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Magnetic charges and local duality symmetry - II

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Abstract

The notion of magnetic charges is intimately linked with the global duality symmetry exhibited by the extended Maxwell equations. We show that duality symmetry is meaningful only in 3+1 dimensional space-times, implying thereby that magnetic monopoles as fundamental particles can be postulated only in 3+1 dimensions. We also study the consequences of elevating the status of duality symmetry to a local symmetry. Local duality symmetry is achieved by introducing a complex scalar field in the theory. The new theory is a generalization of the extended Maxwell theory. In this formalism, the electromagnetic charges arise due to a spontaneous symmetry breaking. By choosing a suitable gauge one can recover ordinary electrodynamics without the magnetic charges.

1. Introduction

Magnetic charge as a concept is very interesting because of several reasons. Firstly, they make the structure of classical electrodynamics more symmetric. The second reason is that existence of a single magnetic monopole in the universe can explain charge quantization [1]. Furthermore, 't Hooft [2] and Polyakov [3] showed that magnetic monopoles are generic in grand unified theories.

This work is an extension of our earlier work [4] which directed one's attention to the duality symmetry that exists when magnetic charges are allowed in the theory. In section 2, we show that the duality symmetry is meaningful only in 3+1 dimensional space-times, implying that the notion of magnetic charge is linked with the dimensionality of space-time. Then, we gauge the duality symmetry by invoking a complex scalar field in section 3. The resultant theory is a generalization of standard electrodynamics, which reduces to the usual Maxwell equations (without magnetic charges) when there is a spontaneous symmetry breaking in the scalar field sector. This is shown in section 4.

2. Duality symmetry and dimensionality of space-time

In 3 + 1 dimensional flat space-time, when magnetic monopoles are present, electromagnetic theory is described in terms of extended Maxwell-Lorentz equations [5],

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j_e^\nu, \quad (1)$$

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{4\pi}{c} j_m^\nu, \quad (2)$$

and,

$$\frac{dp^\mu}{d\tau} = \frac{1}{c} \left[q_e F^{\mu\nu} + q_m \tilde{F}^{\mu\nu} \right] \frac{dx_\nu}{d\tau}, \quad (3)$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu}_{\alpha\beta}F^{\alpha\beta}$ is the dual of the electromagnetic field tensor $F^{\mu\nu}$, while j_e^μ and j_m^μ are the 4-current densities corresponding to electric and magnetic charges, respectively.

It can be easily verified that under the following transformation,

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = \cos\theta F_{\mu\nu} - \sin\theta \tilde{F}_{\mu\nu}, \quad (4)$$

and,

$$q_e \rightarrow q'_e = \cos\theta q_e - \sin\theta q_m, \quad (5)$$

$$q_m \rightarrow q'_m = \sin\theta q_e + \cos\theta q_m, \quad (6)$$

extended Maxwell equations (1) - (3) are invariant.

The parameter θ characterizes the transformation (4) - (6) and is usually taken to be a constant. This is the usual duality transformation.

Is the duality rotation (4) - (6) meaningful for electrodynamics in space-times of arbitrary dimensions ? To answer this, we consider electrodynamics without magnetic charges in $D+1$ dimensional flat space-time. The action corresponding to a particle of charge q interacting with electromagnetic fields is given by,

$$\mathcal{A} = -mc \int \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} - \frac{q}{c} \int A_\mu dx^\mu - \frac{1}{16\pi} \int F_{\mu\nu} F^{\mu\nu} d^{D+1}x, \quad (7)$$

where $\mu, \nu = 0, 1, 2, \dots, D$.

The equations of motion that follow from (7) are,

$$E^i, i = 4\pi\rho, \quad (8)$$

$$F^{ij}, j = -\frac{1}{c} \frac{\partial E^i}{\partial E} - \frac{4\pi}{c} j^i, \quad (9)$$

