

Quasinormal modes in Schwarzschild–de Sitter spacetime: A simple derivation of the level spacing of the frequencies

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It is known that the imaginary parts of the quasinormal mode (QNM) frequencies for the Schwarzschild black hole are evenly spaced with a spacing that depends only on the surface gravity. On the other hand, for massless minimally coupled scalar fields, there exist no QNMs in the pure de Sitter spacetime. It is not clear what the structure of the QNMs would be for the Schwarzschild–de Sitter (SDS) spacetime, which is characterized by two different surface gravities. We provide a simple derivation of the imaginary parts of the QNM frequencies for the SDS spacetime by calculating the scattering amplitude in the first Born approximation and determining its poles. We find that, for the usual set of boundary conditions in which the incident wave is scattered off the black hole horizon, the imaginary parts of the QNM frequencies have an equally spaced structure with the level spacing depending on the surface gravity of the black hole. Several conceptual issues related to the QNM are discussed in the light of this result and a comparison with previous work is presented.

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I. INTRODUCTION

It is well known that the quasinormal modes (QNMs) are crucial in studying the gravitational and electromagnetic perturbations around black hole spacetimes (for comprehensive reviews and an exhaustive list of references see [1,2]). The QNMs also seem to have an observational significance as the gravitational waves produced by the perturbations can, in principle, be used for unambiguous detection of black holes. The early studies of the QNMs concentrated on numerical computations (see, for example, [3,4]) as analytical solutions of the perturbation equations were difficult to obtain. Essentially, one solved the perturbation equations numerically for different initial conditions in a given black hole spacetime. It was found that the results were, in general, independent of the initial conditions and depended mostly on the parameters characterizing the black hole horizon. For example, for the Schwarzschild black hole, the spectrum of the QNMs was found to have the structure [5–10]

$$k_n = i\kappa \left(n + \frac{1}{2} \right) + \frac{\ln 3}{2\pi} \kappa + O(n^{-1/2}), \quad (1)$$

which depends only on the surface gravity of the black hole κ . Similarly, for charged (Reissner-Nordström) and rotating (Kerr) black holes, the spectrum was found to depend only on the parameters characterizing the horizon, i.e., the mass, charge and angular momentum of the black hole [11–23].

The numerical studies were followed by a series of analytical calculations which were mainly based upon some approximation scheme, such as, for example, (i) approximating the scattering potential with some simple functions so that the problem becomes exactly solvable [24–26], (ii) using

WKB-like techniques [27–38], (iii) the Born approximation [39,40] and (iv) others [41–54]. Nearly all of these schemes came with their own limitations and rarely reproduced the exact form of the QNM spectrum as given in Eq. (1)—in particular, there was hardly any explanation for the $\ln 3$ term. Analytical proofs of Eq. (1) were provided only recently, using the methods of continued fractions [5,12,55,56] and also by studying the monodromy of the perturbation continued to the complex plane [57,58].

While most of the analytical work concentrated on the real part of QNM (due to the interest in the $\ln 3$ factor), the structure of the imaginary part is equally intriguing and requires physical understanding. In particular, we need to understand why the imaginary part of k_n in Eq. (1) has a simple, equally spaced structure, with κ determining the level spacing. Physical quantities with a quantized spectrum are always of interest when they have constant spacing. In the case of horizon area, for example, one can attempt to relate an equally spaced spectrum (obtained in several investigations, e.g., see [59] and references therein) to the intrinsic limitations in measuring length scales smaller than Planck length [60]. But the uniform spacing of QNM frequencies is a purely *classical* result and hence is harder to understand physically.

It turns out, however, that one can make *some* progress in this direction. There are some simple derivations, based on the Born approximation, which reproduce this structure for the imaginary part of the QNM frequencies for a large class of spherically symmetric spacetimes such as the Schwarzschild black hole [39,40]. The relation obtained, valid for large k (i.e., for large n), has the form

$$k_n = in\kappa \quad (\text{for } n \gg 1). \quad (2)$$

[In the imaginary part of k_n , one obtains a factor n instead of $n + (1/2)$ since the Born approximation is expected to be

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valid only for $n \gg 1$.] The important point to note is that—though it is not possible to obtain the real part of the QNM frequencies using the Born approximation—this simple derivation does give the correct level spacing for imaginary values of k_n . Hence the possibility of extending the formalism to spacetimes with more complicated structure is worth examining. Following this approach, we use the Born approximation to examine the level spacing for spacetimes with two horizons.

