

A possible Newtonian interpretation of relativistic cosmological perturbation theory

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Cosmological perturbations with wavelengths smaller than Hubble radius can be handled in the context of Newtonian theory with very high accuracy. The application of this Newtonian approximation, however, is restricted to nonrelativistic matter and cannot be used for relativistic matter. Recently, by modifying the continuity equation, Lima, et. al., extended the domain of applicability of Newtonian cosmology to radiation dominated phase. We adopted this continuity equation to re-examine linear cosmological perturbation theory for a two fluid universe with uniform pressure. We study the evolution equations for density contrasts and their validity in different epochs and on scales larger than Hubble radius and compare the results with the full relativistic approach. The comparison shows the high accuracy of this approximation.

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I. INTRODUCTION

The real universe contains inhomogeneous structures like galaxies, clusters etc and, in any theory of the formation of these structures, it is essential to understand the evolution of small inhomogeneities in the early universe. In principle, it is straightforward to work out the general relativistic theory of linear perturbations [1,2]. By linearizing Einstein's equations, we can obtain a second-order differential equation of the form

$$\hat{\mathcal{L}}(g_{\alpha\beta}) \delta g_{\alpha\beta} = \delta T_{\alpha\beta}, \quad (1)$$

where $\hat{\mathcal{L}}$ is a linear differential operator depending on the background metric, $g_{\alpha\beta}$, and the set $(\delta g_{\alpha\beta}, \delta T_{\alpha\beta})$ denotes the perturbations in the metric and stress tensor about an expanding FRW background.

In practice, however, there are many complications and conceptual difficulties which make this analysis highly nontrivial. One issue is the so-called "gauge problem" (extensively discussed in the literature; see [3-13]) which arises due to nonuniqueness of splitting the metric and matter variables into a zeroth order background and small, first-order perturbations. Since a relabeling of coordinates $x^\alpha \rightarrow x^{\alpha'}$ can make a small $\delta T_{\alpha\beta}$ large or even generate a component which was originally absent, one must care to factor out effects due to coordinate transformations, when analyzing relativistic perturbations. There are two different ways of handling these difficulties in general relativity. One approach is to analyze a perturbation in a particular gauge, say, synchronous gauge [14]. In this case, one specifically identifies the points of a fictitious background spacetime with those of real spacetime, and treat δT^0_0 to be the perturbed mass density etc. In this method, however, we cannot fix the gauge completely and the residual gauge ambiguities can create some problems. The second method is to construct the perturbed physical variables in a gauge-invariant manner. The gauge-invariant approach is conceptually more attractive since there is no need for specific identification of the points between the two spacetimes, though it is more complicated and the physical meaning of variables do not, in general, possess any simple interpretation and becomes obvious only for specific observers.

It is convenient to divide the cosmological perturbations into two subclasses: (i) Perturbations with wavelengths larger than Hubble scale ($\lambda > d_H$), for which we have to use some form of a general relativistic theory of perturbations and (ii) small-scale perturbations ($\lambda < d_H$) for which the evolution of mass density can be studied using Newtonian theory. In this context, all physical quantities can be defined unambiguously to the order of accuracy needed. In general, the application of Newtonian equations is further restricted to nonrelativistic matter and cannot be used for relativistic component even for scales much smaller than Hubble radius ($\lambda \ll d_H$).

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Recently, Lima [15], et al., re-examined the basic equations describing a Newtonian universe with *uniform pressure* and found a way for obtaining the same evolution equation for density contrast as could be obtained by the full relativistic approach. They achieved this goal by modifying the continuity equation in an expanding background. Using this result, they argued that one can extend the domain of validity of Newtonian cosmology [16] in order to analyze some problems of formation of structures even in the radiation dominated phase.

In this paper we extend the result of reference [15] to a multi-component universe with different equations of state. We shall then consider a two fluid universe in the context of “*pseudo Newtonian*” cosmology. Comparison with the fully relativistic two-component universe reveals the high accuracy of density contrast equations in the pseudo Newtonian cosmology.

The organization of this paper is as follows: In section II we will use the modified continuity equation in order to find out the evolution equation for density contrast of a multi-component universe with arbitrary equations of state. This is actually the generalization of paper by Lima [15], et. al. In section III we will examine the result of section II for the two fluid system with radiation and dark matter. In this section we will try to give approximate solutions to coupled equations of density contrast of radiation and dark matter. We shall compare the result in this context with those ones as obtained in fully relativistic approach, for example given in ref [17]. We give the conclusions in section IV.