$$\tilde{F}^{\mu_1 \mu_2 \dots \mu_{D-1}}, \mu_1 = 0, \quad (10)$$

where $E^i \equiv -F^{0i}$ and $\tilde{F}^{\mu_1 \mu_2 \dots \mu_{D-1}} = \frac{1}{2} \epsilon^{\mu_1 \mu_2 \dots \mu_{D-1} \mu_D \mu_{D+1}} F_{\mu_D \mu_{D+1}}$ with $i, j = 1, 2, \dots, D$ and $\mu, \nu = 0, 1, 2, \dots, D$. In order to extend eqs (8) - (10) by adding magnetic monopoles, one needs to modify eq (10) so that, $\tilde{F}^{\mu_1 \mu_2 \dots \mu_{D-1}}, \mu_1 = \frac{4\pi}{c} j_m^{\mu_2 \mu_3 \dots \mu_{D-1}}$. But this immediately brings an asymmetry between electric and magnetic charges, because the electric charge current density is only a $(D + 1)$ -vector. There is symmetry only when $\tilde{F}^{\mu_1 \mu_2 \dots \mu_{D-1}}$ and $F^{\mu_1 \mu_2}$ are of same rank, implying $D = 3$. Therefore, we conclude that duality transformation is a meaningful symmetry only when the dimensionality of space-time is $3 + 1$, implying that only in such space-times electric and magnetic charges have similar status.

3. Local duality transformation

We come back to classical electrodynamics in 3+1 dimensional flat space-time. By defining, complex electromagnetic field tensor, complex charge and current density, respectively, as

$$G_{\mu\nu} = F_{\mu\nu} + i\tilde{F}_{\mu\nu}, \quad (11)$$

$$Q = q_e + iq_m, \quad (12)$$

and,

$$J^\mu = j_e^\mu + ij_m^\mu, \quad (13)$$

we can re-write the extended Maxwell equations (1) and (2) as,

$$\partial_\mu G^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad (14)$$

and the generalized Lorentz force equation (3) as,

$$\frac{dp^\mu}{d\tau} = \frac{1}{2c} [Q^* G^{\mu\nu} + Q G^{*\mu\nu}] \frac{dx_\nu}{d\tau}. \quad (15)$$

Because of (4) - (6), the duality rotation now reads as,

$$G_{\mu\nu} \rightarrow G'_{\mu\nu} = e^{i\theta} G_{\mu\nu}, \quad (16)$$

$$Q \rightarrow Q' = e^{i\theta} Q, \quad (17)$$

and,

$$J^\mu \rightarrow J'^\mu = e^{i\theta} J^\mu. \quad (18)$$

It is obvious that the equations (14) and (15) are invariant under the transformation (16) - (18). So far we had been assuming that the transformation parameter θ is constant in space-time, implying that the duality transformation is global. We wish, now, to extend the hitherto global symmetry to a local one, by making θ depend on space-time coordinates. With $\theta = \theta(x^\mu)$, the generalized Lorentz force equation (15) is still invariant although the field equations (14) are not. Thus, it is clear that gauging the duality symmetry requires modification of the field equation. **More significantly, local duality transformation makes the electromagnetic charge Q space-time dependent! This is an unusual feature suggesting a different way of looking at the concept of electromagnetic charge. In the next paragraph we elaborate on this.**

To begin with, we introduce a complex scalar field $\phi(x)$ which under local duality transformation changes as follows,

$$\phi(x) \rightarrow \phi'(x) = e^{i\theta(x)} \phi(x) \quad (19)$$

In this new picture, the electromagnetic charge arises due to the interaction between the charged particle and the scalar field ϕ so that,

$$Q(x) = \alpha\phi(x) \quad , \quad (20)$$

where α is a coupling constant that solely depends on the particle, and is truly a **constant**. The transformation (17) is guaranteed for Q because of (19). This way of viewing at the electromagnetic charge is reminiscent of the origin of mass in electroweak theories through Higgs field. In fact, in the next section we will incorporate most of the features associated with the Higgs sector in the dynamics of ϕ .

We now make use of the scalar field ϕ to define a gauge covariant derivative,

$$\mathcal{D}_\mu \equiv \partial_\mu - i\psi_\mu \quad (21),$$

where,

$$i\psi_\mu = \partial_\mu [\ln \phi(x)] \quad (22)$$

From (21) and (22), it is easy to see that,

$$\mathcal{D}_\mu G^{\alpha\beta} \rightarrow e^{i\theta(x)} \mathcal{D}_\mu G^{\alpha\beta}. \quad (23)$$

In (21) ψ_μ acts apparently like a gauge field, but (22) makes it obvious that this is a pure gauge. And, hence, the definition (21) does not introduce any new gauge interaction.

Modifying (14) to ,

$$\mathcal{D}_\mu G^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad (24)$$

we find that the equations of motion given by (15) and (24) are invariant under the local duality transformation, and these form the generalized version of the extended Maxwell equations. In the following section, we will derive these equations as well as the equations of motion for the scalar field from an action.