The simplest spacetime with two horizons is that of a black hole in a spacetime with a cosmological constant, described by the Schwarzschild–de Sitter (SDS) metric. The metric is characterized by the presence of a black hole event horizon and a cosmological horizon. In recent times, studying such a spacetime has acquired further significance because of the cosmological observations suggesting the existence of a nonzero positive cosmological constant ([61]; for reviews, see [62,63]). While the observations can be explained by a wide class of models (see, e.g. [64]), including those in which the cosmic equation of state can depend on spatial scale [65,66], virtually all these models approach the de Sitter (DS) spacetime at late times and at large scales. As in the other cases, the QNMs for the SDS spacetime have been studied both numerically (see, for example, [67–72]) and analytically (see, for example, [73–75]). However, since the SDS spacetime is characterized by two different surface gravities (corresponding to the two horizons), the dependence of the level spacing of the imaginary part of the QNM frequencies on the surface gravities is not obvious. The numerical studies fail to give a “clean” result like Eq. (1) as they seem to vary depending on the relative values of the two surface gravities. A calculation based on the monodromy of the perturbation continued to the complex plane [76], similar to what is done for the Schwarzschild case, gives a result of the form $k_n = i\kappa_-(n + \frac{1}{2}) + \text{real part}$, where κ_- is the surface gravity of the black hole horizon. The real part has a term analogous to $\ln 3$ whose actual form depends on the values of the surface gravities. The above result is found to be in qualitative agreement with numerical results for the near-extremal case (i.e., when the surface gravities are very close to each other), although the exact numerical coefficients seem to differ [72]. It is not clear whether the problem is with the numerics, or the analytical derivation misses out on some issues. The above calculation indicates that the QNM spectrum is independent of the surface gravity of the cosmological horizon κ_+ . On the other hand the first version of [39], based on Born approximation, claimed that the spectrum of the imaginary part of the QNMs should behave as $k_n = i(n_-\kappa_- + n_+\kappa_+)$, where n_{\pm} are two integers. (This arXiv submission has been revised subsequently, after the appearance of the current work, removing this claim.) Analytical calculation based on approximating the scattering potential by some simple form gives QNMs proportional to both the surface gravities, depending on the time scale one is interested in [75]. Currently, we do not seem to have any general consensus on this particular issue!

There are a few more conceptual issues because of which such a study (using a prototype metric with two horizons) is important. First is the possible connection between the

QNMs and thermodynamics of horizons. It was pointed out in [40] that the Born approximation result arises from integrals which are very similar to those which arise in the case of horizon thermodynamics [see e.g., Eq. (23) of [77]]. In the case of spacetimes with multiple horizons (like SDS), there is no unique temperature except when the ratio of surface gravities is a rational number [78]. If the QNMs are related to the horizon thermodynamics (in some manner which is not yet clearly understood), then it would be interesting to see whether the level spacing in the SDS spacetime can give any idea about the temperature of spacetimes with multiple horizons. Second, it is well known that thermodynamics of gravitating systems depend crucially on the ensemble which is used (see e.g., [79]) which translates into the boundary conditions on the horizon (see e.g., [80]). If QNMs are related to horizon thermodynamics, we will expect some similar kind of dependence on the boundary condition for the wave modes in the case of SDS spacetime. We shall see that this expectation is indeed borne out. Finally, there were also some discussions in the literature as to whether the QNMs depend on the region beyond the horizons. We will see that we can make some comments regarding this issue.

The structure of the paper is as follows. In Sec. II, we briefly review the results for the Schwarzschild metric. The main problem of interest, the SDS metric, is taken up in Sec. III. We derive the explicit form of the scattering amplitude using the Born approximation and discuss the structure of the QNM spectrum. The main conclusions are summarized and compared with other results in Sec. IV.

II. WARM UP: QNMS FOR THE SCHWARZSCHILD METRIC

In this section we review the derivation of the QNMs for the Schwarzschild metric using the first Born approximation. Let us start with a general class of spherically symmetric metrics of the form

$$ds^2 = f(r)dt^2 - [f(r)]^{-1}dr^2 - r^2d\Omega^2 \quad (3)$$

with $f(r)$ having the simple zero at $r=r_0$, i.e., $f(r) \simeq f'(r_0)(r-r_0)$. It was shown in [81] that spacetimes described by the above class of metrics have a fairly straightforward thermodynamic interpretation and—in fact—Einstein’s equations can be expressed in the form of a thermodynamic relation $TdS = dE - PdV$ for such spacetimes, with the temperature being determined by the surface gravity of the horizon:

$$\kappa = \frac{1}{2} |f'(r_0)|. \quad (4)$$

Let us consider a massless scalar field ϕ satisfying the wave equation $\square\phi = 0$ in this spacetime. We look for solutions to the wave equation in the form

$$\phi = \frac{1}{r} F(r) Y_{lm}(\Omega) e^{ikt}; \quad \text{Re}(k) > 0. \quad (5)$$

Straightforward algebra now leads to a ‘‘Schrödinger equation’’ for F given by

$$\left[-\frac{d^2}{dr_*^2} + V(r) \right] F(r) = k^2 F(r) \quad (6)$$

where the potential is given by

$$V(r) = f(r) \left[\frac{l(l+1)}{r^2} + \frac{f'(r)}{r} \right] \quad (7)$$

and the tortoise coordinate is defined as

$$r_* \equiv \int \frac{dr}{f(r)}. \quad (8)$$

One can, in principle, solve the differential equation (6) given a particular set of boundary conditions. However, the equation, in general, does not have an exact solution and one has to solve it either numerically or by using some approximation scheme. In this paper, we shall be using one such approximation scheme, namely, the first Born approximation.