II. THE LINEAR PERTURBATION THEORY FOR A MULTI-COMPONENT UNIVERSE

The set of equations describing a several component universe, in the proper coordinates (t, \mathbf{r}) with $\mathbf{r} = a(t)\mathbf{x}$, can be written as, (i) “*Continuity*” equation, (ii) *Euler* equation, and (iii) *Poisson* equation,

$$\frac{\partial \rho}{\partial t} + \nabla_r \cdot (\rho \mathbf{u}) + p \nabla_r \cdot \mathbf{u} = 0, \quad (2)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla_r) \mathbf{u} = -\nabla_r \psi - (\rho + p)^{-1} \nabla_r p, \quad (3)$$

$$\nabla_r^2 \psi = 4\pi G(\rho + 3p), \quad (4)$$

where the “*continuity*” equation is the one that has been used by Lima [15], et. al., for an expanding universe, to study the evolution of cosmological perturbation using linear regime. Here, the velocity of light, c , assumed to be unity and the quantities

$$\rho = \sum_{i=1}^N \rho_i, \quad p = \sum_{i=1}^N p_i, \quad \mathbf{u} = \sum_{i=1}^N \mathbf{u}_i \quad \text{and} \quad \psi = \sum_{i=1}^N \psi_i \quad (5)$$

denote the total density, total pressure, total field velocity, and the total generalized gravitational potential of the cosmic fluid.

In an expanding Newtonian universe the evolution of small fluctuations can be studied in the usual perturbation theory, i.e, by perturbing the background density, pressure, field velocity, and gravitational potential. That is, we take

$$\rho(\mathbf{r}, t) = \rho_{bg}(t)[1 + \delta(\mathbf{r}, t)], \quad (6)$$

$$p(\mathbf{r}, t) = p_{bg}(t) + \delta p(\mathbf{r}, t), \quad (7)$$

$$\psi(\mathbf{r}, t) = \psi_{bg}(\mathbf{r}, t) + \varphi(\mathbf{r}, t), \quad (8)$$

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{u}_{bg} + \mathbf{v}(\mathbf{r}, t), \quad (9)$$

where the second term is a small correction. By inserting the above expressions into Eqs (1), linearizing the resulting equations to first order in perturbations, and transforming to comoving coordinates by $\mathbf{x} = (\mathbf{r}/a)$, we get,

$$\begin{aligned} \dot{\delta}^{(X)} + \frac{1 + \nu^{(X)}}{a} \nabla_x \cdot \mathbf{v}^{(X)} &= 0 \\ \dot{\mathbf{v}}^{(X)} + \frac{\dot{a}}{a} \mathbf{v}^{(X)} &= -\frac{1}{a} \nabla_x \varphi - \frac{v_s^2(X)}{(1 + \nu^{(X)})a} \nabla_x \delta^{(X)} \\ \nabla_x^2 \varphi &= 4\pi G a^2 \sum_i (1 + 3\nu_i) \rho_{bg_i} \delta_i \end{aligned} \quad (10)$$

where we have assumed that the expansion is adiabatic and thus $v_s^{(X)^2} = (\delta p / \delta \rho) = \nu^{(X)}$, is the sound velocity. Here (X) denotes the species under consideration and $p_i = \nu_i \rho_i$. The sum on the right hand side is over all components. Note that ν_{tot} and $v_{s_{tot}}^2$ are not constant any more and thus the relation $v_{s_{tot}}^2 = \nu_{tot}$ does not hold for the full system.

By eliminating the peculiar velocity between the above equations, and Fourier transforming the perturbation such that $\nabla_x^2 \delta^{(X)} = -k^2 \delta^{(X)}$ etc., we get

$$\ddot{\delta}_k^{(X)} + 2\frac{\dot{a}}{a}\dot{\delta}_k^{(X)} + \left(\frac{k^{(X)}v_s^{(X)}}{a}\right)^2 \delta_k^{(X)} = 4\pi G(1 + \nu^{(X)}) \sum_i (1 + 3\nu_i)\rho_{bg_i}\delta_{ki}. \quad (11)$$

This is a peculiar equation in the sense that there is no counterpart for this equation in the fully relativistic treatment, unless we impose the synchronous gauge and comoving gauge simultaneously. (one cannot impose the two gauge conditions simultaneously even in large scale limit in the presence of pressure, since we get $\delta = 0$ system when we ignore the entropic and anisotropic pressures [18].) In the case of single component medium [15] and $(k^2 v_s^2 / a^2) \ll 1$, however, equation (11) leads to the same equation derived in full relativistic approach by imposing the two gauge conditions simultaneously (see for example, eq. (15.10.57) in ref. [14], eq. (10.118) in ref. [19] and eq. (6.136) in ref. [20]).

