is easily found to be

$$\frac{e^2}{\hbar^2}\left(\frac{E_f - E_i}{\hbar}\right)^2 M[\mathbf{a}(t)]\, M^*[\mathbf{a}'(t)], \tag{44}$$

where

$$M[\mathbf{a}(t)] = \int_{\Delta_+} \exp\left[\frac{i}{\hbar}(E_f - E_i)\,t\right] dt \int \psi_f^*(\mathbf{b})\, \mathbf{b} \cdot \mathbf{A}_{\text{ret}}^{(a)}(\mathbf{b})\, \psi_i(\mathbf{b})\, d^3\mathbf{b}, \tag{45}$$

$\mathbf{A}_{\text{ret}}^{(a)}$ being calculated for the path $\mathbf{a}(t)$. The functional $M^*[\mathbf{a}'(t)]$ is the complex conjugate of $M[\mathbf{a}(t)]$ except that $\mathbf{A}_{\text{ret}}^{(a)}$ is calculated for the path $\mathbf{a}'(t)$. Next we must obtain $\mathbf{A}_{\text{ret}}^{(a)}$.

If there were no dispersion of frequencies and no absorption in the intergalactic medium we should have

$$\mathbf{A}_{\text{ret}}^{(a)}(\mathbf{b}) = \frac{e\dot{\mathbf{a}}}{r - \mathbf{r} \cdot \dot{\mathbf{a}}}, \tag{46}$$

where $\mathbf{b}$ is taken at $t^{(b)}$ and $\dot{\mathbf{a}}$ at $t^{(a)}$; $t^{(a)}$, $t^{(b)}$ being related by (33). In the Coulomb gauge,[1] which we prefer to use here, we have

$$\mathbf{A}_{\text{ret}}^{(a)}(\mathbf{b}) = \frac{e}{r - \mathbf{r} \cdot \dot{\mathbf{a}}} \sum_{j=1,2} [\dot{\mathbf{a}} \cdot \boldsymbol{\alpha}^{(j)}]\, \boldsymbol{\alpha}^{(j)}, \tag{47}$$

where $\boldsymbol{\alpha}^{(1)}$, $\boldsymbol{\alpha}^{(2)}$ are unit vectors which together with $\mathbf{r}/r$ form a mutually orthogonal triad.

Consider $\mathbf{A}_{\text{ret}}^{(a)}$ at a distance ϵR (from the origin near particle a). Choose $\epsilon \ll 1$ so that there is no cosmological absorption or dispersion but such that ϵR is nevertheless large compared to atomic dimensions. Putting $\mathbf{r} = \epsilon\mathbf{R}$, we can use (47). Expanding in a Fourier series

$$\mathbf{A}_{\text{ret}}^{(a)}(\epsilon\mathbf{R}, t) = \sum_{l=-\infty}^{\infty} \mathbf{A}_l e^{-2\pi i l t/T'}, \tag{48}$$

[1] In the direct particle theory the Lorentz gauge must be used. We can transform to the Coulomb gauge, however, provided the transformation makes no difference to the calculation. In fact $\mathbf{A} = \mathbf{A}' - \nabla\chi, \phi' = \phi + \dot{\chi}$ has the effect in the path integral method of introducing a factor $\exp(ie\,\chi/\hbar)$ in the propagator. For $r \gg |\,\mathbf{a}\,|, |\,\mathbf{b}\,|$, χ is effectively constant over atomic dimensions, so that transformation to the Coulomb gauge only introduces a constant phase factor in the wavefunction.

where

$$\mathbf{A}_l = \frac{e}{\epsilon RT'} \int \frac{\dot{\mathbf{a}}}{1 - \dot{\mathbf{a}} \cdot \mathbf{R}/R} e^{2\pi i l t'/T'} \, dt'. \tag{49}$$

The range T' of t' in (49) corresponds to $0 \leqslant t \leqslant T$ at $\mathbf{a}$ in accordance with

$$t = t' - \epsilon R + \frac{\mathbf{a} \cdot \mathbf{R}}{R}. \tag{50}$$

Since

$$\frac{dt'}{dt} = 1 - \frac{\dot{\mathbf{a}} \cdot \mathbf{R}}{R}, \tag{51}$$

changing the variable from t' to t in (49) gives

$$\mathbf{A}_l = \frac{e \exp(2\pi i \epsilon l R/T')}{\epsilon RT'} \int_0^T \dot{\mathbf{a}} \exp\left[\frac{2\pi i l}{T'}\left(t - \frac{\mathbf{a} \cdot \mathbf{R}}{R}\right)\right] dt. \tag{52}$$

Integration of (51) leads immediately to

$$T' = T - \frac{\mathbf{R}}{R}[\mathbf{a}(T) - \mathbf{a}(0)], \tag{53}$$

so that T' is determined in terms of the segment $0 \leqslant t \leqslant T$ of the path $\mathbf{a}(t)$.

Now put $\epsilon = 1$ in (48) and (52), but introduce a phase correction $\exp(i\chi_l)$ and an absorption factor $\exp(-\tau_l/2)$ for each term in the Fourier series, so that (52) is changed to

$$\mathbf{A}_l = \frac{e}{RT'} \exp\left[i\left(\frac{2\pi l R}{T'} + \chi_l\right) - \frac{1}{2}\tau_l\right] \int_0^T \dot{\mathbf{a}} \exp\left[\frac{2\pi i l}{T'}\left(t - \frac{\mathbf{a} \cdot \mathbf{R}}{R}\right)\right] dt. \tag{54}$$

Then (48), with $\epsilon = 1$, and (54) give $\mathbf{A}_{\text{ret}}^{(a)}$ at the origin near $\mathbf{b}$—i.e. at $\mathbf{R}$ relative to the origin near $\mathbf{a}$. We require the field not at $\mathbf{R}$, however, but at $\mathbf{R} + \mathbf{b}$. All we need do to make this change is to replace R by $R + (\mathbf{R} \cdot \mathbf{b})/R$ in the first exponential factor of (54). Explicitly,

$$\mathbf{A}_{\text{ret}}^{(a)}(\mathbf{b}, t) = \sum_{l=-\infty}^{\infty} \mathbf{A}_l e^{-2\pi i l t/T'}, \tag{55}$$

$$\mathbf{A}_l = \frac{e}{RT'} \exp\left[-\frac{1}{2}\tau_l + i\left\{\frac{2\pi l}{T'}\left(R + \frac{\mathbf{R} \cdot \mathbf{b}}{R}\right) + \chi_l\right\}\right] \times \int_0^T \dot{\mathbf{a}} \exp\left[\frac{2\pi i l}{T'}\left(t - \frac{\dot{\mathbf{a}} \cdot \mathbf{R}}{R}\right)\right] dt. \tag{56}$$

Since χ_l, τ_l only change appreciably over cosmological distances we regard them as constant over the path $\mathbf{b}(t)$.

In Section V, when we come to deal with the cosmological situation, we shall find that χ_l is an exceedingly large angle—indeed that χ_l changes by a very large angle between successive terms, l and $l + 1$, of the Fourier series. The situation is that, while $| \mathbf{A}_l |$ changes only slowly with l, a large jump of the phase of $\mathbf{A}_l$ occurs from one term to the next. This means that products in the components of $\mathbf{A}_l$, $\mathbf{A}_{l'}$, which appear when (55) is substituted in (44), contain large phase angles $\chi_l + \chi_{l'}$, unless $l' = -l$. Moreover $\chi_l + \chi_{l'}$, $l' \neq -l$, also changes by an angle $\gg 2\pi$ as we pass from one absorbing particle to another. This means that although product terms in the components of $\mathbf{A}_l$, $\mathbf{A}_{l'}$, $l' \neq -l$ appear for a single absorbing particle, any such term added for the whole distribution of absorbing particles averages to zero—we have an addition of points distributed randomly on the unit circle in the complex plane.