For a pure Schwarzschild black hole, we have $f(r) = 1 - 2M/r$ and the horizon is at $r_0 = 2M$. The potential $V(r)$ in Eq. (7) vanishes at the horizon ($r_* \rightarrow -\infty$) and at spatial infinity ($r_* \rightarrow \infty$), which means that the wave function can be taken to be plane waves in the two regions. A class of physically acceptable solutions for this system has the asymptotic form

$$F(r) \sim \begin{cases} e^{ikr_*} & (\text{at } r_* \rightarrow -\infty), \\ A_{\text{in}} e^{ikr_*} + A_{\text{out}} e^{-ikr_*} & (\text{at } r_* \rightarrow \infty). \end{cases} \quad (9)$$

The ‘‘incident’’ wave ($A_{\text{in}} e^{ikr_*}$) for this class of solutions is propagating towards the black hole, and hence the solutions of the above form are appropriate for studying *scattering off the black hole*. The scattering amplitude for the above solution is simply given by

$$S(k) \propto \frac{A_{\text{out}}}{A_{\text{in}}}. \quad (10)$$

Now, the QNMs are defined to be those for which one has a purely ingoing plane wave at the horizon and a purely outgoing wave at spatial infinity, i.e., which satisfy the boundary conditions

$$F(r) \sim \begin{cases} e^{ikr_*} & (\text{at } r_* \rightarrow -\infty), \\ e^{-ikr_*} & (\text{at } r_* \rightarrow \infty). \end{cases} \quad (11)$$

It is clear from Eq. (9) that the QNMs actually correspond to the case where $A_{\text{in}} = 0$ and are obtained by calculating the poles of the scattering amplitude $S(k)$ [see Eq. (10)]. Obtaining the exact form of the scattering amplitude for the potential (7) is nontrivial—hence one has to obtain it through some approximation scheme. It turns out that one can obtain the explicit form of this amplitude using the first Born approximation.

In general, the scattering amplitude in the Born approximation is given by the Fourier transform of the potential $V(\mathbf{x})$ with respect to the momentum transfer $\mathbf{q} = \mathbf{k}_f - \mathbf{k}_i$:

$$S(\mathbf{q}) = \int d\mathbf{x} V(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}}. \quad (12)$$

In one dimension, \mathbf{k}_i and \mathbf{k}_f should be parallel or antiparallel; further, we can take their magnitudes to be the same for scattering in a fixed potential. Then nontrivial momentum transfer occurs only for $\mathbf{k}_f = -\mathbf{k}_i$ so that $\mathbf{q} = -2\mathbf{k}_i$. From Eq. (9), the ‘‘incident’’ wave is seen to be of form e^{ikr_*} , giving the scattering amplitude as

$$S(k) = \int_{-\infty}^{\infty} dr_* V[r(r_*)] e^{2ikr_*} \\ = \int_{r_0}^{\infty} dr \left[\frac{l(l+1)}{r^2} + \frac{f'(r)}{r} \right] e^{2ikr_*(r)} \quad (13)$$

where we have omitted irrelevant constant factors. (Our original problem was three-dimensional and we are *not* working out the three-dimensional scattering amplitude in, say, the s -wave limit. Rather, we first map the problem to a one-dimensional Schrödinger equation and study the scattering amplitude in one dimension.) This integral picks up significant contribution *only* near the horizon where $r_* \approx (1/2)\kappa^{-1} \ln(r/r_0 - 1)$, and it can be shown that the approximate form of the scattering amplitude is given by [40]

$$S(k) \approx \text{constant factors} \times \Gamma \left(1 + i \frac{k}{\kappa} \right). \quad (14)$$

It is clear from the above expression that the poles of the amplitude are given by the poles of the Gamma function, which occur at

$$k_n = in\kappa \quad (\text{for } n \geq 1). \quad (15)$$

It also turns out that the integral (13) can be solved *exactly* for a pure Schwarzschild metric [40], and the imaginary part of the QNM frequencies are found to be exactly identical to what is obtained above. As discussed in Section 1, the Born approximation fails to reproduce the real part of the QNM spectrum.

As an aside, we would like to comment on the issue of whether QNMs depend on the form of the metric inside the horizon and in particular on the singularity of the Schwarzschild metric at $r=0$. (These comments do not depend on the Born approximation but it is easy to see the result in this limit.) The answer is essentially ‘‘no’’ in the sense that if the Schwarzschild metric is modified in a small region around $r=0$, making it nonsingular, but leaving the form of the metric unchanged for $r \geq 2M$, the QNMs do not change; but there is subtlety in this issue. It is easy to see that, in the Born approximation, we are dealing with a scattering problem in the r_* coordinates with boundary conditions at $r_* = \pm \infty$. This scattering problem only depends on $V[r(r_*)]$

which—in turn—depends only on $f(r)$ for $r \geq 2M$. So if we modify the $f(r)$ for $r < 2M$, it does not change the Born approximation results. What happens when we go beyond the Born approximation and consider the real part of QNMs, for example? Here we need to analytically continue to the complex values of r and r_* . Now the original definition of the problem, posed as a Schrödinger equation in Eq. (6), again cares only for the r_* coordinate and is well-defined if boundary conditions are specified at $r_* = \pm \infty$. But the relation between r and r_* [which again depends only on $f(r)$ at $r \geq 2M$] can be *analytically continued for all r including near $r=0$* . This leads to a unique analytic structure in the complex plane and even for $r < 2M$ through analytic continuation, *as though* the form of $f(r)$ is valid all the way to $r=0$! More explicitly, this analytical structure is obtained by (i) defining $f(r)$ for $r \geq 2M$; (ii) defining r_* using it; (iii) analytically continuing this relation to define r_* and r in the complex plane; and (iv) defining $f(r)$ for all r including near $r=0$ by analytic continuation. This is independent of the *actual* form of $f(r)$ for $r < 2M$, and previous results like those in [57,58] depend only on this structure. This is gratifying since we do not know the effects of quantum gravity, which could modify the spacetime structure near the singularity.

