

Concept of temperature in multi-horizon spacetimes: analysis of Schwarzschild–De Sitter metric

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Abstract In case of spacetimes with single horizon, there exist several well-established procedures for relating the surface gravity of the horizon to a thermodynamic temperature. Such procedures, however, cannot be extended in a straightforward manner when a spacetime has multiple horizons. In particular, it is not clear whether there exists a notion of global temperature characterizing the multi-horizon spacetimes. We examine the conditions under which a global temperature can exist for a spacetime with two horizons using the example of Schwarzschild–De Sitter (SDS) spacetime. We systematically extend different procedures (like the expectation value of stress tensor, response of particle detectors, periodicity in the Euclidean time etc.) for identifying a temperature in the case of spacetimes with single horizon to the SDS spacetime. This analysis is facilitated by using a global coordinate chart which covers the entire SDS manifold. We find that all the procedures lead to a consistent picture characterized by the following features: (a) In general, SDS spacetime behaves like a non-equilibrium system characterized by two temperatures. (b) It is not possible to associate a global temperature with SDS spacetime except when the ratio of the two surface gravities is rational. (c) Even when the ratio of the two surface gravities is rational, the thermal nature depends on the coordinate chart used. There exists a global coordinate chart in which there is global equilibrium temperature while there exist other charts in which SDS behaves as though it has two different temperatures. The coordinate dependence of the thermal nature is reminiscent of the flat spacetime in Minkowski and Rindler coordinate charts. The implications are discussed.

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1 Introduction

It is possible to associate a notion of a thermodynamic temperature with metrics having a single horizon. For example, the general class of spacetimes described by spherically symmetric metrics of the form

$$ds^2 = f(r)dt^2 - [f(r)]^{-1}dr^2 - r^2[d\theta^2 + \sin^2\theta d\phi^2] \quad (1)$$

with $f(r)$ having a single, simple zero at $r = r_0$ [i.e., $f(r) \simeq f'(r_0)(r - r_0)$ near $r = r_0$], have a fairly straightforward thermodynamic interpretation. In fact, it can be shown that the Einstein's equations can be expressed in the form of a thermodynamic relation $TdS = dE - PdV$ for such spacetimes [1–5], with the temperature being determined by the surface gravity of the horizon:

$$\kappa = \frac{1}{2}|f'(r_0)| \quad (2)$$

The most familiar metric amongst the above class is, of course, the Schwarzschild metric with a black hole event horizon, with the temperature being directly related to the mass of the black hole. The De Sitter metric can also be analysed in a similar manner and one can again identify a unique temperature for the metric [6]. However, since (i) De Sitter spacetime is not asymptotically flat and (ii) the De Sitter horizon is observer dependent, certain new difficulties arise in this case. In particular, concept like “evaporation” of the cosmological horizon is not very obvious unlike in the case of ordinary black holes (some of these issues are discussed in several of the references cited in the bibliography). But these issues have nothing to do with the existence of *multiple* horizons in spacetimes and hence are beyond the scope of this paper.

A completely different class of conceptual and mathematical difficulties arise while dealing with spacetimes having multiple horizons which we shall analyse in this paper. The simplest spacetime with multiple horizons is that of a black hole in a spacetime with a cosmological constant, described by the Schwarzschild–De Sitter (SDS) metric [7–46]. The metric is characterized by the presence of a black hole event horizon and a cosmological horizon. In recent times, studying such a spacetime has acquired further significance because of the cosmological observations suggesting the existence of a non-zero positive cosmological constant ([47]; for reviews, see [48–51]). While the observations can be explained by a wide class of models (see, e.g. [52, 53]), including those in which the cosmic equation of state can depend on spatial scale (see e.g., [54–56]), virtually all these models approach the De Sitter (DS) spacetime at late times and at large scales. Thus any black hole which forms in the real universe with a cosmological constant provides an idealization of a SDS spacetime.