For the reason just explained we omit all terms $l' \neq -l$ in the double sum, $\sum_l \sum_{l'}$, obtained by substituting (55) in (44). It is easy to show that we then have

$$\frac{1}{3}\frac{e^4}{R^2T'^2}\left(\frac{E_f - E_i}{\hbar}\right)^2 \sum_{l=-\infty}^{\infty} \left| \mathbf{b}_{if}\left(\frac{2\pi l}{T'}\right)\right|^2 e^{-\tau_l} \left| \int_0^{T'} \exp\left\{\frac{i}{\hbar}\left(E_f - E_i - \frac{2\pi l\hbar}{T'}\right)t'\right\} dt' \right|^2$$
$$\cdot \sum_{j=1,2} \int_0^T \boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}} \exp\left[\frac{2\pi l i}{T'}\left(t - \frac{\mathbf{a}\cdot\mathbf{R}}{R}\right)\right] dt$$
$$\cdot \int_0^T \boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}}' \exp\left[\frac{2\pi i l}{T'}\left(\frac{\dot{\mathbf{a}}'\cdot\mathbf{R}}{R} - t\right)\right] dt, \tag{57}$$

in which we have averaged with respect to the orientation of $\mathbf{b}$ in order to remove product terms in the components of $\boldsymbol{\alpha}^{(1)}$, $\boldsymbol{\alpha}^{(2)}$. Because of dispersion the different Fourier components reach $\mathbf{b}(t)$ over different intercepts Δ_+ (cf. Fig. 1). However, dispersion does not affect the time interval T' over which each Fourier component is experienced at $\mathbf{R}$. The time interval over which each Fourier component is experienced along the path $\mathbf{b}(t)$ is not exactly T' but $T' + (\mathbf{R}\cdot\Delta\mathbf{b})/R$, where $\Delta\mathbf{b}$ is the change of $\mathbf{b}$ over Δ_+. But since we are dealing with a cold universe the important paths $\mathbf{b}(t)$ are very nonrelativistic so that $\Delta\mathbf{b}$ can be regarded as small. The range 0 to T' for the variable t' in (57) implies that $\Delta\mathbf{b}$ has been neglected.

Next, we note that

$$\left| \int_0^{T'} \exp\left[\frac{i}{\hbar}\left(E_f - E_i - \frac{2\pi\hbar l}{T'}\right)t'\right] dt' \right|^2 = \frac{T'^2 \sin^2(E_f - E_i)\,T'/2\hbar}{[(E_f - E_i)\,T'/2\hbar - \pi l]^2}$$
$$\simeq 2\pi T'\hbar\delta\left(E_f - E_i - \frac{2\pi l\hbar}{T'}\right). \tag{58}$$

Defining

$$\hbar k = E_f - E_i\,, \qquad \mathbf{k} = k\mathbf{R}/R, \tag{59}$$

the approximate delta-function property of (58) allows (57) to be written in the form

$$\frac{e^4k^2}{3R^2}\, e^{-\tau(k)}\, |\, \mathbf{b}_{if}(k)|^2 \sum_{j=1,2} \int_0^T \boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}} e^{-i\mathbf{k}\cdot\mathbf{a}+ikt}\, dt \int_0^T \boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}}' e^{i\mathbf{k}\cdot\mathbf{a}'-ikt}\, dt$$

$$\cdot \sum_{l=-\infty}^{\infty} \frac{\sin^2(E_f - E_i)\, T'/2\hbar}{[(E_f - E_i)\, T'/2\hbar - \pi l]^2}\,. \tag{60}$$

The last factor in (60) can be omitted since

$$\sum_{l=-\infty}^{\infty} \frac{\sin^2 \alpha}{(\pi l - \alpha)^2} = 1$$

for all α.

Suppose that at distance R in a particular element of solid angle $d\Omega$ there are $n(k)\, dk$ particles per unit volume with states f and i such that $(E_f - E_i)/\hbar$ lies between k and $k + dk$. Writing $\overline{|\, \mathbf{b}_{if}(k)|^2}$ for the average value of $|\, \mathbf{b}_{if}(k)|^2$ for all systems satisfying this requirement (the wavefunctions ψ_f, ψ_i can be different for different b provided $(E_f - E_i)/\hbar$ lies between k and $k + dk$), the contribution to $F[\mathbf{a}(t), \mathbf{a}'(t)]$ in (38) from all absorbers between R and $R + dR$ in $d\Omega$ is

$$[1 + \text{Expression (60)}]^{n(k)dk\cdot R^2 dR d\Omega}. \tag{61}$$

The contribution from each individual absorber is exceedingly small, so that we can regard the index in (61) as being a large number, in which case (61) can be written in the form

$$\exp \Big[\frac{e^4}{3\hbar^2}\, \overline{|\, \mathbf{b}_{if}(k)|^2} \sum_{j=1,2} \int_0^T (\boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}})\, e^{-i\mathbf{k}\cdot\mathbf{a}+ikt}\, dt \int_0^T (\boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}}')\, e^{i\mathbf{k}\cdot\mathbf{a}'-ikt}\, dt$$

$$\cdot\, k^2 \cdot e^{-\tau(k)} n(k)\, dk\, dR\, d\Omega\Big]. \tag{62}$$

The final value of $F[\mathbf{a}(t), \mathbf{a}'(t)]$ is obtained by integrating the exponent of (62) with respect to k, Ω, and R. Consider first the integral with respect to R. The function $\tau(k)$ is just the opacity difference at frequency k taken along the vector $\mathbf{R}$ from $\mathbf{a}$ to $\mathbf{b}$.

From (31)

$$\frac{d\tau(k)}{dk} = \frac{4\pi^2}{3} \cdot \frac{e^2 k}{\hbar}\, \overline{|\, \mathbf{b}_{if}(k)|^2}\, n(k),$$

and (62) can be rewritten as

$$\exp\left[\frac{e^2}{4\pi^2\hbar}\sum_{j=1,2}\int_0^T \boldsymbol{\alpha}^{(j)}\cdot\dot{\mathbf{a}}e^{-i\mathbf{k}\cdot\mathbf{a}+ikt}\,dt\int_0^T \boldsymbol{\alpha}^{(j)}\cdot\dot{\mathbf{a}}'e^{i\mathbf{k}\cdot\mathbf{a}'-ikt}\,dt\right.$$
$$\left.\cdot\, k\,dk\,d\Omega\,e^{-\tau(k)}\frac{d\tau(k)}{dR}\,dR\right]. \tag{63}$$

The introduction of the opacity has removed $\overline{|\mathbf{b}_{if}(k)|^2}$.

From (63) it is seen that integration with respect to R has the effect of an integration with respect to τ. We have to consider $\int_0 e^{-\tau}\,d\tau$, the lower limit of the integral being zero since τ is defined to be zero at $R = 0$. For a completely absorbing universe the upper limit of the integral is ∞ for all k. Hence for such a universe the integral is unity for all k. Integrating finally with respect to Ω and k, we obtain

$$F[\mathbf{a}(t), \mathbf{a}'(t)] = \exp\left[\frac{e^2}{4\pi^2\hbar}\int d\Omega\int_0^\infty k\,dk\sum_{j=1,2}\int_0^T (\boldsymbol{\alpha}_{\mathbf{k}}^{(j)}\cdot\dot{\mathbf{a}})\,e^{-i\mathbf{k}\cdot\mathbf{a}+ikt}\,dt\right.$$
$$\left.\cdot\int_0^T (\boldsymbol{\alpha}_{\mathbf{k}}^{(j)}\cdot\dot{\mathbf{a}}')\,e^{i\mathbf{k}\cdot\mathbf{a}-ikt}\,dt\right]. \tag{64}$$

The subscript $\mathbf{k}$ has been added to $\boldsymbol{\alpha}^{(j)}$ since we are now integrating with respect to Ω, and the vectors $\boldsymbol{\alpha}^{(j)}$ change as $d\Omega$ changes—$\mathbf{k}$ is a vector in $d\Omega$.

