As a result, the two most prominent experimental evidences of general relativity are duplicated by this much simpler theory. The experiments are the measurement of the precession of the orbit of Mercury and the deflection of light near the Sun. The result for gravitational red shift also matches with the present theory. But it has not been worked out here on grounds of simplicity.

The aim of this article is not to prove the inadequacy of general relativity, but rather to demonstrate the lack of strong experimental backing. If a relatively simple theory like the present one can produce the same experimental consequences, then quite obviously greater effort should be made to test general relativity more rigorously.

The other purpose of this article is to provide a more physical insight into the results of general relativity.

[1]P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Englewood Cliffs, NJ, 1942), pp. 217–218 (precession from special relativity); pp. 212–217 (precession from general relativity); and pp. 218–221 (deflection of light near the Sun).
[2]T. E. Phipps, Jr., Am. J. Phys. **54**, 245 (1986).
[3]W. E. Thirring, Ann. Phys. **16**, 96 (1961).
[4]R. P. Feynman, *Lectures on Gravitation* (California Institute of Technology, Pasadena, 1971).

On Feynman's formula for the electromagnetic field of an arbitrarily moving charge

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A simple derivation of Feynman's formula for the electromagnetic field of an arbitrarily moving charge, starting from Maxwell's equations, is presented. Feynman's formula is also related to the standard expressions for the fields of a moving charge.

I. INTRODUCTION

In his lectures on the electromagnetic field,[1] Feynman gives a remarkable formula for the electric and magnetic fields of a charge q moving along an arbitrary trajectory,

$$\mathbf{E}(\mathbf{x},t) = q\left[\frac{\mathbf{n}}{R^2} + \frac{R}{c}\frac{d}{dt}\left(\frac{\mathbf{n}}{R^2}\right) + \frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}\right], \tag{1}$$

$$\mathbf{B}(\mathbf{x},t) = \mathbf{n}\times\mathbf{E}.$$

Here, R is the distance from the *retarded* position of the charge to the point P where the field is being evaluated (see Fig. 1), and $\mathbf{n}$ is a unit vector pointing from the retarded position to the field point P. Recall that the retarded position is the location of the charge at an earlier time t', such that $(t - t')$ is the time light would take to travel the distance R from the charge to the field point. That is,

$$R(t')/c = t - t'. \tag{2}$$

Thus the fields at time t are determined by the position and motion of the charge at the retarded time t'. Feynman also gives a simple interpretation to his formula for the electric field. The first term is the Coulomb field of the charge at its retarded position. The second term takes account of the motion of the charge and is roughly the time rate of change of the retarded Coulomb field multiplied by the time the charge takes to travel from the retarded to the present position. The third term is in no obvious way a higher-order correction to the electric field. However, it can be easily shown[2] that at large distances from the charge the first two terms fall off inversely as the square of the distance R, whereas the third term falls off as $1/R$. Thus the third term is actually the radiative field of the charge and, far away from the charge it is proportional to the component of the acceleration at right angles to the line of sight.

Feynman's formula provides beautiful insight into the way the fields depend on the motion of the charge. Equation (1) can help a student visualize electromagnetic fields in a nice intuitive manner. Unfortunately, standard texts on electromagnetism[3] do not seem to provide a derivation of this formula. Feynman provides a partial derivation of this formula (see Ref. 1, Vol. II); but it does not go far enough. It is probably worthwhile to derive this formula from Maxwell's equations and indicate its relationship with more conventional formulas (see, e.g., Landau and

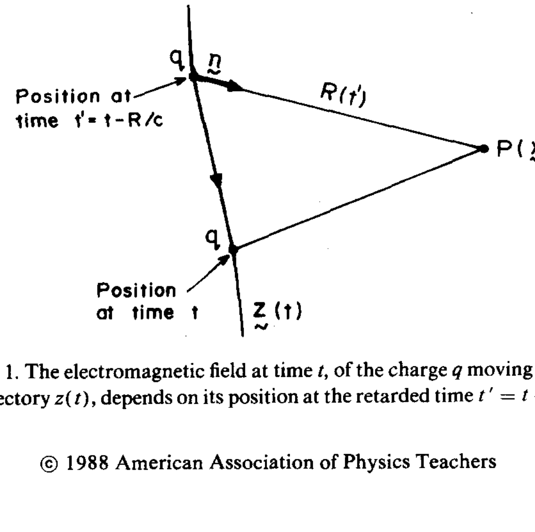

Fig. 1. The electromagnetic field at time t, of the charge q moving along a trajectory $z(t)$, depends on its position at the retarded time $t' = t - R/c$.