4. Lagrangian Formulation

In this section, we derive the equations of motion for fields and particles from an action. We begin with few definitions. Let $a_\mu(x)$ be a complex 4-vector field that under duality transformation behaves in the following way :

$$a_\mu(x) \rightarrow a'_\mu(x) = e^{i\theta(x)} a_\mu(x). \quad (25)$$

The complex electromagnetic field tensor $G_{\mu\nu}$ is related to a_μ in the following way,

$$G_{\mu\nu} = (\partial_\mu - i\psi_\mu^*) a_\nu - (\partial_\nu - i\psi_\nu^*) a_\mu, \quad (26)$$

where ψ_μ is related to ϕ according to (22). However, not all the components of a_μ are independent. This is because of the definition (11) for $G_{\mu\nu}$ that requires the following constraint to be satisfied,

$$G_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu}^{\alpha\beta} G_{\alpha\beta}. \quad (27)$$

The action for the electromagnetic field is given by,

$$\mathcal{A}_0 = -\frac{1}{16\pi} \int G_{\mu\nu}^* G^{\mu\nu} d^4x. \quad (28)$$

For particles, we label the world-lines $y^\mu(\tau)$ with latin indices $i, j, \dots = 1, 2, \dots$, and denote the world-line and the 4-velocity of the j^{th} particle as $y^\mu(\tau_j) \equiv y_j^\mu$ and $\frac{dy^\mu}{d\tau_j} \equiv \dot{y}_j^\mu$. The portion of the total action relevant for the equations of motion corresponding to particles is given by,

$$\mathcal{A}_1 = -\sum_j m_j c \int \sqrt{\eta_{\mu\nu} \dot{y}_j^\mu \dot{y}_j^\nu} d\tau_j - \frac{1}{2c} \sum_j \alpha_j \int [\phi^*(y_j) a^\mu(y_j) + \text{c.c.}] \dot{y}_{\mu j} d\tau_j. \quad (29)$$

where c.c. denotes the complex conjugate, and α_j is the coupling constant (see (20)) corresponding to the j -th particle.

We now come to the scalar field ϕ . Because of (19), the scalar field sector has to be invariant under local U(1) group suggesting the existence of a abelian gauge field χ_μ that interacts with ϕ . The corresponding gauge covariant derivative then can be written as,

$$\nabla_\mu = \partial_\mu + ig\chi_\mu \quad , \quad (30)$$

where g is the gauge coupling constant.

Under local duality transformation, the abelian gauge field transforms as,

$$\chi_\mu \rightarrow \chi'_\mu = \chi_\mu + \frac{1}{g}\partial_\mu\theta \quad . \quad (31)$$

The action for the scalar field sector is taken to be,

$$\mathcal{A}_2 = \int d^4x [\mathcal{L}_\phi - \frac{1}{16\pi}\Sigma_{\mu\nu}\Sigma^{\mu\nu}] \quad , \quad (32)$$

where,

$$\mathcal{L}_\phi = \frac{1}{2}(\nabla_\mu\phi)^*(\nabla^\mu\phi) - \frac{\lambda}{4}(\phi^*\phi - \eta^2)^2 \quad , \quad (33)$$

and,

$$\Sigma_{\mu\nu} = \partial_\mu\chi_\nu - \partial_\nu\chi_\mu \quad , \quad (34)$$

It is well known that the ground state of this sector is described by the following solutions,

$$\phi_{vac}(x) = \eta e^{i\psi(x)} \quad , \quad (35)$$

and,

$$(\chi_\mu)_{vac} = 0 \quad . \quad (36)$$

It is evident from (28), (29) and (32) that \mathcal{A}_0 , \mathcal{A}_1 and \mathcal{A}_2 are invariant under local duality transformation respectively. Variation of the total action with respect to the

particle trajectory y_j^μ and the complex 4-vector field a^μ leads to the following equations of motion :

$$\frac{dp_j^\mu}{d\tau_j} = \frac{\alpha_j}{2c} [\phi^*(y_j)G^{\mu\nu}(y_j) + \phi(y_j)G^{*\mu\nu}(y_j)] \frac{dy_{\nu j}}{d\tau_j}, \quad (37)$$

and,

$$\mathcal{D}_\mu G^{\mu\nu}(x) = \frac{4\pi}{c} \sum_j \alpha_j \phi(x) \frac{dy_j^\nu}{dt} \delta^3(\bar{x} - \bar{y}_j(t)), \quad (38)$$

respectively.