III. TWO FLUID UNIVERSE

Now, we shall assume that our universe consists of radiation [R] and dark matter [DM] with $\nu_R = 1/3 = v_R^2$ and $\nu_{DM} = v_{DM}^2 = 0$. Then the evolution of $\delta_k^R(t)$ and $\delta_k^{DM}(t)$ can be determined from eq. (11), to give

$$\ddot{\delta}_R + 2\frac{\dot{a}}{a}\dot{\delta}_R + \left[\frac{k^2 v_R^2}{a^2} - \frac{32\pi G}{3}\rho_R \left(1 + \frac{\delta\rho_{DM}}{2\delta\rho_R}\right)\right] \delta_R = 0. \quad (12)$$

$$\ddot{\delta}_{DM} + 2\frac{\dot{a}}{a}\dot{\delta}_{DM} - 4\pi G\rho_{DM} \left(1 + \frac{2\delta\rho_R}{\delta\rho_{DM}}\right) \delta_{DM} = 0. \quad (13)$$

Though these equations cannot be solved exactly in closed form all the important properties of the solutions can be obtained by suitable approximations. We shall now discuss these properties by introducing the new variable $x \equiv (a/a_{eq})$ where a_{eq} is the expansion factor at the time $t = t_{eq}$. Transforming from t to x as independent variable we can re-express all the quantities in terms of x ,

$$\frac{\rho_{DM}}{\rho_{eq}} = \frac{1}{2x^3}; \quad \frac{\rho_R}{\rho_{eq}} = \frac{1}{2x^4}; \quad \frac{\rho_{tot}}{\rho_{eq}} = \frac{1}{2x^4}(x+1). \quad (14)$$

Since $(p_{tot}/\rho_{eq}) = (p_{DM} + p_R)/\rho_{eq} = (1/6x^4)$ we get the following result for the full two component system

$$\nu = \nu_{tot} = \frac{p_{tot}}{\rho_{tot}} = \frac{p_{DM} + p_R}{\rho_{DM} + \rho_R} = \frac{1}{3(1+x)}, \quad (15)$$

and

$$H^2(x) = H_{eq}^2 \frac{1}{2x^4}(1+x). \quad (16)$$

On the other hand, one can express the combination $(k^2/H^2 a^2)$ as

$$\frac{k^2}{H^2 a^2} = \frac{2x^2}{(1+x)}\omega^2, \quad (17)$$

where ω is defined by the ratio $2\pi\omega = [d_H(a_{eq})/\lambda(a_{eq})]$. In terms of this variable, x , the time derivatives can be given by

$$\begin{aligned} \frac{d}{dt} &= Ha \frac{d}{da} = H(x)x \frac{d}{dx} \equiv H\hat{D} \\ \frac{d^2}{dt^2} &= H^2 a \frac{d}{da} \left(a \frac{d}{da} \right) - \frac{3}{2}H^2(1+\nu)a \frac{d}{da} \equiv H^2\hat{D}^2 - \frac{3}{2}H^2(1+\nu)\hat{D}, \end{aligned} \quad (18)$$

where $\hat{D} = x(d/dx)$ and we have used the new continuity equation (2). With these modifications, the coupled equations for δ_R and δ_{DM} become

$$\left[\hat{D}^2 + \frac{1}{2} \frac{x}{(1+x)} \hat{D} + \left(\frac{2}{3} \frac{\omega^2 x^2}{(1+x)} - \frac{4}{(1+x)} \right) \right] \delta_R = \frac{2x}{(1+x)} \delta_{DM}, \quad (19)$$

and

$$\left[\hat{D}^2 + \frac{1}{2} \frac{x}{(1+x)} \hat{D} - \frac{3}{2} \frac{x}{(1+x)} \right] \delta_{DM} = \frac{3}{(1+x)} \delta_R. \quad (20)$$

A particular mode, labeled by the parameter ω , enters the Hubble radius in the radiation dominated phase if $\omega > 1$ and in matter dominated phase if $\omega < 1$. The ω and x_{ent} are related to each other through the relation $\omega^2 x_{ent}^2 = 2\pi^2(1+x_{ent})$. We shall consider the case $\omega^2 \gg 1$ which has more features and is more relevant to this paper. Then three cases need to be discussed. (i) When $x\omega \ll 1$, in which case the wave length of the mode is bigger than the Hubble radius; (ii) when $\omega^{-1} \ll x \ll 1$, the mode has entered the Hubble radius in the radiation dominated phase; and finally the case, (iii) when $\omega x \gg 1$, the mode is inside the Hubble radius in a matter dominated phase. Let us now consider each case.