The last part of the calculation is similar to the procedure that led to (12). Expand the exponential in (64) and retain only the first-order term in $e^2/\hbar$. Previously we had

$$\frac{e^2}{\hbar^2}\int_0^T \dot{\mathbf{a}}\cdot\mathbf{A}\,dt\int_0^T \dot{\mathbf{a}}'\cdot\mathbf{A}\,dt, \tag{65}$$

in which $\mathbf{A}$ was a specified field. Now we have the exponent of (64). Noting that if $\mathbf{A}$ in (65) had not been a real field

$$\frac{e^2}{\hbar^2}\int_0^T \dot{\mathbf{a}}\cdot\mathbf{A}\,dt\int_0^T \dot{\mathbf{a}}'\cdot\mathbf{A}^*\,dt \tag{66}$$

would still have led to (12), we see that provided we replace $e^2/\hbar^2$ by

$$\frac{e^2}{4\pi^2\hbar}\int d\Omega\int_0^\infty k\,dk\sum_{j=1,2},$$

and provided we write $\mathbf{A} = \boldsymbol{\alpha}_{\mathbf{k}}^{(j)} e^{-i\mathbf{k}\cdot\mathbf{a}+ikt}$, the present case is the same as the previous one. We obtain

$$P(m \to n)$$

$$= \frac{e^2}{4\pi^2\hbar}\left(\frac{E_m - E_n}{\hbar}\right)^2 \int d\Omega \int_0^\infty k\, dk \sum_{j=1,2} \left| \int \phi_n^*(\mathbf{a})\, \mathbf{a} \cdot \boldsymbol{\alpha}_{\mathbf{k}}^{(j)} e^{-i\mathbf{k}\cdot\mathbf{a}} \phi_m(\mathbf{a})\, d^3\mathbf{a} \right|^2$$

$$\cdot \left| \int_0^T \exp\left[\frac{i}{\hbar}(E_n - E_m + \hbar k)\, t\right] dt \right|^2. \tag{67}$$

Using (15), we see that $E_m > E_n$ is necessary to obtain a nonzero result, and that the spontaneous emission probability per unit time is

$$\frac{e^2}{2\pi\hbar}\left(\frac{E_m - E_n}{\hbar}\right)^3 \int d\Omega \int_0^\infty dk$$

$$\cdot \sum_{k=1,2} \left| \int \phi_n^*(\mathbf{a})\, \mathbf{a} \cdot \boldsymbol{\alpha}_k^{(j)} e^{-ik\cdot\mathbf{a}} \phi_m(\mathbf{a})\, d^3\mathbf{a} \right|^2 \delta\left(k - \frac{E_m - E_n}{\hbar}\right) \tag{68}$$

in agreement with the usual expression.

In order to compare (68) with the stimulated emission rate given by (27), average (68) with respect to solid angle, giving

$$\frac{e^2}{3\pi\hbar}\left(\frac{E_m - E_n}{\hbar}\right)^3 \int_0^\infty dk\, |\mathbf{a}_{mn}(k)|^2\, \delta\left(k - \frac{E_m - E_n}{\hbar}\right) \int d\Omega. \tag{69}$$

Now equate the contribution from $d\Omega$ to (27) to $\bar{q}(\mathbf{k})$ times the contribution from $d\Omega$ to (69). This gives the following definition of $\bar{q}(\mathbf{k})$

$$\bar{q}(\mathbf{k}) = \frac{1}{\hbar}\left(\frac{2\pi}{k}\right)^3 \cdot \frac{1}{2} \sum_{j=1,2} I^{(j)}(\mathbf{k}), \tag{70}$$

where $\hbar k = E_m - E_n$ and $I(\mathbf{k})$ is separated into the two polarizations defined in (29). In the usual quantum electrodynamics (70) is the relation between the field intensity and the average number of quanta per vacuum oscillator in the frequency range k to $k + dk$. Although quanta do not appear explicitly in the present theory, it is interesting that we obtain the usual formulae by taking the spontaneous transition rate as a reference standard. It follows that, if $I(\mathbf{k})$ were to have the value appropriate to a thermodynamic radiation field at temperature T, $\bar{q}(\mathbf{k})$ would follow Planck's law,

$$\bar{q}(\mathbf{k}) \equiv \bar{q}(k) = \frac{1}{\exp\left(\frac{\hbar k}{T}\right) - 1} \tag{71}$$

in which the temperature scale has been chosen so that the Boltzmann constant is unity.

The delta-function in (68) gives an asymmetry between emission and absorption. Spontaneous transitions are downward because we have taken the absorber particles as being in their ground levels, $E_i \leqslant E_f$ for all f. We see therefore that the asymmetry of spontaneous emission arises from the assumption of a cold universe. We shall return to this point in Section V.

IV. THE LEVEL SHIFT FORMULA

We return to (43) and work out the second-order term belonging to $f = i$. From (39) and (40) the second-order term is easily found to be

$$
\begin{aligned}
&- \frac{1}{2\hbar^2} \iiint S_{\text{int}}^2[\mathbf{a}(t), \mathbf{b}(t)] \exp\{iS_0[\mathbf{b}(t)]/\hbar\}\, \psi_i^*(\mathbf{b}_f)\, \psi_i(\mathbf{b}_i)\, d^3\mathbf{b}_f\, d^3\mathbf{b}_i \mathscr{D}^3\mathbf{b}(t), \\
&- \frac{1}{2\hbar^2} \iiint S_{\text{int}}^2[\mathbf{a}'(t), \mathbf{b}'(t)] \exp\{-iS_0[\mathbf{b}'(t)]/\hbar\}\, \psi_i(\mathbf{b}_f')\, \psi_i^*(\mathbf{b}_i')\, d^3\mathbf{b}_f'\, d^3\mathbf{b}_i' \mathscr{D}^3\mathbf{b}'(t).
\end{aligned} \tag{72}
$$

It will be sufficient to work out the first of these terms, since the second can then be written down by inspection.

Inserting (55) for $\mathbf{A}_{\text{ret}}^{(a)}$ in the expression (41) for S_{int} leads to

$$
\begin{aligned}
&- \frac{e^2}{2\hbar^2} \sum_{l=-\infty}^{\infty} \iiint\int_0^{T'} (\mathbf{A}_l \cdot \dot{\mathbf{b}})\, e^{-2\pi i l t'/T}\, dt' \int_0^{T'} (\mathbf{A}_l^* \cdot \dot{\mathbf{b}})\, e^{2\pi i l t'/T}\, dt' \\
&\quad \cdot \exp\{iS_0[\mathbf{b}(t)]/\hbar\}\, \psi_i^*(\mathbf{b}_f)\, \psi_i(\mathbf{b}_i)\, d^3\mathbf{b}_f\, d^3\mathbf{b}_i \mathscr{D}^3\mathbf{b}(t).
\end{aligned} \tag{73}
$$

Using ordinary perturbation methods for second-order transitions, (73) can be reduced to

$$
\begin{aligned}
&- \frac{e^2}{\hbar^2} \sum_{l=-\infty}^{\infty} \sum_g \left(\frac{E_g - E_i}{\hbar}\right)^2 \left| \int \psi_i^*(\mathbf{b})(\mathbf{b} \cdot \mathbf{A}_l)\, \psi_g(\mathbf{b})\, d^3\mathbf{b} \right|^2 \\
&\quad \cdot \int_0^{T'} \exp\left[\frac{i}{\hbar}\left(E_i - E_g - \frac{2\pi l\hbar}{T'}\right) t'\right] dt' \int_0^{t'} \exp\left[\frac{i}{\hbar}\left(E_g - E_i + \frac{2\pi l\hbar}{T'}\right) \tilde{t}'\right] d\tilde{t}',
\end{aligned} \tag{74}
$$

where the summation with respect to g is over all intermediate states of b.