Lifshitz, Ref. 3) describing the electromagnetic field of a charged particle in arbitrary motion. Here, we present such a derivation, which we believe is quite simple and instructive. We also relate Feynman's formula to the usual expressions for the electromagnetic field of a charge. The derivations essentially depend on some of the properties of the Dirac delta function, which should be familiar to the student from a first course in quantum mechanics.

II. DERIVATION OF THE FEYNMAN FORMULA

We start with the Maxwell equations for the electromagnetic fields (written in standard notation),

$$\nabla\cdot\mathbf{E} = 4\pi\rho, \quad \nabla\cdot\mathbf{B} = 0,$$
$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \quad \nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}. \tag{3}$$

Introducing the usual definitions for the scalar potential $\phi(\mathbf{x},t)$ and the vector potential $\mathbf{A}(\mathbf{x},t)$,

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla\times\mathbf{A} \tag{4}$$

and imposing the Lorentz gauge condition

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0, \tag{5}$$

Maxwell equations can be rewritten as wave equations for the potentials

$$\Box\,\phi(\mathbf{x},t) = 4\pi\rho(\mathbf{x},t),$$
$$\Box\,\mathbf{A}(\mathbf{x},t) = (4\pi/c)\,\mathbf{j}(\mathbf{x},t). \tag{6}$$

Here, $\Box$ stands for the d'Alembertian operator,

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \tag{7}$$

We now specialize Eqs. (6) to the case of a point charge q moving along a trajectory $\mathbf{z}(t)$, for which the charge density and the current density are given by

$$\rho(\mathbf{x},t) = q\,\delta(\mathbf{x}-\mathbf{z}(t)), \quad \mathbf{j} = q\mathbf{v}\,\delta(\mathbf{x}-\mathbf{z}(t)), \tag{8}$$

where $\mathbf{v}$ is the three-velocity of the charge. As shown in the Appendix, the potentials for such a source are given by

$$\phi(\mathbf{x},t) = \int d^3\mathbf{y}\,dt'\,\frac{q}{R}\,\delta(\mathbf{y}-\mathbf{z}(t'))\delta\left(t'-t+\frac{R}{c}\right), \tag{9}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{c}\int d^3\mathbf{y}\,dt'\,\frac{q\mathbf{v}(t')}{R} \times\delta(\mathbf{y}-\mathbf{z}(t'))\delta\left(t'-t+\frac{R}{c}\right), \tag{10}$$

where $R = |\mathbf{x}-\mathbf{y}|$. From the presence of the delta functions in Eqs. (9) and (10), it is clear that the potentials depend on the retarded position and velocity of the charge.

At this stage, one could carry out the integrations in Eqs. (9) and (10) and obtain the usual expressions for the Lienard–Wiechert potentials. The electromagnetic fields are then obtained using Eq. (4). However, it turns out to be convenient for our purpose to find $\partial\mathbf{A}/\partial t$ and $\nabla\phi$, using ϕ and $\mathbf{A}$ as in (9) and (10). We first evaluate $\nabla\phi$. Using the basic result,

$$\int f(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})d^3\mathbf{x} = f(\mathbf{y}), \tag{11}$$

we can carry out the spatial integrations in (9) and obtain

$$\phi(\mathbf{x},t) = \int dt'\,\frac{q}{R}\delta\left(t'-t+\frac{R}{c}\right), \tag{12}$$

where now $R(t') = |\mathbf{x}-\mathbf{z}(t')|$ and, hence, $\mathbf{v}(t') = -d\mathbf{R}/dt'$. Therefore,