The SDS metric has the same form (in one coordinate chart) as the metric in (1) but with a $f(r)$ that has two simple zeros at $r = r_{\pm}$ and two surface gravities $\kappa_{\pm} = (1/2)|f'(r_{\pm})|$. Naively, one could associate two different temperatures to the black hole and cosmological horizon using the two different surface gravities. Since the surface gravities are (generically) different, the spacetime behaves like a system with two temperatures—somewhat like a solid with its edges kept at two thermal

baths of two temperatures. In that case, there will be no well-defined notion of global temperature associated with the spacetime [45, 57]. While this sounds plausible, one must bear in mind that the notion of a temperature in spacetimes with horizon is neither local nor coordinate independent. Hence it is not clear whether associating two separate temperatures with the two horizons is a coordinate independent statement. On the other hand, there are also some indications that one can associate a single, effective temperature with the SDS spacetime at least in some special cases and possibly in specific coordinate charts [7–12, 45]. This suggests that *both* the viewpoints could be correct but in different coordinate charts. It should be stressed that the temperature is not a property of the spacetime geometry in general but depends on the coordinate chart used by a class of observers [1–5, 58]. The best known example is the flat spacetime itself which acquires an observer dependent horizon and temperature in the Rindler coordinate chart. In view of these complications, we feel it is worth analysing the temperature of SDS spacetime in some detail, which we attempt to do in this paper.

Our approach will be as follows. We will simplify mathematical complexities by working with $1 + 1$ spacetimes since it is well known that the issues we are attempting to address exist even in two dimensions. We shall then use the standard procedures which lead to the concept of a temperature in the case of single horizon metrics to the SDS metric. This analysis shows that the naive notion of associating two different temperatures with the two horizons is indeed justifiable at least in some approximate sense. One of the approaches, based on periodicity of Euclidean time indicates that there could exist a notion of global temperature in SDS when the ratio of surface gravities is a rational number. To investigate this issue carefully, we use a global coordinate chart which covers the entire SDS manifold and show that *in this coordinate chart* there does exist a global notion of temperature for the SDS metric when the ratio of surface gravities is rational. However, even in this case, the spacetime behaves as though it has two different temperatures in certain coordinate charts while it has one equilibrium temperature in another global coordinate chart. This is reminiscent of flat spacetime which exhibits different thermal characteristics in different coordinate chart. The implications of this result are discussed in the last section.

2 General expressions for a metric with single horizon

In this section, we shall briefly review the basic results for a spherically symmetric spacetime with single horizon, described by the metric (1). In what follows, the angular coordinates (θ, ϕ) do not play any important role, thus allowing us to work in the $1 + 1$ dimensional (t, r) subspace. The metric can then be written in a conformally flat form by introducing the “tortoise coordinate” (r^*) and the associated null coordinates (u, v) defined by:

$$r^* = \int \frac{dr}{f(r)}; \quad u = t - r^*, \quad v = t + r^*. \quad (3)$$

The metric becomes the form

$$ds^2 = f[r(u - v)] du dv. \quad (4)$$

In this form, the metric—written in terms of the (t, r^*) , or the (u, v) coordinates—is singular at the horizon where $f(r)$ has a simple zero. To regularize the metric at the horizon and to remove the singularity, we need to introduce the conventional Kruskal coordinates defined by

$$U = -\frac{1}{\kappa} e^{-\kappa u}, \quad V = \frac{1}{\kappa} e^{\kappa v} \quad (5)$$

where

$$\kappa \equiv \frac{1}{2} f'(r_0) \quad (6)$$

(The parameter κ can either be positive or negative—for example, it is positive for the Schwarzschild metric but is negative for the De Sitter metric. On the contrary, the surface gravity, which is defined in Eq. (2) as $\kappa = |\kappa|$, is positive definite.) The metric in terms of the Kruskal coordinates is

$$ds^2 = f[r(U - V)] e^{-2\kappa r^*(U-V)} dU dV, \quad (7)$$

which, near the horizon, is free from singularities as $ds^2 \approx 2\kappa r_0 dU dV$.

We shall now briefly summarize the different procedures that can be used to associate the notion of a temperature with the above metric. (Detailed discussion of these and similar procedures for probing the vacuum structure can be found, for example, in [58,59].)

2.1 Expectation value of the energy-momentum tensor

One can study some of the thermodynamic properties of the spacetime by computing the expectation values of the energy-momentum tensor T_k^i for the corresponding metric. In order to calculate the expectation value of T_k^i of the matter field in a given spacetime, one needs to define a quantum state of the system. Even if one takes it to be the vacuum state, there still remains an ambiguity in choosing the vacuum in curved spacetime, and thus the expectation value of T_k^i will depend on the choice of the vacuum state.