Now insert (56) for $\mathbf{A}_l$ in (74), to give

$$
\begin{aligned}
&-\frac{e^2}{3\hbar^2R^2T'^2}\sum_g\left(\frac{E_g-E_i}{\hbar}\right)^2\sum_{l=-\infty}^{\infty}|\mathbf{b}_{ig}(2\pi l/T')|^2\,e^{-\tau_l}\\
&\cdot\int_0^{T'}\exp\left[\frac{i}{\hbar}\left(E_i-E_g-\frac{2\pi\hbar l}{T'}\right)t'\right]dt'\int_0^{t'}\exp\left[\frac{i}{\hbar}\left(E_g-E_i+\frac{2\pi\hbar l}{T'}\right)\tilde{t}'\right]d\tilde{t}'\\
&\cdot\sum_{j=1,2}\int_0^T\boldsymbol{\alpha}^{(j)}\cdot\dot{\mathbf{a}}\exp\left[\frac{2\pi il}{T'}\left(t-\frac{\mathbf{a}\cdot\mathbf{R}}{R}\right)\right]dt\\
&\cdot\int_0^T\boldsymbol{\alpha}^{(j)}\cdot\dot{\mathbf{a}}\exp\left[-\frac{2\pi il}{T'}\left(\tilde{t}-\frac{\mathbf{a}\cdot\mathbf{R}}{R}\right)\right]d\tilde{t}
\end{aligned}
\tag{75}
$$

after averaging with respect to the orientation of b.

Although (75) appears complicated we shall find simplifications. For fixed ψ_g, the main contributions come from $2\pi l\hbar/T' \simeq E_i - E_g$. Only when this condition is satisfied can the integrals in the second line of (75) yield a contribution that behaves as T'^2. Defining

$$\mathbf{k}=\frac{E_i-E_g}{\hbar}\cdot\frac{\mathbf{R}}{R},\tag{76}$$

we can therefore write

$$
\begin{aligned}
&-\frac{e^2}{3\hbar^2R^2T'^2}\sum_g k^2\,|\mathbf{b}_{ig}(k)|^2\cdot e^{-\tau(k)}\int_0^{T'}e^{-ikt'}\,dt'\int_0^{t'}e^{ik\tilde{t}'}\,d\tilde{t}'\\
&\cdot\sum_{j=1,2}\int_0^T\boldsymbol{\alpha}^{(j)}\cdot\dot{\mathbf{a}}e^{-i\mathbf{k}\cdot\mathbf{a}}\,dt\int_0^T\boldsymbol{\alpha}^{(j)}\cdot\dot{\mathbf{a}}e^{i\mathbf{k}\cdot\mathbf{a}}\,d\tilde{t}\sum_{l=-\infty}^{\infty}\exp\left[\frac{2\pi il}{T'}(t-\tilde{t}-t'+\tilde{t}')\right].
\end{aligned}
\tag{77}
$$

Furthermore

$$\sum_{l=-\infty}^{\infty}\exp\left[\frac{2\pi il}{T'}(t-\tilde{t}-t'+\tilde{t}')\right]=T'\delta(t-\tilde{t}-t'+\tilde{t}').\tag{78}$$

We must therefore have $t \geqslant \tilde{t}$ since $t' \geqslant \tilde{t}'$, and (77) becomes

$$-\frac{e^2}{3\hbar^2R^2}\sum_g k^2e^{-\tau(k)}\,|\mathbf{b}_{ig}(k)|^2\sum_{j=1,2}\int_0^T\boldsymbol{\alpha}^{(j)}\cdot\dot{\mathbf{a}}e^{-i\mathbf{k}\cdot\mathbf{a}-ikt}\,dt\int_0^t\boldsymbol{\alpha}^{(j)}\cdot\dot{\mathbf{a}}e^{i\mathbf{k}\cdot\mathbf{a}+ik\tilde{t}}\,d\tilde{t}.\tag{79}$$

This contribution to $F^{(b)}$ must be added to (60).

As before it is sufficient to consider a single state g, since all states were automatically included in the discussion that followed (60). Indeed the summation with respect to all absorbers proceeds exactly as before, and in place of (64) we now have

$$F[\mathbf{a}(t), \mathbf{a}'(t)] = \exp \Big[\frac{e^2}{4\pi^2\hbar} \int d\Omega \int_0^\infty k\, dk$$

$$\times \sum_{j=1,2} \Big\{ \int_0^T (\boldsymbol{\alpha}_{\mathbf{k}}^{(j)} \cdot \dot{\mathbf{a}})\, e^{-i\mathbf{k}\cdot\mathbf{a}+ikt}\, dt \int_0^T (\boldsymbol{\alpha}_{\mathbf{k}}^{(j)} \cdot \dot{\mathbf{a}}')\, e^{+i\mathbf{k}\cdot\mathbf{a}'-ikt'}\, dt'$$

$$- \int_0^T (\boldsymbol{\alpha}_{\mathbf{k}}^{(j)} \cdot \dot{\mathbf{a}})\, e^{-i\mathbf{k}\cdot\mathbf{a}-ikt}\, dt \int_0^t (\boldsymbol{\alpha}_{\mathbf{k}}^{(j)} \cdot \dot{\mathbf{a}})\, e^{i\mathbf{k}\cdot\mathbf{a}+ik\tilde{t}}\, d\tilde{t}$$

$$- \int_0^T (\boldsymbol{\alpha}_{\mathbf{k}}^{(j)} \cdot \dot{\mathbf{a}}')\, e^{-i\mathbf{k}\cdot\mathbf{a}'+ikt}\, dt \int_0^t (\boldsymbol{\alpha}_{\mathbf{k}}^{(j)} \cdot \dot{\mathbf{a}}')\, e^{i\mathbf{k}\cdot\mathbf{a}'-ik\tilde{t}}\, d\tilde{t} \Big\} \Big], \tag{80}$$

in which we have also included the second term of (72).

The expression (80) is an influence functional and it obeys the general rules discussed by Feynman and Hibbs ((*9*), page 347). If we identify the paths, $\mathbf{a}(t) \equiv \mathbf{a}'(t)$, the exponent in (80) vanishes. The paths do not act on themselves via the response of the Universe.

The new terms in (80) have no effect on the calculation of $P(m \to n)$, $m \neq n$, but they are necessary to obtain $P(m \to m)$. We now show that

$$P(m \to m) = 1 - \sum_{n \neq m} P(m \to n). \tag{81}$$

$P(m \to m)$ is given by writing m for n in (36). Expanding the exponential in (80) to first order we have

$$P(m \to m) = 1 - \frac{e^2}{4\pi^2\hbar} \iiint \phi_m^*(\mathbf{a}_f) \exp \Big\{\frac{i}{\hbar} S_0[\mathbf{a}(t)]\Big\}\, \phi_m(\mathbf{a}_i) \int d\Omega \int_0^\infty k\, dk$$

$$\cdot \sum_{j=1,2} \int_0^T (\dot{\mathbf{a}} \cdot \boldsymbol{\alpha}_{\mathbf{k}}^{(j)})\, e^{-i\mathbf{k}\cdot\mathbf{a}-ikt}\, dt \int_0^t (\dot{\mathbf{a}} \cdot \boldsymbol{\alpha}_{\mathbf{k}}^{(j)})\, e^{i\mathbf{k}\cdot\mathbf{a}+ik\tilde{t}}\, d\tilde{t}\, \mathscr{D}^3\mathbf{a}(t)\, d^3\mathbf{a}_f\, d^3\mathbf{a}_i$$

$$- \frac{e^2}{4\pi^2\hbar} \iiint \phi_m(\mathbf{a}_f') \exp \Big\{-\frac{i}{\hbar} S_0[\mathbf{a}'(t)]\Big\}\, \phi_m^*(\mathbf{a}_i') \int d\Omega \int_0^\infty k\, dk$$

$$\cdot \sum_{j=1,2} \int_0^T (\dot{\mathbf{a}}' \cdot \boldsymbol{\alpha}_{\mathbf{k}}^{(j)})\, e^{-i\mathbf{k}\cdot\mathbf{a}'+ikt}\, dt$$

$$\cdot \int_0^t (\dot{\mathbf{a}}' \cdot \boldsymbol{\alpha}_{\mathbf{k}}^{(j)})\, e^{i\mathbf{k}\cdot\mathbf{a}'-ik\tilde{t}}\, d\tilde{t}\, \mathscr{D}^3\mathbf{a}'(t)\, d^3\mathbf{a}_f'\, d^3\mathbf{a}_i' . \tag{82}$$