Case (i) $x\omega \ll 1$

In this case the equation (19) and (20) give

$$\left(\hat{D}^2 - 4 \right) \delta_R \cong 0; \quad \hat{D}^2 \delta_{DM} \cong 3\delta_R. \quad (21)$$

The growing solution is

$$\frac{3}{4} \delta_R = \delta_{DM} = x^2 \quad (22)$$

Thus both radiation and dark matter grow as $x^2 \propto a^2$. Obviously, since the universe is radiation dominated δ_{DM} is driven by δ_R and its back reaction on δ_R is negligible. This result matches with the fully relativistic one!

Case (ii) $\omega x \gg 1$ and $x \ll 1$

The concerned equations in this approximation are

$$\left[\hat{D}^2 + \frac{2}{3} \omega^2 x^2 \right] \delta_R \cong 0; \quad \hat{D}^2 \delta_{DM} \cong 3\delta_R \quad (23)$$

The solution for δ_R equation would be of the form

$$\delta_R = C_1 J_0 \left[\sqrt{\frac{2}{3}} \omega x \right] + C_2 N_0 \left[\sqrt{\frac{2}{3}} \omega x \right], \quad (24)$$

where J_0 and N_0 are the Bessel functions of the first and second kind, respectively. If we use the fact that $\omega x \gg 1$ the approximate solution to δ_R becomes

$$\delta_R \simeq \frac{C}{\sqrt{\omega x}} \sin \left(\sqrt{\frac{2}{3}} \omega x \right), \quad (25)$$

which is oscillating rapidly since $\omega x \gg 1$. Substituting for δ_R in the second equation and using the fact that $x \ll 1$, we find that

$$\delta_{DM} \simeq -\frac{6C}{\sqrt{\omega x}} \sin \left(\sqrt{\frac{2}{3}} \omega x \right) + A \ln x + B. \quad (26)$$

Therefore the perturbation in dark matter grows essentially logarithmically during the period $\omega^{-1} \ll x \ll 1$, which also known from previous work. Though the wavelength of the dark matter perturbation is bigger than effective Jeans length, the growth of dark matter perturbation is prevented by the rapid expansion of the universe. The similarities of these results with their fully relativistic ones come from using the modified continuity equation, which-in turn-permits one to enlarge the domain of applicability of Newtonian cosmology even to radiation dominated phase.

Case (iii) $\omega x \gg 1$

This range corresponds to the matter dominated phase with mode inside the Hubble radius. The coupled equations now become

$$\left[\hat{D}^2 + \frac{1}{2} \hat{D} - \frac{3}{2} \right] \delta_{DM} \cong 0; \quad \left[\hat{D}^2 + \frac{1}{2} \hat{D} + \frac{2}{3} \omega^2 x \right] \delta_R \cong 2\delta_{DM} \quad (27)$$

Solving for δ_{DM} we get

$$\delta_{DM} = Ax + Bx^{-2/3} \cong Ax. \quad (28)$$

Plugging this into the δ_R equation we find that

$$\frac{d}{dx} \left(x \frac{d\delta_R}{dx} \right) + \frac{1}{2} \frac{d\delta_R}{dx} + \frac{2}{3} \omega^2 \left(\delta_R - \frac{3A}{\omega^2} \right) = 0 \quad (29)$$

Choosing, $\delta_R = (3A/\omega^2) + x^{-3/4} f(x)$, we get the following differential equation for $f(x)$

$$f''(x) = - \left(\frac{3}{16} \frac{1}{x^2} + \frac{2}{3} \frac{\omega^2}{x^2} \right) f(x) \cong - \frac{2}{3} \frac{\omega^2}{x} f(x). \quad (30)$$

By applying the WKB approximation, we may solve the f equation and get the final result for δ_R to be

$$\delta_R = \frac{3A}{\omega^2} + B \frac{1}{\sqrt{\omega x}} \exp \left(\pm i \sqrt{\frac{8}{3}} \omega x^{1/2} \right). \quad (31)$$

The oscillations presented by the second term, in δ_R , continue to dominate over the driving by the δ_{DM} term. Hence, the radiation density does not grow in the matter dominated universe, i.e., when $\lambda_R < d_H$. These solutions are very similar to the solutions to the fully relativistic equations. The reasons are clear during this phase: First of all, we are dealing with the modes which are well within the Hubble radius and secondly the modes are in the matter dominated phase.