In an $1 + 1$ dimensional conformally flat spacetime (which is the case we are interested in), the mode functions are simple plane waves, thus simplifying the calculations substantially. For a spacetime with single horizon, there can be (at least) three natural choices of the vacuum state corresponding to different sets of ingoing and outgoing modes. The outgoing and incoming modes of the form $(4\pi\omega)^{-1/2} [e^{-i\omega u}, e^{-i\omega v}]$ define the Boulware vacuum (B) [60], while $(4\pi\omega)^{-1/2} [e^{-i\omega U}, e^{-i\omega V}]$ define the Hartle–Hawking vacuum (H) [61]. The third vacuum is defined as $(4\pi\omega)^{-1/2} [e^{-i\omega U}, e^{-i\omega v}]$ and is called the Unruh vacuum (U) [62].

Usually one is interested in the expectation values of the stress tensor as will be measured by a freely falling inertial observer. Since near the horizon, the coordinate system (u, v) is singular, the coordinates appropriate for the inertial observer will

be the Kruskal coordinates (U, V) . Following standard calculations, the expectation values of the T_{UU}, T_{VV}, T_{UV} components of the stress tensor in the three vacua states are [63]

$$\begin{aligned}
 \langle B|T_{UU}|B\rangle &= \frac{1}{96\pi\kappa^2U^2} \left[ff'' - \frac{1}{2}f'^2 \right] \\
 \langle H|T_{UU}|H\rangle = \langle U|T_{UU}|U\rangle &= \langle B|T_{UU}|B\rangle + \frac{1}{48\pi U^2} \\
 \langle B|T_{VV}|B\rangle = \langle U|T_{VV}|U\rangle &= \frac{1}{96\pi\kappa^2V^2} \left[ff'' - \frac{1}{2}f'^2 \right] \\
 \langle H|T_{VV}|H\rangle = \langle B|T_{VV}|B\rangle &+ \frac{1}{48\pi V^2} \\
 \langle B|T_{UV}|B\rangle = \langle H|T_{UV}|H\rangle = \langle U|T_{UV}|U\rangle &= -\frac{1}{96\pi\kappa^2UV} ff''
 \end{aligned}
 \tag{8}$$

It is easy to show that $\langle B|T_{UU}|B\rangle$ diverges (as U^{-2}) near the horizon, while $\langle H|T_{UU}|H\rangle$ is finite there. Furthermore, the difference between the Boulware and Hartle–Hawking vacua signifies a presence of a thermal bath in the latter vacuum. This point is clear from the relation

$$\langle H|T_t^t|H\rangle - \langle B|T_t^t|B\rangle = -(\langle H|T_r^r|H\rangle - \langle B|T_r^r|B\rangle) = \frac{\kappa^2}{24\pi f}.
 \tag{9}$$

The temperature of the thermal bath in the Hartle–Hawking vacuum can simply be read off from the above expression and is seen to be $\kappa/2\pi$. On the other hand, the Unruh vacuum possesses a completely new property which is not present in the other two vacua. Note that for Unruh vacuum, $\langle U|T_{UU}|U\rangle \neq \langle U|T_{VV}|U\rangle$, which indicates the presence of a non-zero flux of energy. In fact, a straightforward calculation will show that

$$\begin{aligned}
 \langle B|T_t^r|B\rangle = \langle H|T_t^r|H\rangle &= 0 \\
 \langle U|T_t^r|U\rangle &= \frac{\kappa^2}{48\pi}
 \end{aligned}
 \tag{10}$$

The form of the energy flux in the Unruh vacuum suggests a blackbody emission (in $1 + 1$ dimensions) at a temperature $\kappa/2\pi$.

The above calculations, although done in only $1 + 1$ dimensions, show that there is a clear association between the surface gravity of the horizon and a temperature defined in the spacetime. A full $3 + 1$ dimensional calculation, which is technically much more difficult to tackle, is however not expected to alter the above conclusions in general.