The term involving $\mathbf{a}(t)$ separates from the term involving $\mathbf{a}'(t)$. The path integrals are not hard to evaluate. We obtain

$$P(m \to m) = 1 - \frac{e^2}{4\pi^2\hbar} \sum_{j=1,2} \int d\Omega \int_0^\infty k\, dk$$
$$\cdot \sum_n \left(\frac{E_n - E_m}{\hbar}\right)^2 \left| \int \phi_m^*(\mathbf{a})\, \boldsymbol{\alpha}_{\mathbf{k}}^{(j)} \cdot \mathbf{a} e^{-i\mathbf{k}\cdot\mathbf{a}} \phi_n(\mathbf{a})\, d^3\mathbf{a} \right|^2$$
$$\cdot \int_0^T dt \int_0^t \Big[\exp\Big\{\frac{i}{\hbar}(E_m - E_n - \hbar k)(t - \tilde{t})\Big\}$$
$$+ \exp\Big\{\frac{i}{\hbar}(E_m - E_n - \hbar k)(\tilde{t} - t)\Big\}\Big]\, d\tilde{t}. \tag{83}$$

The term in $\exp\{i/\hbar(E_m - E_n - \hbar k)(t - \tilde{t})\}$ in (83) comes from the quadratic term in the path $\mathbf{a}(t)$ in $F[\mathbf{a}(t), \mathbf{a}'(t)]$, and the term in $\exp\{i/\hbar(E_m - E_n - \hbar k)(\tilde{t} - t)\}$ comes from the quadratic term in $\mathbf{a}'(t)$.

The last two integrals of (83) give

$$2 \int_0^T \frac{\sin[(E_m - E_n)/\hbar - k]\, t}{(E_m - E_n)/\hbar - k}\, dt \cong 2\pi T \delta\left(k - \frac{E_m - E_n}{\hbar}\right). \tag{84}$$

Hence (83) is just $1 - \sum_{E_n < E_m} P(m \to n)$. Since $P(m \to n)$ is zero for $E_n > E_m$ we obtain (81). Probability is therefore conserved.

The system a is by hypothesis in the state m at $t = 0$. The effect of the response of the universe is to change the amplitude for the system to be in the state m at time T from $\phi_m \exp(-iE_m T/\hbar)$ to

$$\phi_m \exp\left[-\frac{i}{\hbar}(E_m + \Delta E_m) - \frac{1}{2}\gamma\right] T \cong \left(1 - \frac{\gamma T}{2} - \frac{i\Delta E_m T}{\hbar}\right) \phi_m e^{-iE_m T/\hbar} \tag{85}$$

for T not too large. The probability of the system being in the state m at time T is therefore

$$P(m \to m) \cong \left(1 - \frac{\gamma T}{2} - \frac{i\Delta E_m T}{\hbar}\right)\left(1 - \frac{\gamma T}{2} + \frac{i\Delta E_m T}{\hbar}\right) \tag{86}$$
$$= 1 - \left(\frac{\gamma}{2} + \frac{i\Delta E_m}{\hbar}\right) T - \left(\frac{\gamma}{2} - \frac{i\Delta E_m}{\hbar}\right) T + O(\Delta E_m^2). \tag{86}$$

Suppose we identify the second and third terms on the right hand side of (86) with the second and third terms on the right hand side of (82). Then

$$\left(\frac{1}{2}\gamma + i\frac{\Delta E_m}{\hbar}\right) T = \frac{e^2}{4\pi^2\hbar} \iiint \phi_m^*(\mathbf{a}_f)\, \phi_m(\mathbf{a}_i) \exp\left\{\frac{i}{\hbar} S_0[\mathbf{a}(t)]\right\} \int d\Omega \int_0^\infty k\, dk$$
$$\cdot \sum_{j=1,2} \int_0^T \dot{\mathbf{a}} \cdot \boldsymbol{\alpha}_{\mathbf{k}}^{(j)} e^{-i\mathbf{k}\cdot\mathbf{a}-ikt}\, dt \int_0^t \dot{\mathbf{a}} \cdot \boldsymbol{\alpha}_{\mathbf{k}}^{(j)} e^{i\mathbf{k}\cdot\mathbf{a}+ik\bar{t}}\, d\bar{t}\, \mathscr{D}^3\mathbf{a}(t)\, d^3\mathbf{a}_f\, d^3\mathbf{a}_i . \tag{87}$$

Care is needed in evaluating the path integral because a "cross-over" term arises at $t = \bar{t}$ ((9), page 191). This term yields

$$\frac{ie^2T}{\pi m} \int_0^\infty k\, dk, \tag{88}$$

while the main term in the reduction is

$$\frac{e^2}{4\pi^2\hbar} \sum_{j=1,2} \int d\Omega \int_0^\infty k\, dk \sum_n \left(\frac{E_n - E_m}{\hbar}\right)^2 \left| \int \phi_n^*(\mathbf{a})\, \mathbf{a} \cdot \boldsymbol{\alpha}_{\mathbf{k}}^{(j)} e^{-i\mathbf{k}\cdot\mathbf{a}} \phi_m(\mathbf{a})\, d^3\mathbf{a} \right|^2$$
$$\cdot \int_0^T dt \int_0^t \exp\left\{\frac{i}{\hbar}(E_m - E_n - \hbar k)(t - \bar{t})\right\} d\bar{t}$$
$$= \frac{1}{2} \sum_{E_n<E_m} P(m \to n) + \frac{ie^2}{4\pi^2\hbar} \sum_{j=1,2} \int d\Omega \int_0^\infty k\, dk$$
$$\cdot \sum_n \left(\frac{E_n - E_m}{\hbar}\right)^2 \left| \int \phi_n^*(\mathbf{a})\, \mathbf{a} \cdot \boldsymbol{\alpha}_{\mathbf{k}}^{(j)} e^{-i\mathbf{k}\cdot\mathbf{a}} \phi_m(\mathbf{a})\, d^3\mathbf{a} \right|^2$$
$$\cdot \int_0^T dt \int_0^t \sin[(E_m - E_n - \hbar k)(t - \bar{t})/\hbar]\, d\bar{t}. \tag{89}$$

Collecting terms

$$\frac{\gamma}{2} + \frac{i}{\hbar}\Delta E_m = \frac{1}{2T} \sum_{E_n<E_m} P(m \to n) + \frac{ie^2}{\pi m} \int_0^\infty k\, dk$$
$$+ \frac{ie^2}{4\pi^2} \sum_n \left(\frac{E_m - E_n}{\hbar}\right)^2 \sum_{j=1,2} \text{P.P.} \int_0^\infty \frac{k\, dk}{E_m - E_n - \hbar k}$$
$$\cdot \left| \int \phi_n^*(\mathbf{a})\, \boldsymbol{\alpha}_{\mathbf{k}}^{(j)} \cdot \mathbf{a} e^{-i\mathbf{k}\cdot\mathbf{a}} \phi_m(\mathbf{a})\, d^3\mathbf{a} \right|^2. \tag{90}$$

The summation in the last term is not restricted to $E_n < E_m$.

The imaginary part of (90) gives the usual expression for the level shift, while γ is the usual expression for the natural width of the state m. In the present theory both the level shift and the width arise from the response of the Universe.