$$\nabla\phi \equiv \frac{\partial\phi}{\partial\mathbf{x}} = q\int dt'\left[-\frac{\mathbf{R}}{R^3}\delta\left(t'-t+\frac{R}{c}\right)\right] + \frac{\mathbf{R}}{cR^2}\frac{d\delta}{df}, \tag{13}$$

where $f(t') = t'-t+R/c$. The first term in the above integral can be simplified by using the formula

$$\delta[f(t')] = \sum_i \frac{\delta(t'-t_i)}{|df/dt'|_{t'=t_i}}, \tag{14}$$

where t_i are the zeros of $f(t')$.

To integrate the second term, we note that

$$\frac{d\delta}{df} = \left(\frac{df}{dt'}\right)^{-1}\frac{d\delta}{dt'} \tag{15}$$

and that

$$\frac{df}{dt'} = 1 - \frac{\mathbf{v}\cdot\mathbf{R}}{Rc} \equiv 1 - v_R. \tag{16}$$

Making the above substitutions, and carrying out a partial integration on the second term, finally gives

$$\nabla\phi = -\frac{q}{(1-v_R)}\frac{\mathbf{R}}{R^3} - \frac{q}{(1-v_R)} \times\frac{d}{c\,dt'}\left(\frac{\mathbf{R}}{R^2(1-v_R)}\right), \tag{17}$$

where now t' is the retarded time and is given by Eq. (2).

Similarly, it can be shown that

$$\frac{\partial\mathbf{A}}{\partial t} = \frac{1}{1-v_R}\frac{d}{c\,dt'}\left(\frac{q\mathbf{v}}{R(1-v_R)}\right). \tag{18}$$

Thus, from (4), we get

$$\mathbf{E}(\mathbf{x},t) = \frac{\mathbf{n}}{R^2}\frac{q}{(1-v_R)} + \frac{q}{(1-v_R)}\frac{d}{c\,dt'}\left(\frac{\mathbf{R}-(\mathbf{v}/c)R}{R^2(1-v_R)}\right). \tag{19}$$

Equation (19) is our main result. From this point, one can follow two different routes. If we carry out the differentiations in (19) we will arrive at the standard results quoted in textbooks. To arrive at Feynman's formula, we should proceed differently. To begin with, we note from Eq. (2) that

$$\frac{dt'}{dt} = 1 - \frac{dR}{c\,dt} = (1-v_R)^{-1} \equiv 1 - \frac{\dot{R}}{c}. \tag{20}$$

With this substitution, Eq. (19) becomes

$$\mathbf{E} = \frac{q\mathbf{n}}{R^2}\left(1-\frac{\dot{R}}{c}\right) + q\frac{d}{c\,dt}\left[\frac{(1-\dot{R}/c)}{R}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\right]$$
$$= \frac{q\mathbf{n}}{R^2} - \underbrace{\frac{q\mathbf{n}}{R^2}\frac{\dot{R}}{c} + \frac{q}{c}\frac{d}{dt}\frac{\mathbf{n}}{R}}\; \underbrace{-\frac{q}{c}\frac{d}{dt}\left(\frac{\mathbf{n}\dot{R}}{cR}\right) + \frac{q}{c^2}\frac{d}{dt}\left(\frac{1}{R}\frac{d}{dt}(R\mathbf{n})\right)}$$
$$= q\frac{\mathbf{n}}{R^2} + q\frac{R}{c}\frac{d}{dt}\left(\frac{\mathbf{n}}{R^2}\right) + \frac{q}{c^2}\frac{d^2\mathbf{n}}{dt^2}. \tag{21}$$

Thus Eq. (19) is related in a rather simple way to Feynman's formula!