It might seem that the above formalism can be trivially applied to the case of a spacetime with a cosmological horizon, described by the pure DS metric with $f(r) = 1 - H^2 r^2$. The potential term for calculating the scattering amplitude, given by Eq. (7), becomes

$$V(r) = (1 - H^2 r^2) \left[\frac{l(l+1)}{r^2} - 2H^2 \right] \quad (16)$$

which, like in the pure Schwarzschild case, vanishes at the horizon $r = H^{-1}$. However, in contrast to the Schwarzschild case, it does *not* vanish at the other boundary, i.e., at $r=0$. It is easy to check that near the origin, $V(r) \approx -2H^2$ when $l=0$, while it blows up as r^{-2} for $l > 0$. This implies that one cannot take the boundary conditions as simple plane waves like in Eq. (9). In fact, the set of boundary conditions used for studying QNMs in the pure DS space is that the wave function should be outgoing at the horizon, but should vanish at the origin, i.e.,

$$F(r) \sim \begin{cases} 0 & (\text{at } r \rightarrow 0), \\ e^{-ikr_*} & (\text{at } r \rightarrow H^{-1}). \end{cases} \quad (17)$$

It turns out that the wave equation can be solved *exactly* for the DS spacetime [82,83], and for scalar fields satisfying the wave equation $\square \phi = 0$, there exist no modes for which the above boundary condition is satisfied. This result is implicit in [68] and we include the details of the calculation in Appendix A for completeness.

III. QNMS FOR THE SCHWARZSCHILD–de SITTER METRIC

We next consider the Schwarzschild–de Sitter (SDS) spacetime, which is described by a spherically symmetric metric of the form (3), with

$$f(r) = 1 - \frac{2M}{r} - H^2 r^2. \quad (18)$$

Let us denote the black hole event horizon and the cosmological horizon by r_- and r_+ respectively. The corresponding surface gravities are denoted as κ_- and κ_+ respectively. Note that, by definition, both κ_- and κ_+ are positive definite. The tortoise coordinate is given by

$$r_* = \frac{1}{2\kappa_-} \ln \left| \frac{r}{r_-} - 1 \right| - \frac{1}{2\kappa_+} \ln \left| 1 - \frac{r}{r_+} \right| - \frac{1}{2} \left(\frac{1}{\kappa_-} - \frac{1}{\kappa_+} \right) \ln \left| \frac{r}{r_- + r_+} + 1 \right|. \quad (19)$$

With this definition, the regions $r \leq r_-$ and $r \geq r_+$ are mapped to $r_* \leq -\infty$ and $r_* \geq \infty$ respectively, and we will not require the regions beyond the two horizons. [In particular, the form of $f(r)$ inside the Schwarzschild horizon and the singularity at $r=0$ are irrelevant in what follows.] The potential in Eq. (7) reduces to

$$V(r) = f(r) \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} - 2H^2 \right] \quad (20)$$

which, because of the $f(r)$ factor, vanishes at both the horizons. Thus one can take the boundary conditions to be simple plane waves at the two horizons. The usual set of solutions used in a scattering problem is identical to Eq. (9) which, as mentioned earlier, is appropriate for studying scattering off the black hole. Let us assume that the boundary conditions which define the QNMs are still given by Eq. (11), which imply that one has purely ingoing plane waves at the black hole horizon ($r = r_-$) and purely outgoing waves at the cosmological horizon ($r = r_+$). One should realize that this set of boundary conditions implies that the waves are propagating “into” the horizons at both the boundaries and is probably the most reasonable set of conditions to be used.

We now apply the first Born approximation, with the “incident” wave being taken as e^{ikr_*} as before. The scattering amplitude is then given by

$$S(k) = \int_{-\infty}^{\infty} dr_* V(r) e^{2ikr_*} \quad (21)$$

which can be simplified to

$$S(k) = \int_{r_-}^{r_+} dr \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} - 2H^2 \right] \left(\frac{r}{r_-} - 1 \right)^{ik/\kappa_-} \times \left(1 - \frac{r}{r_+} \right)^{-ik/\kappa_+} \left(1 + \frac{r}{r_+ + r_-} \right)^{ik(1/\kappa_+ - 1/\kappa_-)}. \quad (22)$$

As in the pure Schwarzschild case, this integral will pick up a contribution only near the horizons. However, there are some crucial differences which need to be taken care of. Near the black hole horizon, we have the usual relation