It is time now to show that in the low energy limit, (37) and (38) are equivalent to the usual Lorentz force equation and ordinary Maxwell equations, respectively. In the low energy region, the scalar field takes the vacuum configuration (35) so that the gauge covariant derivative \mathcal{D}_μ takes the form,

$$\mathcal{D}_\mu = \partial_\mu - i\partial_\mu \psi \quad (39)$$

By virtue of (19), under a local duality transformation the phase ψ transforms as,

$$\psi(x) \rightarrow \psi'(x) = \psi(x) + \theta(x) \quad (40)$$

Since the entire theory is invariant under local duality transformation, we are free to choose a gauge $\theta(x) = -\psi(x)$ so that $\psi'(x) = 0$ because of (40). This immediately makes the gauge covariant derivative (in the new gauge) reduce to ordinary partial derivative (see (39)),

$$\mathcal{D}'_\mu = \partial_\mu \quad (40)$$

Furthermore, in this gauge the electromagnetic charge of the j-th particle is given by,

$$Q'_j = \alpha_j \phi'(x) = \alpha_j \eta \quad , \quad (41)$$

implying that the charges for all particles are constant and are real (corresponding to electric charge alone). It is easy to see making use of (40) and (41) that (37) and (38) reduce to,

$$\frac{dp_j^\mu}{d\tau_j} = \frac{1}{c}(q_e)_j F^{\mu\nu} \frac{dx_{\nu j}}{d\tau_j}, \quad (42)$$

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j_e^\nu, \quad (43)$$

and,

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \quad (44)$$

where,

$$(q_e)_j = Q'_j, \quad (45)$$

and,

$$j_e^\nu = \sum_j Q'_j \frac{dy_j^\nu}{dt} \delta^3(\bar{x} - \bar{y}_j(t)). \quad (46)$$

Equations (42) - (44) are the usual Maxwell-Lorentz equations in the absence of magnetic monopoles. Thus, in the low energy region the electromagnetic sector of this theory is identical to the conventional classical electrodynamics. The new physics arise out of the scalar field sector which has in it the entire machinery of theories incorporating spontaneous breaking of local U(1) symmetry. By expanding the scalar field ϕ around the vacuum (35),

$$\phi(x) = [\eta + h(x)] e^{i\psi}, \quad (47)$$

one can very easily show that $h(x)$, which is real, is the only physical field, and due to the Higgs mechanism, $h(x)$ and the abelian gauge field χ_μ acquire masses $\sqrt{2\lambda\eta^2}$ and

$g\eta$, respectively. Clearly, if η is very large, these predictions can only be realized at high energies. One very interesting consequence of this theory is that of having ‘cosmic string’ like solution for the scalar field $\phi(x)$ where this field vanishes along a curve and in general depends on spatial coordinates within the ‘cosmic string’. In such a case, a charge particle will have a spatially dependent electromagnetic charge inside the ‘cosmic string’. These issues will be addressed in a separate paper.

Before ending this section, we wish to draw attention to an additional local symmetry of the theory. Consider the following transformation,

$$a_\mu \rightarrow a'_\mu = a_\mu + \partial_\mu \beta(\phi^*), \quad (48)$$

where β is any complex differentiable function of ϕ^* . It can be easily shown that (48) leaves $G_{\mu\nu}$ invariant, and causes the action (29) pick up just boundary terms. Thus, equations of motion are left invariant under the transformation (48).

5. Summary and Discussion

Duality symmetry is central to the notion of magnetic charge. Maxwell electrodynamics can incorporate duality symmetry only in 3+1 dimensional space-time. This indirectly links the concept of an elementary magnetic monopole with the dimensionality of space-time.

Most symmetry in nature are local symmetries e.g. gauge symmetries in electrodynamics and electro-weak theories, general covariance in Einstein’s theory of gravitation, etc. It is therefore interesting to study the consequences of a local duality symmetry. Gauging this symmetry requires invoking a complex scalar field $\phi(x)$ that exhibits spontaneous breaking of duality symmetry. In the low energy limit, there is a gauge

in which this theory automatically leads to conventional electrodynamics without magnetic charges. Vortex solutions in the scalar field sector may have interesting consequences for the charge particle - vortex interaction, and this problem is presently being studied.

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