For the sake of completeness, we also mention the derivation of the usual expression for the electromagnetic field due to a charged particle, starting from (19). All that one has to do is to carry out the time differentiation in (19), noting that

$$\frac{d}{dt'}v_R = \frac{1}{c}\frac{d}{dt'}(\mathbf{v}\cdot\mathbf{n}) . \qquad (22)$$

The final result can be written in the form

$$\mathbf{E} = \frac{q(1-v^2/c^2)}{R^2(1-v_R)^3}\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right) + \frac{q}{R^2(1-v_R)^3} \times \mathbf{R}\times\left[\left(\mathbf{n}-\frac{\mathbf{v}}{c}\right)\times\frac{d\mathbf{v}}{c^2\,dt'}\right], \qquad (23)$$

where all quantities on the right-hand side refer to the retarded time t'. Since we have derived both (21) and (23) from (19), the equivalence of these formulas is apparent.

It would be interesting to examine the generalization of Feynman's formula for a charged particle moving in a geodesic in a curved space-time.

APPENDIX

We solve Eqs. (6) by first writing them as relativistic equations for the four-potential $A^\mu = (\phi,\mathbf{A})$

$$\Box A^\mu = (4\pi/c)J^\mu , \qquad (A1)$$

with $J^\mu = (c\rho, \mathbf{j})$. The solution for A^μ may then be obtained with the help of a Green's function $G(x,x')$, which satisfies the equation

$$\Box_x G(x,x') = \delta(x-x') . \qquad (A2)$$

Here, $x \equiv (ct,\mathbf{x})$ and $\delta(x-x') = \delta(ct-ct')\delta(\mathbf{x}-\mathbf{x}')$. Translational invariance implies that $G(x,x') = G(x-x') \equiv G(X)$. Introducing the Fourier transform $G(k)$

$$G(X) = \frac{1}{(2\pi)^4}\int d^4k\, G(k)e^{-ikX} \qquad (A3)$$

and using the integral representation for the delta function

$$\delta(x) = \frac{1}{(2\pi)^4}\int d^4k\, e^{-ikx}, \qquad (A4)$$

we find that

$$G(k) = -1/k^2 , \qquad (A5)$$

where $k^2 = k_0^2 - |\mathbf{k}|^2$. Thus $G(X)$ is given by

$$G(X) = \frac{-1}{(2\pi)^4}\int d^4k (e^{-ikX}/k^2) . \qquad (A6)$$

The integrand in (A6) has two simple poles at $k_0 = \pm|\mathbf{k}|$. The retarded Green's function is obtained by treating k_0 as the limit of a complex variable $k_0 = \lim_{\epsilon\to 0}(k_0 + i\epsilon)$; $\epsilon > 0$, and closing the contour of integration in the lower half of the complex k_0 plane. The result of the integration is

$$G_{\rm ret}(x-x') = [\theta(t-t')/4\pi Rc]\delta(t-t'+R/c), \qquad (A7)$$

where $R = |\mathbf{x}-\mathbf{x}'|$. Assuming there are no incoming fields, the solution of (A1) can be written as

$$A^\mu(x) = \frac{4\pi}{c}\int d^4x'\, G_{\rm ret}(x-x')J^\mu(x') . \qquad (A8)$$

Using $G_{\rm ret}$ from above, assuming $t > t'$, and taking $J^\mu(x)$ to be of the form given in Eq. (8), we find ϕ and $\mathbf{A}$ to be as given in Eqs. (9) and (10).

[1]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures in Physics* (Addison-Wesley, Reading, MA, 1963), Vol. II, Chap. 21.
[2]Reference 1, Vol. I, Chap. 28.
[3]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1979), Chap. 8; J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), Chap. 14; and E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1965), Berkeley Physics Course, Vol. 2.

PROBLEM: THREE MOVING CLOCKS IN A GRAVITATIONAL FIELD

Assume that we have three particles A, B, and C: A is at rest on the surface of the Earth (at one of the poles of the Earth); B and C are launched simultaneously from point A in the horizontal direction and vertical direction, respectively. The velocity of B is the orbital velocity and the initial velocity of C is so adjusted that when C falls back onto the ground it can be reunited with B at point A after B moves around the Earth one circle. Find the times T_A, T_B, and T_C measured by the three clocks associated with three bodies A, B, and C, respectively, during this period. (In Newtonian physics, these three times should all be identical.) (Solution is on p. 1048.)