$r_* \approx (1/2)\kappa_-^{-1}\ln(r/r_- - 1)$, and the contribution is exactly similar to Eq. (14). On the other hand, near the cosmological horizon, we have $r_* \approx -(1/2)\kappa_+^{-1}\ln(1 - r/r_+)$, which differs from the other case in the sign of the surface gravity term. The scattering amplitude will then be a sum of two contributions given by

$$S(k) \approx \text{constant factors} \times \Gamma\left(1 + i\frac{k}{\kappa_-}\right) + \text{constant factors} \times \Gamma\left(1 - i\frac{k}{\kappa_+}\right). \quad (23)$$

In this case, the poles of the amplitude are given by the poles of *both* the Gamma functions:

$$k_n = in\kappa_-, \quad k_n = -in\kappa_+ \quad (n \gg 1). \quad (24)$$

The fact that the amplitude would pick up contributions from both the horizons was pointed out earlier in [39], and—in the first version of [39] that appeared in the arXiv—it was suggested that the poles would occur at $k_n = i(n_- \kappa_- + n_+ \kappa_+)$. However, we have shown by the explicit calculation above that this is not correct; summing up two contributions to get the value of an integral is not the same as adding the arguments for poles. (We note that [39] has since been revised and this particular claim has been withdrawn.) Further, there is a *crucial* sign difference in the surface gravity of the cosmological horizon.

This conclusion can be explicitly verified since, fortunately, one can evaluate the integral in Eq. (22) *exactly*. Essentially we have to solve integrals of the form

$$I_n = \int_{r_-}^{r_+} dr r^{-n} \left(\frac{r}{r_-} - 1\right)^{ik/\kappa_-} \left(1 - \frac{r}{r_+}\right)^{-ik/\kappa_+} \times \left(1 + \frac{r}{r_+ + r_-}\right)^{ik(1/\kappa_+ - 1/\kappa_-)} \quad (25)$$

with the scattering amplitude being given by the sum

$$S(k) = 2MI_3 + l(l+1)I_2 - 2H^2I_0. \quad (26)$$

The expression for I_n turns out to be [84] an integral representation of the Appell hypergeometric function F_1 ,

$$I_n = (r_+ - r_-)^{1-ik(1/\kappa_+ - 1/\kappa_-)} (r_+ + r_-)^{ik(1/\kappa_- - 1/\kappa_+)} \times r_-^{-ik/\kappa_-} r_+^{ik/\kappa_+} (r_+ + 2r_-)^{ik(1/\kappa_+ - 1/\kappa_-)} \times \frac{\Gamma\left(1 + \frac{ik}{\kappa_-}\right)\Gamma\left(1 - \frac{ik}{\kappa_+}\right)}{\Gamma\left(2 + ik\left[\frac{1}{\kappa_-} - \frac{1}{\kappa_+}\right]\right)} \times r_-^{-n} F_1\left(1 + \frac{ik}{\kappa_-}, n, ik\left[\frac{1}{\kappa_-} - \frac{1}{\kappa_+}\right], 2, + ik\left[\frac{1}{\kappa_-} - \frac{1}{\kappa_+}\right]; -\frac{r_+ - r_-}{r_-}, -\frac{r_+ - r_-}{r_+ + 2r_-}\right). \quad (27)$$

This expression can be further simplified and written in terms of the usual hypergeometric function ${}_2F_1$ for n

$= 0, 2, 3$ —we give the relevant expressions in Appendix B for completeness. The pole structure of I_n can, however, be determined from Eq. (27) itself. The combinations of the form

$$\frac{F_1\left(\alpha, n, \beta, 2 + ik\left[\frac{1}{\kappa_-} - \frac{1}{\kappa_+}\right]; r_1, r_2\right)}{\Gamma\left(2 + ik\left[\frac{1}{\kappa_-} - \frac{1}{\kappa_+}\right]\right)} \quad (28)$$

which occur in the expression for I_n , do *not* have any poles. (Even though both the denominator and numerator have poles, the ratio does not.) The only poles of I_n occur at the poles of the two Gamma functions $\Gamma[1 + (ik/\kappa_-)]$ and $\Gamma[1 - (ik/\kappa_+)]$. It is then straightforward to see that the poles of the scattering amplitude $S(k)$ are given by Eq. (24)—exactly identical to what we obtained by evaluating the integral near the horizons.

Let us now discuss the QNM spectrum as obtained in Eq. (24). Note that the QNMs are identified as the positive imaginary values of k , which implies that the QNMs in this case are given by $k_n = in\kappa_-$; the modes which are dependent on κ_+ correspond to negative imaginary values of k and hence do *not* represent QNMs. [In general, the poles given by negative imaginary values of k correspond to bound states of the system. However, since the potential for $l > 0$, Eq. (20), is positive everywhere and vanishes at the boundaries, it can be shown that there cannot exist any bound states for the system [85]. Thus, the poles given by $k_n = -in\kappa_+$ are physically irrelevant as far as this problem is concerned.]

The above analysis indicates that the QNMs obtained through the first Born approximation are independent of the cosmological horizon. This conclusion agrees, in the large n limit, with the imaginary part of the QNM spectrum obtained through the monodromy of the perturbation continued to the complex plane [76]. In view of the fact that there exists no QNMs for the pure DS spacetime, “adding” a black hole near the origin merely introduces the QNMs corresponding to the black hole. Interpreted in the above manner, this result should not be surprising. It is also clear that this result gives the two correct limits, i.e., when $H \rightarrow 0$, the level spacing reduces to that corresponding to a Schwarzschild black hole, while for $M \rightarrow 0$, we have $\kappa_- \rightarrow \infty$, and hence there exist no QNMs for the pure DS spacetime.