Both parts of the level shift formula diverge as $k \to \infty$. It might at first sight seem disturbing to find such a result appearing in the direct particle theory, since it has always been supposed that if the direct particle theory could be made to "work," in the sense of producing the practical results of quantum electrodynamics, no divergence would be encountered. Here we seem to have recovered the usual formulae only too well—divergences and all. However, to obtain (90) we made an association that is not logically demanded by the theory, namely we identified *separately* the second and third terms on the right hand side of (82) with the second and third terms on the right hand side of (86). Strictly we can only identify the right hand sides of these equations as a whole. No divergence then arises. Our separation of these terms has been given here in order to set out the logical position of the level shift formula in the present theory. Only when a practical result involving the level shift is calculated can the question of divergence arise. We shall return to this issue in paper II.

V. COSMOLOGICAL CONSIDERATIONS AND THE PHYSICAL NATURE OF THE ABSORBER

We have assumed implicitly in the above work that we are dealing with the steady-state cosmology, the line element for which in the usual notation is

$$ds^2 = d\tau^2 - \exp(2H\tau)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)]. \tag{91}$$

Writing

$$Ht = 1 - \exp(-H\tau) \tag{92}$$

enables (91) to be expressed in the conformally flat form

$$ds^2 = \exp(2H\tau)[dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)]. \tag{93}$$

Any constant on the right hand side of (92) permits the same reduction—the unit constant has been chosen so that $t = 0$ corresponds to $\tau = 0$. The range $0 \leqslant \tau < \infty$ then corresponds to $0 \leqslant t < H^{-1}$. The Hubble constant H is related to the average mass density ρ by

$$\rho = 3H^2/4\pi G, \tag{94}$$

where G is the gravitational constant. The observationally determined value of H^{-1} is $\sim 3.10^{17}$ sec, and with this value for H (94) gives $\rho \cong 4.10^{-29}\, \text{g} \cdot \text{cm}^{-3}$.

Since Maxwell's equations are conformally invariant, the electromagnetic field radiated in the cosmological case has exactly the same mathematical form in the r, θ, ϕ, t coordinates as the field radiated in the case of the Minkowski metric

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2). \tag{95}$$

This means that the potentials of the radiated field for (93) can differ from the potentials for (95) only to the extent of a gauge transformation. But we have already seen that a gauge transformation has no importance in our problem (cf. footnote 1—the gauge function χ must not vary appreciably through atomic dimensions but this is certainly the case here). It follows that working with (95), which is what we have done so far, is equivalent to working with (93). Hence all our work can be taken over to the cosmological case.

There is an interesting point of interpretation to be noted, however. In considering the spontaneous transition probability $P(m \to n)$, $E_m > E_n$, the important frequencies in the influence functional were those for which $\hbar k \simeq E_m - E_n$. Now the frequency k in the influence functional (80) comes from absorbers with $\hbar k \simeq E_f - E_i$. Hence the important frequencies come from absorbers with $E_f - E_i \simeq E_m - E_n$. This correspondence applies in the r, θ, ϕ, t coordinates. The relevant point here is that while the t coordinate represents proper time at particle a ($t \simeq \tau$ for $t \simeq 0$) t does not represent proper time for a cosmologically distant absorbing system. τ gives proper time in all cases. Hence the frequency $(E_f - E_i)/\hbar$ appropriate to our analysis is not proper frequency, but is related to proper frequency by

$$\begin{aligned} \frac{1}{\hbar}(E_f - E_i) &= \exp(H\tau) \quad \text{(Proper frequency)} \\ &= (1 - Ht)^{-1} \quad \text{(Proper frequency).} \end{aligned} \tag{96}$$

An absorbing system with coordinate r is reached by radiation from $r = 0$, $t \simeq 0$ at time $t \simeq r$. Hence the appropriate value of $(E_f - E_i)/\hbar$ to be used for such an absorber is not the usual proper frequency but the proper frequency increased by the factor $(1 - Hr)^{-1}$. It follows that two similar absorbers—i.e. with the same proper frequency—contribute different values of $E_f - E_i$ if their r coordinates are different. It is therefore possible for an identical set of absorbing systems to contribute all frequencies greater than their common proper frequency simply by being placed at suitably different values of r. In particular, absorbers with a sufficiently low proper frequency can contribute all values that are of interest in radiation problems.

So far we have been concerned with relating our previous work to the cosmological case and for this it is convenient to use the t coordinate. However in order to discuss the likely physical nature of the absorbers themselves it is preferable to use the τ coordinate, since τ gives proper time for all absorbers. From the point of view of an observer at rest with respect to an absorbing system the frequencies of the field radiated from $r = 0$, $t = 0$ are *lowered* by the factor

$$\exp(-H\tau) = 1 - Hr. \tag{97}$$

Hence an absorbing system of proper frequency ω is in tune with a radiated frequency $k(>\omega)$ provided the system has an r-coordinate satisfying

$$(1 - Hr)\, k = \omega. \tag{98}$$

No matter how small ω/k may be (98) can always be satisfied for r near enough to H^{-1}.

It is important in relation to (98) to notice that r does not measure proper spatial distance. The proper spatial distance dl corresponding to dr is given by

$$dl = \exp(H\tau) \cdot dr = (1 - Hr)^{-1}\, dr. \tag{99}$$

The variation $d\omega$ of the absorber in tune with radiation of original frequency k produced by the coordinate displacement dr is

$$d\omega = -Hk\, dr = -\omega H(1 - Hr)^{-1}\, dr = -\omega H\, dl. \tag{100}$$

In general, absorbing systems will operate not just at one frequency but over some frequency range. Then (100) gives the corresponding range of the r coordinate over which the absorber can be effective, and also the corresponding range dl. The range dl is important because the number of effective absorbers along the line of sight is obtained by multiplying dl by the proper density of the absorbers in question, and by then integrating with respect to l.

If the absorbers were to operate over a frequency range

$$0 < \omega_{\text{min}} \leqslant \omega \leqslant \omega_{\text{max}} < k,$$

the corresponding range of l would be given by

$$l = \int dl = H^{-1} \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \frac{d\omega}{\omega} = H^{-1} \ln \frac{\omega_{\text{max}}}{\omega_{\text{min}}}. \tag{101}$$

Since we require the total absorption to be complete, l cannot be finite. Since moreover we cannot increase the absorption by increasing ω above k, we must evidently have $\omega_{\text{min}} \to 0$. So long as the absorption probability per absorber remains finite as $\omega \to 0$ we then have complete absorption. This requirement is satisfied by collisional absorption in an ionised gas.

It is sufficient to treat the absorption classically, since we shall be concerned with low proper frequencies. The classical attenuation factor for a wave of frequency ω travelling a distance dl is

$$\exp\left[-\frac{2\pi\nu_{\text{eff}}Ne^2}{m\omega^2}\,dl\right], \tag{102}$$

where N is the free electron density and ν_{eff} is the effective collision frequency, given for ionised hydrogen by

$$\nu_{\text{eff}} = 2\pi Nv\left(\frac{e^2}{mv^2}\right)^2 \ln\left(\frac{mv^2}{\hbar\omega}\right), \tag{103}$$

v being an appropriate average for the electron velocity.

Using (98) and (100), (102) can be written in the form

$$\exp\left[-\frac{2\pi\nu_{\text{eff}}Ne^2}{mk^2}\cdot\frac{dr}{(1-Hr)^3}\right]. \tag{104}$$

Neglecting the weak dependence of ν_{eff} on ω, the total attenuation for $0 \leqslant r \leqslant R$ is

$$\exp\left[-\frac{2\pi\nu_{\text{eff}}Ne^2}{mk^2}\int_0^R\frac{dr}{(1-Hr)^3}\right]. \tag{105}$$

The absorption is evidently complete so long as N remains finite—as it does in the steady-state cosmology—as $R \to H^{-1}$. We obtained $\rho \cong 4.10^{-29}$ g. cm^{-3} from (94). In the form of hydrogen, the main constituent of the Universe, the average density is about 2.10^{-5} atoms cm^{-3}. The value of N must be somewhat lower than this, to allow for neutral atoms and for condensed matter, but N is unlikely to be much less then 10^{-6} cm^{-3}.