IV. CONCLUSIONS

We have shown that the result of the linear cosmological perturbation of a two fluid universe can be obtained with high accuracy from Newtonian cosmology even in the presence of pressure and for scales larger than Hubble radius. This is obtained by using the modified Newtonian equations. In fact, by using the modified continuity equation one can get rid of a misleading pressure gradient term which is obtained in the semi-classical formulation and more over obtain the time evolution of the density contrast for any value of parameter ν . Comparison of (19) and (20) with fully relativistic equations, say, in the comoving gauge (see e.g. ref. [17]) given by:

$$\left[\hat{D}^2 + \frac{1}{2} \frac{x}{(1+x)} \hat{D} + \left(\frac{2}{3} \frac{\omega^2 x^2}{(1+x)} - \frac{4}{(1+x)} \right) \right] \delta_R - \frac{2x}{(1+x)} \delta_{DM} = F(\delta_R, \delta_{DM}), \quad (32)$$

and

$$\left[\hat{D}^2 + \frac{1}{2} \frac{x}{(1+x)} \hat{D} - \frac{3}{2} \frac{x}{1+x} \right] \delta_{DM} - \frac{3}{1+x} \delta_R = G(\delta_R), \quad (33)$$

with

$$F(\delta_R, \delta_{DM}) = \left[\hat{D} - \frac{4}{9} \frac{6x^2 + 13x + 8}{(x + 4/3)^2 (1+x)} \right] \delta_R - \frac{4}{3} \frac{x}{(x + 4/3)} \hat{D} \delta_{DM}, \quad (34)$$

and

$$G(\delta_R) = \frac{1}{x + 4/3} \left[\hat{D} - \frac{1}{3} \frac{6x^2 + 13x + 8}{(1+x)(x + 4/3)} \right] \delta_R, \quad (35)$$

will allow us to estimate the accuracy. Left hand sides of Eqs. (32) and (33) are the same as their pseudo Newtonian counterparts Eqs. (19) and (20), while F and G are the corrections coming from general relativity. Numerical comparison of F and G with their left hand sides in Eqs. (19) and (20) reveals that these corrections are very small and almost negligible indeed. This can be better seen in Figs. 1 and 2. In each of these figures the vertical axis is the ratio of pseudo Newtonian density contrast to its exact relativistic value at the same epoch. Both the figures show that this ratio for the radiation component is almost unity and the value of dark matter density contrast in pseudo Newtonian case differs at most by a factor 2. As we discussed earlier, the solutions of Eqs. (19) and (20) for three different cases are in a very good agreement with the solutions of Eqs. (32) and (33). This agreement for $\omega \gg 1$ is more profound (see Fig. 1 and Fig. 2). In other words, the ratio $R = (\delta_{Newton}/\delta_{exact})$ for radiation tends to unity for all epochs as we increase the value of ω , though remains off for dark matter by a lesser factor. It seems that the Friedmann's equations have strong correspondence with Newtonian theory, even more than one might have naively expected.

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Figure captions

Fig. 1. Comparison of perturbation amplitudes in dark matter (DM) in *pseudo Newtonian cosmology* (PNC) and *relativistic cosmology* (RC). The amplitude of DM in NC is more than RC with an error of less than factor 2 for $\omega = 100$. After $z = 390$ up to present time the ratio becomes constant. The radiation amplitude in RC is closer to that in NC. The behavior of two perturbations are almost the same except for some small phase difference. The ratio $R = (\delta_{Newton}/\delta_{exact})$ for radiation is almost one except for some tiny fluctuations.

Fig. 2 This is same as Fig. 1 with $\omega = 200$. Note that the agreement between the modes are more pronounced. The ratio for dark matter in this case is less than 2 and it becomes a constant after $z = 39$. In radiation dominated phase when modes are inside the Hubble radius, modes are almost in phase and again amplitude of radiation is more in RC compare to PNC at any instant.