As an aside, one can also consider a different set of boundary conditions, where the incident wave is scattered off the cosmological horizon. It turns out that such conditions will give QNMs proportional to κ_+ (the details of the calculation are given in Appendix C). However, these boundary conditions may not be physically relevant and are considered here just as a mathematical possibility.

IV. DISCUSSION

We have used the first Born approximation to obtain the QNM spectrum for the SDS spacetime. The approximation gives the correct level spacing for the imaginary values of the QNM frequencies for the Schwarzschild black hole, and the spacing is related to the temperature corresponding to the

horizon. On the other hand, there exist no QNMs for a massless minimally coupled scalar field in a pure DS spacetime. It turns out that the situation is more complicated in the SDS spacetime and depends on the type of scattering one is interested in, i.e., on the type of boundary conditions one imposes. One can start with the usual set of conditions where an incident wave is propagating towards the black hole and calculate the scattering amplitude. The poles of the amplitude will then represent boundary conditions appropriate for QNMs. It turns out that for this case the QNM level spacing depends only on the surface gravity of the black hole κ_- , as expected. Thus the introduction of a black hole in the DS spacetime brings along the appropriate QNMs.

However, there exists another set of boundary conditions in which one starts with an incident wave propagating towards the cosmological horizon. As shown in Appendix C, it is possible to choose the boundary conditions and the definition of the scattering amplitude such that the QNM level spacing in this case depends only on the surface gravity of the cosmological horizon κ_+ . We do not believe these boundary conditions are physically relevant.

It was found earlier, based on the monodromy of the perturbation continued to the complex plane [76] that the imaginary part of the QNM frequencies has an equally spaced structure, with the spacing dependent only on κ_- . It is not clear whether there exist any extensions of the above procedure for obtaining the other set of QNMs which are dependent on κ_+ . Analytical calculations, based on approximating the potential by a Poschl-Teller form [75], gave a QNM spectrum which depends on *both* the surface gravities κ_- and κ_+ depending on the time-scale one is interested in. Since we are studying a time-independent situation, it is difficult to comment on time-dependence of the QNM spectrum—however, our analysis indicates that if one starts with an incident wave packet which is composed of monochromatic waves propagating in both directions (i.e., terms of the form e^{ikr_*} and e^{-ikr_*}), then one might obtain a QNM spectrum which depends on both the surface gravities. There is one more crucial difference between our results and those obtained by the Poschl-Teller potential [75]—the level spacing in the latter case is $2\kappa_+$ rather than κ_+ . Such a difference was noted in the case of QNMs obtained by the Born approximation for the Schwarzschild black hole [40] while comparing with results obtained by approximating the potential [43]; the difference is probably related to the incorrect use of $q=k_i$ rather than $q=2k_i$ for momentum transfer in the Born approximation.

Most of the numerical computations regarding the SDS spacetime concentrate on the near extremal case (where $\kappa_+ \approx \kappa_-$), and it is found that the imaginary part of the QNM frequencies has an equally spaced structure with the spacing given by either of the surface gravities (which are anyway equal to the lowest order) [69,70,72]. This means that, to the lowest order, we do not find any disagreement between our results and numerical computations. In other numerical computations, where the values of the two surface gravities are taken to be widely different [67,68], one obtains two sets of QNM spectra proportional to the two surface gravities, each valid at different time-scales. At this stage, we are unable to

make any direct comparison with such results since the issue of time-scales is difficult to settle in our approach.

In future studies for the SDS spacetime, it would be interesting to calculate the scattering amplitude using some more rigorous technique (like, say, what is done for the pure Schwarzschild case [58]) and see how the real parts of the QNM frequencies depend on the different surface gravities and on the type of scattering.

APPENDIX A: QNMS FOR THE PURE de SITTER METRIC

In this appendix, we give the details of the calculations for calculating the QNMs for the DS spacetime. Although most of the mathematical apparatus already exists in the literature [68,82,83,86–88], we include the details for completeness and for emphasizing the conclusion.

The radial wave equation for the DS metric [see Eqs. (6) and (7)]

$$\left[- \left\{ (1-H^2 r^2) \frac{d}{dr} \right\}^2 + (1-H^2 r^2) \left\{ \frac{l(l+1)}{r^2} - 2H^2 \right\} \right] F(r) = k^2 F(r) \quad (\text{A1})$$

can be reduced to the hypergeometric form by introducing a new variable $z=r^2 H^2$. The solution, which is regular at the origin, can be written as

$$F(r) = r^{l+1} (1-H^2 r^2)^{ik/2H} \times {}_2F_1 \left(\frac{l}{2} + \frac{ik}{2H}, \frac{l}{2} + \frac{3}{2} + \frac{ik}{2H}, l + \frac{3}{2}; H^2 r^2 \right) \quad (\text{A2})$$

where the normalization is arbitrary. The behavior of this solution near the horizon $r=H^{-1}$ is given by

$$F(r) \propto \Gamma \left(l + \frac{3}{2} \right) \left[(1-H^2 r^2)^{-ik/2H} \times \frac{\Gamma \left(\frac{ik}{H} \right)}{\Gamma \left(\frac{l}{2} + \frac{ik}{2H} \right) \Gamma \left(\frac{l}{2} + \frac{3}{2} + \frac{ik}{2H} \right)} + (1-H^2 r^2)^{ik/2H} \times \frac{\Gamma \left(-\frac{ik}{H} \right)}{\Gamma \left(\frac{l}{2} - \frac{ik}{2H} \right) \Gamma \left(\frac{l}{2} + \frac{3}{2} - \frac{ik}{2H} \right)} \right]. \quad (\text{A3})$$