As R increases, an appreciable fraction of the absorption has occurred when

$$(1 - HR)^2 \cong \frac{2\pi\nu_{\text{eff}}Ne^2}{mk^2H}, \tag{106}$$

i.e. at a frequency ω_{eff} given by

$$\omega_{\text{eff}}^2 \cong \frac{2\pi\nu_{\text{eff}}Ne^2}{mH}. \tag{107}$$

Putting $v \cong 1/300$ (of the velocity of light), $e^2/m \cong 2.8 \cdot 10^{-13}$ cm, $N \cong 10^{-5}$ cm^{-3}, $H^{-1} \cong 10^{28}$ cm, in (103) and (107) gives

$$\omega_{\text{eff}} \cong 10^6 \text{ sec}^{-1}, \qquad \nu_{\text{eff}} \cong 10^{-10} \text{ sec}^{-1}, \tag{108}$$

where we have used the fact that 1 sec $= 3.10^{10}$ cm. From (100) we also see that ω remains of order ω_{eff} over a range $\sim H^{-1} \cong 10^{28}$ cm of l.

In what may be more familiar terms, the original frequency is red-shifted by the expansion of the Universe until it falls to $\omega_{\text{eff}} \cong 10^6\ \text{sec}^{-1}$. Then effective absorption occurs over a proper distance of $\sim 10^{28}$ cm. This applies to all initial values of k, except very low values $< \sim\omega_{\text{eff}}$, for which absorption takes place immediately at proper distances $< \sim 10^{28}$ cm.

We turn next to the phase shift problem discussed in the paragraph following (56). The sum of phases $\chi_l + \chi_{l'}$, $l' \neq -l$ was stated to be a very large angle, and we shall now show that this is so. From (107) we have

$$\frac{4\pi Ne^2}{m\omega_{\text{eff}}^2} \cong \frac{2H}{\nu_{\text{eff}}} \cong 10^{-7}. \tag{109}$$

Since the real part of the refractive index n is given by

$$n = \left(1 - \frac{4\pi Ne^2}{m\omega^2}\right)^{1/2} \cong 1 - \frac{2\pi Ne^2}{m\omega^2}, \tag{110}$$

it follows that $1 - n \cong 10^{-7}$ when $\omega \cong \omega_{\text{eff}}$. Consider

$$k = \frac{2\pi l}{T'}, \qquad k + \Delta k = \frac{2\pi(l+1)}{T'}. \tag{111}$$

For an observer at rest relative to an absorber, two such Fourier components have frequencies ω and $\omega + \Delta\omega$ such that

$$\frac{\Delta\omega}{\omega} = \frac{\Delta k}{k} = \frac{1}{l}. \tag{112}$$

The difference Δn of the refractive index for these adjacent frequencies is

$$\Delta n = \frac{4\pi Ne^2}{m\omega^3}\,\Delta\omega. \tag{113}$$

When $\omega \cong \omega_{\text{eff}}$, (108) and (112) give

$$\Delta n \cong 10^{-7}\,\frac{\Delta\omega}{\omega} = \frac{10^{-7}}{l}. \tag{114}$$

Since this difference of refractive index is maintained over a proper distance $\sim H^{-1}$, the phase difference is in general of order

$$\Delta n \cdot \omega_{\text{eff}} \cdot H^{-1} \cong 10^{-7}\omega_{\text{eff}} l^{-1} H^{-1} \cong 10^{-7}\omega_{\text{eff}} H^{-1}\left(\frac{2\pi}{kT'}\right). \tag{115}$$

Because this is the phase difference between adjacent frequencies, and because $\chi_l = -\chi_{-l}$, (115) gives the minimum value of $|\chi_l + \chi_{l'}|$ apart from $l' = -l$.

Now T' must exceed k^{-1} by an appreciable factor, but for an allowed transition not by more than $\sim(e^2/\hbar)^3$. For an allowed transition it follows therefore that the phase difference between adjacent terms in the Fourier series is $> \sim 10^{-12}\omega_{\text{eff}}H^{-1} \cong 10^{11}$ radians. Even for a forbidden transition the appropriate phase difference is still exceedingly large.

It is important that the phase difference is not the same for every absorber. Indeed so far from being the same the phase changes in the main region of absorption by $\sim 10^{11}$ radians, so that an average taken with respect to all absorbers is equivalent to summing a very large number of points distributed at random on the unit circle in the complex plane. This confirms the statement following (56). Instead of needing to assume random phasing, as one has to do for the quantised oscillators of the usual theory, we have proved it.

Throughout the work of the preceding sections it was supposed that absorption takes place between discrete states, while now we have been led to consider a collision process involving a continuum of states. It is to be expected that the continuum behaves as a closely packed set of states and that the previous work remains applicable. This is shown to be so in the Appendix—only a straightforward rewriting of the previous work is needed.

We turn now to the asymmetry of the spontaneous transition formula; $P(m \to n) > 0$ for $E_m > E_n$, $P(m \to n) = 0$ for $E_m < E_n$. This result came from taking the initial state for the absorbers as the ground level. In relation to the present discussion we know that a low frequency wave is always absorbed in an ionised gas unless there is some "population inversion" in the gas—i.e. unless the gas has upper levels more populated than would be the case in thermodynamic equilibrium. The gas can then behave as a maser, amplifying instead of attenuating the original wave. Such conditions do not exist in the extragalactic medium. If they did, the possibility would exist for obtaining the opposite asymmetry, $P(m \to n) > 0$ for $E_m < E_n$, $P(m \to n) = 0$ for $E_m > E_n$.

For absorption to take place in the present sense it is not really sufficient to consider transitions $\psi_i \to \psi_f$, $E_i < E_f$ without also considering what happens subsequently to the absorber. If the problem were discussed from the point of view of resonance fluorescence, with the absorber returning to ψ_i through a radiative emission, there would be no genuine absorption of the original wave. We would have effectively a scattering, which contributes to χ_l but not to τ_l. We require instead a collision of the absorber with another particle. The energy of the original wave then disappears as heat in the gas.

Although the distinction between scattering and absorption is clear enough in a classical discussion, one can ask at what point in the quantum treatment does the difference show itself. We assumed above that

$$F[\mathbf{a}(t), \mathbf{a}'(t)] = \prod_{b \neq a} F^{(b)}[\mathbf{a}(t), \mathbf{a}'(t)],$$

which requires the particles $b \neq a$ to be independent of each other. In fluorescence the particles remain independent throughout, and $f \to i$ cancels $i \to f$ so far as absorption is concerned. When collisions take place, however, further interaction terms $S_{\text{int}}[\mathbf{b}(t), \mathbf{c}(t)]$ must be considered between particles b and c. These further terms contain random phases which prevent $f \to i$ contributing to the effect on a. This leaves $i \to f$ for the effect on a, which is what we have calculated.

Absorption in an ionised gas changes the equilibrium Boltzmann distribution of the particles from f_0, say, to $f_0 + f_1$ where f_1 is a small perturbation of the distribution. So long as f_1 remains, the particle motions contain a "memory" of the absorbed wave, and the cancellation due to random phases mentioned in the previous paragraph would not apply. But the distribution f_1 decays due to collisions as $\exp[-\nu_{\text{eff}} t]$. It is this decay that constitutes the absorption process.

APPENDIX

The discussion following (56) needs rewording when the absorbers have a continuum of states instead of a discrete set. We take the continuum to have $\rho(E)\, dE$ states in the energy range E to $E + dE$. Although the terms of the Fourier series with respect to l are closely spaced they are not so close now as the continuum states. We show that this mathematical difference has no physical effect.