According to the boundary conditions (17), the solution should be purely outgoing near the horizon, i.e., $F(r) \sim (1-H^2 r^2)^{ik/2H}$. [This follows from the fact that $r_* = H^{-1} \tanh^{-1}(Hr)$ for the DS metric which, near the horizon,

gives $1 - H^2 r^2 = \text{sech}^2(Hr_*) \sim e^{-2Hr_*}$.] This implies that the QNMs are given by the poles of the expression

$$\frac{\Gamma\left(\frac{l}{2} + \frac{ik}{2H}\right)\Gamma\left(\frac{l}{2} + \frac{3}{2} + \frac{ik}{2H}\right)}{\Gamma\left(\frac{ik}{H}\right)}. \quad (\text{A4})$$

The numerator has two sets of poles at $k_{n,l} = iH(2n+l)$ and $k_{n,l} = iH(2n+l+3)$ for $n=0,1,2,\dots$. However, each of these poles is canceled by a similar pole of the Gamma function in the denominator [68]. Hence, there exist no QNMs for the pure DS spacetime which obey the boundary condition (17).

Note that this conclusion is only true for a *massless, minimally coupled* scalar field with wave equation $\square\phi=0$. There do exist well-defined QNMs for a massive scalar field, or for a scalar field coupled to the Ricci scalar.

APPENDIX B: SIMPLIFIED EXPRESSIONS FOR THE SCATTERING AMPLITUDE

In this appendix, we shall write the scattering amplitude given by Eqs. (26) and (27) in terms of the more familiar hypergeometric functions. For notational convenience, let us define

$$a_k \equiv \frac{k}{\kappa_-}, \quad b_k \equiv \frac{k}{\kappa_+},$$

$$c_k \equiv a_k - b_k = k \left(\frac{1}{\kappa_-} - \frac{1}{\kappa_+} \right). \quad (\text{B1})$$

Also define

$$A_k = (r_+ - r_-)^{1+ic_k} (r_+ + r_-)^{ic_k} \times r_-^{-ia_k} r_+^{ib_k} (r_+ + 2r_-)^{-ic_k}. \quad (\text{B2})$$

Then we have a much simpler expression,

$$I_n = A_k \frac{\Gamma(1+ia_k)\Gamma(1-ib_k)}{\Gamma(2+ic_k)} \times r_-^{-n} F_1\left(1+ia_k, n, ic_k, 2+ic_k; -\frac{r_+ - r_-}{r_-}, -\frac{r_+ - r_-}{r_+ + 2r_-}\right). \quad (\text{B3})$$

For calculating the scattering amplitude, we are only interested in the three quantities I_0, I_2, I_3 . For $n=0$, use the relation

$$F_1(a, 0, b, c, x, y) = {}_2F_1(a, b, c; y) \quad (\text{B4})$$

to obtain

$$I_0 = A_k \Gamma(1+ia_k)\Gamma(1-ib_k) \times \frac{{}_2F_1\left(1+ia_k, ic_k, 2+ic_k; -\frac{r_+ - r_-}{r_+ + 2r_-}\right)}{\Gamma(2+ic_k)}. \quad (\text{B5})$$

Similarly, for $n=2$, use the relation

$$F_1(a, c-b, b, c, x, y) = (1-x)^{-a} {}_2F_1\left(a, b, c; \frac{y-x}{1-x}\right) \quad (\text{B6})$$

to write

$$F_1\left(1+ia_k, 2, ic_k, 2+ic_k; -\frac{r_+ - r_-}{r_-}, -\frac{r_+ - r_-}{r_+ + 2r_-}\right) = \left(\frac{r_+}{r_-}\right)^{-1-ia_k} {}_2F_1\left(1+ia_k, ic_k, 2+ic_k; \frac{r_+^2 - r_-^2}{r_+(r_+ + 2r_-)}\right) \quad (\text{B7})$$

and hence

$$I_2 = A_k \Gamma(1+ia_k)\Gamma(1-ib_k) r_-^{-2} \left(\frac{r_+}{r_-}\right)^{-1-ia_k} \times \frac{{}_2F_1\left(1+ia_k, ic_k, 2+ic_k; \frac{r_+^2 - r_-^2}{r_+(r_+ + 2r_-)}\right)}{\Gamma(2+ic_k)}. \quad (\text{B8})$$

Finally, use

$$F_1(a, c-b+1, b, c, x, y) = (1-x)^{-a} \left[{}_2F_1\left(a, b, c; \frac{y-x}{1-x}\right) + \frac{a}{c} \frac{x}{1-x} \times {}_2F_1\left(a+1, b, c+1; \frac{y-x}{1-x}\right) \right] \quad (\text{B9})$$

to obtain

$$F_1\left(1+ia_k, 3, ic_k, 2+ic_k; -\frac{r_+ - r_-}{r_-}, -\frac{r_+ - r_-}{r_+ + 2r_-}\right) = \left(\frac{r_+}{r_-}\right)^{-1-ia_k} \left[{}_2F_1\left(1+ia_k, ic_k, 2+ic_k; \frac{r_+^2 - r_-^2}{r_+(r_+ + 2r_-)}\right) - (1+ia_k) \frac{r_+ - r_-}{r_+} \times \frac{{}_2F_1\left(1+ia_k, ic_k, 3+ic_k; \frac{r_+^2 - r_-^2}{r_+(r_+ + 2r_-)}\right)}{2+ic_k} \right] \quad (\text{B10})$$

and hence

$$\begin{aligned}
I_3 = & A_k \Gamma(1+ia_k) \Gamma(1-ib_k) r_-^{-3} \left(\frac{r_+}{r_-} \right)^{-1-ia_k} \\
& \times \left[\frac{{}_2F_1\left(1+ia_k, ic_k, 2+ic_k; \frac{r_+^2-r_-^2}{r_+(r_++2r_-)}\right)}{\Gamma(2+ic_k)} \right. \\
& - (1+ia_k) \frac{r_+-r_-}{r_+} \\
& \left. \times \frac{{}_2F_1\left(1+ia_k, ic_k, 3+ic_k; \frac{r_+^2-r_-^2}{r_+(r_++2r_-)}\right)}{\Gamma(3+ic_k)} \right]. \tag{B11}
\end{aligned}$$

To determine the pole structures of the three quantities I_0, I_2, I_3 , note that all of them contain combinations of the form

$$\frac{{}_2F_1(1+ia_k, ic_k, n+ic_k; r_1)}{\Gamma(n+ic_k)}. \tag{B12}$$

One might notice that both the quantities ${}_2F_1(1+ia_k, ic_k, n+ic_k; r_1)$ and $\Gamma(n+ic_k)$ have poles at negative integral values of $n+ic_k$ —however, their ratio turns out to be regular everywhere [84]. This implies that the poles of I_0, I_2, I_3 occur only at the poles of the Gamma functions $\Gamma(1+ia_k)$ and $\Gamma(1-ib_k)$, which is what we have been using in the main text.

APPENDIX C: QNMS FOR A DIFFERENT SET OF BOUNDARY CONDITIONS

One should note that the solutions (9) used are appropriate for describing scattering off the black hole. As a mathematical possibility, it might be interesting to see what happens if one uses a different boundary condition in the form of the solutions which represent the *scattering off the cosmological horizon*, i.e.,

$$F(r) \sim \begin{cases} \bar{A}_{\text{in}} e^{-ikr_*} + \bar{A}_{\text{out}} e^{ikr_*} & (\text{at } r_* \rightarrow -\infty), \\ e^{-ikr_*} & (\text{at } r_* \rightarrow \infty). \end{cases} \tag{C1}$$

Note that in this case, the “incident” wave is propagating towards the cosmological horizon. For the pure black hole case, these solutions are irrelevant as they correspond to the physically unacceptable case where the incident waves propagate towards spatial infinity. These solutions are not relevant for the pure DS case too; this is because the potential is nonzero at $r=0$ and hence it is never possible to have an ingoing plane wave at this region. However, in the case of the SDS spacetime, one might consider the above case for studying scattering off the cosmological horizon at $r_* \rightarrow \infty$. Clearly, the scattering amplitude in this case will be exactly like Eq. (10), i.e., $S(k) \propto \bar{A}_{\text{out}}/\bar{A}_{\text{in}}$, and the QNMs, still defined by the boundary conditions (11), are given by the poles of the scattering amplitude. Setting $\bar{A}_{\text{in}}=0$ in Eq. (C1) leads to the same boundary condition (11) as obtained by setting $A_{\text{in}}=0$ in Eq. (9). In that case, we will obtain the same poles for the scattering matrix as before. If, on the other hand, we also change the sign of \mathbf{k}_i in the definition of the scattering amplitude when we consider the “incident” wave traveling in the opposite direction, then poles flip sign. In this case, the scattering amplitude in the first Born approximation is given by

$$S(k) = \int_{-\infty}^{\infty} dr_* V(r) e^{-2ikr_*}. \tag{C2}$$

Note that because of the different form of the “incident” wave, the sign of k in the above equation is different from the corresponding equation (21) in the previous case. The analysis of Sec. III goes through identically, except that *there is a crucial difference in the sign of k* . The explicit form of $S(k)$ can be calculated exactly as in Sec. III, and because of the change in the sign of k , one would find that the poles occur at

$$k_n = -in\kappa_-, \quad k_n = in\kappa_+ \quad (n \geq 1). \tag{C3}$$

This shows that the QNMs, given by positive imaginary k , are now determined by the surface gravity of the cosmological horizon, as expected. While this is certainly a mathematical possibility, the boundary conditions in Eq. (C1) seems artificial from the physical point of view.

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