The quantity $|\mathbf{b}_{if} \cdot (2\pi l/T')|^2$ was defined by

$$|\mathbf{b}_{if}(\mathbf{k})|^2 = \int \psi_f^*(\mathbf{b})\, \mathbf{b} e^{i\mathbf{k}\cdot\mathbf{b}} \psi_i(\mathbf{b})\, d^3\mathbf{b}, \tag{A1}$$

$$\mathbf{k} = \frac{2\pi l}{T'} \cdot \frac{\mathbf{R}}{R}, \tag{A2}$$

$|\mathbf{b}_{if}(k)|^2$ being a suitable average of $|\mathbf{b}_{if}(\mathbf{k})|^2$ with respect to orientation. We modify this definition slightly. The initial state ψ_i is kept fixed for the moment and $|\mathbf{b}_{if}(2\pi l/T')|^2$ is taken to be $|\mathbf{b}_{if}(k)|^2$ averaged not only with respect to the orientation of the absorber but also with respect to final states ψ_f with energy E_f close to $E_i + 2\pi l\hbar/T'$. In place of (57) we now have

$$\begin{aligned}
&\frac{e^4}{3\hbar^2 R^2 T'^2} \sum_{l=-\infty}^{\infty} e^{-\tau_l} \int \rho(E_f)(E_f - E_i)^2\, |\mathbf{b}_{if}(2\pi l/T')|^2\, dE_f \\
&\cdot \left| \int_0^{T'} \exp\left[\frac{i}{\hbar}\left(E_f - E_i - \frac{2\pi l\hbar}{T'}\right) t'\right] dt' \right|^2 \\
&\cdot \sum_{j=1,2} \int_0^T (\boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}}) \exp\left[\frac{2\pi i l}{T'}\left(t - \frac{\mathbf{a}\cdot\mathbf{R}}{R}\right)\right] dt \\
&\cdot \int_0^T (\boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}}') \exp\left[\frac{2\pi i l}{T'}\left(\frac{\mathbf{a}'\cdot\mathbf{R}}{R} - t\right)\right] dt.
\end{aligned} \tag{A3}$$

Since $\rho(E_f)(E_f - E_i)^2 \mid \mathbf{b}_{if}(2\pi l/T')|^2$ can be taken to vary slowly with respect to E_f, we make immediate use of the approximate delta-function property in (58). Furthermore we take $E_f > E_i$ in accordance with discussion of Section V (see also the remarks below). Then only terms with positive values of l survive and (A3) becomes

$$\frac{2\pi e^4}{3\hbar^2 R^2 T'^2} \sum_{l=0}^{\infty} \exp(-\tau_l) \cdot \left(\frac{2\pi l}{T'}\right)^2 \rho\left(E_i + \frac{2\pi l\hbar}{T'}\right) \mid \mathbf{b}_{if}(2\pi l/T')|^2$$
$$\cdot \sum_{j=1,2} \int_0^T (\boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}})\, e^{-i\mathbf{k}\cdot\mathbf{a}+ikt}\, dt \int_0^T (\boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}}')\, e^{i\mathbf{k}\cdot\mathbf{a}-ikt}\, dt. \tag{A4}$$

Suppose that at distance R in a particular element of solid angle $d\Omega$ there are $n(E_i)\, dE_i$ particles per unit volume with initial states having energies between E_i and $E_i + dE_i$. Writing $\overline{\mid \mathbf{b}_{if}(2\pi l/T')|^2}$ for the average value of $\mid \mathbf{b}_{if}(2\pi l/T')|^2$ for these particles, the contribution to $F[\mathbf{a}(t), \mathbf{a}'(t)]$ in (38) is

$$[1 + \text{Expression (A4)}]^{n(E_i)dE_iR^2dRd\Omega}. \tag{A5}$$

As before we can write (A5) in an exponential form

$$\exp\Bigg[\frac{2\pi e^4}{3\hbar^2 T'} \sum_{l=0}^{\infty} e^{-\tau_l} \left(\frac{2\pi l}{T'}\right)^2 \rho\left(E_i + \frac{2\pi l\hbar}{T'}\right) \overline{\mid \mathbf{b}_{if}(2\pi l/T')|^2}$$
$$\cdot\, n(E_i)\, dE_i\, dR\, d\Omega \sum_{j=1,2} \int_0^T (\boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}})\, e^{-i\mathbf{k}\cdot\mathbf{a}+ikt}\, dt \int_0^T (\boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}}')\, e^{i\mathbf{k}\cdot\mathbf{a}'-ikt}\, dt\Bigg]. \tag{A6}$$

The final value of $F[\mathbf{a}(t), \mathbf{a}'(t)]$ is obtained by integrating the exponent of (A6) with respect to E_i, R, and Ω. The quantity τ_l is the opacity difference at frequency $2\pi l/T'$ taken along the vector $\mathbf{R}$ from $\mathbf{a}$ to $\mathbf{b}$. It is related to absorption in the same way as before and satisfies the equation

$$\frac{d\tau_l}{dR} = \frac{4\pi^2 e^2}{3\hbar} \cdot \left(\frac{2\pi l}{T'}\right) \int \rho\left(E_i + \frac{2\pi l\hbar}{T'}\right) \overline{\mid \mathbf{b}_{if}(2\pi l/T')|^2}\, n(E_i)\, dE_i \tag{A7}$$

The integrals with respect to E_i, R and Ω therefore lead without difficulty to

$$F[\mathbf{a}(t), \mathbf{a}'(t)] = \exp\Bigg[\frac{e^2}{2\pi\hbar T'} \int d\Omega \sum_{l=0}^{\infty} \frac{2\pi l}{T'} \sum_{j=1,2} \int_0^T (\boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}})\, e^{-i\mathbf{k}\cdot\mathbf{a}+ikt}\, dt$$
$$\cdot \int_0^T (\boldsymbol{\alpha}^{(j)} \cdot \dot{\mathbf{a}}')\, e^{i\mathbf{k}\cdot\mathbf{a}'-ikt}\, dt\Bigg], \tag{A8}$$

and converting the sum in (A8) to an integral with respect to $k = 2\pi l/T'$ gives (64).

The restriction of (A4) to positive values of l has the effect of the influence functional being represented by an integral over positive values of k. It is this property of the influence functional which determines the asymmetry of spontaneous transitions; $P(m \to n) > 0$ for $E_m > E_n$, $P(m \to n) = 0$ for $E_m < E_n$. The restriction to positive values of l arose in the above work because E_f was considered to be $\geqslant E_i$. This is the case of absorption. The opposite case, $E_f < E_i$, would correspond to a situation in which the incident wave produced stimulated emission in the absorber. As pointed out in Section V, stimulated emission is not important in this connection unless the absorber acts like a maser—which of course the intergalactic medium does not do.

RECEIVED: January 9, 1969

REFERENCES

1. K. SCHWARZSCHILD, *Göttinger Nachrichten* **128**, 132 (1903).
2. H. TETRODE, *Z. f. Physik* **10**, 317 (1922).
3. A. D. FOKKER, *Z. f. Physik* **58**, 386 (1929), and *Physica* **9**, 83 (1929); *Physica* **12**, 145 (1932).
4. J. A. WHEELER AND R. P. FEYNMAN, *Rev. Mod. Phys.* **17**, 157 (1945); *Rev. Mod. Phys.* **21**, 425 (1949).
5. J. E. HOGARTH, *Proc. Roy. Soc. A* **267**, 365 (1962).
6. F. HOYLE AND J. V. NARLIKAR, *Proc. Roy. Soc. A* **277**, 1 (1963).
7. F. HOYLE, *Proc. Roy. Soc. A*. Bakerian lecture. In press.
8. R. P. FEYNMAN, *Rev. Mod. Phys.* **20**, 367 (1948).
9. R. P. FEYNMAN AND A. R. HIBBS, "Quantum Mechanics and Path Integrals." McGraw-Hill, New York, 1965.