2.2 Response of particle detectors

The results of the above calculations can be compared with the response of a model particle detector in the different vacua states. For a massless scalar field in 1 + 1 dimensions, the response function is related to the standard Wightman functions $D^+(x, x')$. When the detector is at a fixed spatial distance r , the response functions per unit proper time for the three vacua are given by [58]

$$\begin{aligned} \mathcal{F}_B(E)/\text{unit proper time} &= 0, \\ \mathcal{F}_H(E)/\text{unit proper time} &\propto \frac{1}{E \left(e^{2\pi\sqrt{f(r)}E/\kappa} - 1 \right)} \\ \mathcal{F}_U(E)/\text{unit proper time} &\propto \frac{1}{E \left(e^{2\pi\sqrt{f(r)}E/\kappa} - 1 \right)} \end{aligned} \tag{11}$$

These expressions, which are essentially of the Planckian form in 2-dimensions, indicate that the detector detects a thermal bath at apparent temperature $\kappa/2\pi$ in the Hartle–Hawking vacuum, while it detects a flux of particles with the same temperature in the Unruh vacuum. This is what we had concluded in the previous section too.

2.3 Periodicity in the Euclidean time

Another way of relating the notion of temperature with the horizon is by considering the periodicity of the Greens function, which is analytic over the whole spacetime, in the Euclidean time coordinate. The basic idea is to analytically continue the metric to imaginary values of t . Setting $t_E \equiv it$ we get

$$- ds^2 = \left[f(r) dt_E^2 + \frac{dr^2}{f(r)} \right] \tag{12}$$

The periodicity in t_E can be seen from the relation between the (t, r) and the Kruskal coordinates ($T = [U + V]/2, R = [V - U]/2$):

$$T = \frac{e^{\chi r^*}}{\chi} \sinh \chi t, \quad R = \frac{e^{\chi r^*}}{\chi} \cosh \chi t \tag{13}$$

The relations between the Euclidean time coordinates in these two systems are

$$T_E = \frac{e^{\chi r^*}}{\kappa} \sin \kappa t_E \tag{14}$$

which shows that T_E is periodic function of t_E with a period $n\beta$, where

$$\beta = \frac{2\pi}{\kappa} \tag{15}$$

and n is a (positive) integer. The minimum possible period of the Euclidean time t_E , given by $2\pi/\kappa$, is precisely equal to the inverse of the temperature corresponding to the horizon. Similar conclusions can be drawn by considering the spatial coordinate R too. This periodicity will be shown by any analytic function of the coordinates (T_E, R) over the whole manifold. In particular, the Greens function defined over the entire spacetime will be an analytic function of (T_E, R) and hence will be periodic in the imaginary time coordinate. One can then analytically continue this Euclidean Greens function and obtain the Feynman propagator in the original (t, r) space which would also be periodic in the imaginary time. In general, this periodicity of the propagator can be shown to be a characteristic of a thermal state (see, for example, [64]) with a temperature given by the inverse of the period β^{-1} . Thus the period of the Euclidean time t_E is seen to be directly related to the temperature of the spacetime.

We shall now apply the above procedures for associating a temperature with the horizon to the case of SDS metric.

3 Schwarzschild–De Sitter spacetime

We now extend the formalisms of the previous section to the Schwarzschild–De Sitter (S-DS) spacetime, described by the metric

$$ds^2 = \left(1 - \frac{2M}{r} - H^2 r^2\right) dt^2 - \left(1 - \frac{2M}{r} - H^2 r^2\right)^{-1} dr^2 \tag{16}$$

where we have omitted the angular coordinates. This metric has two horizons at r_- and r_+ —they are the black hole event horizon and the cosmological horizon, respectively. Let us denote the surface gravities associated with the two horizons by κ_- and κ_+ respectively. (The detailed expressions for these quantities are given in Appendix A.) The metric can be written in the conformal form by introducing the usual ‘tortoise coordinate’ r^* and the null coordinates $u = t - r^*$, $v = t + r^*$. Until this point, the analysis follows exactly as in the case of the single horizon.

In the case of spacetimes with single horizon, one next introduces the Kruskal coordinates thereby obtaining a non-singular coordinate chart which covers the whole manifold. Note, however, that this transformation necessarily involves the surface gravity which is different from for the two horizons. When the spacetime has more than one horizons, one has to use a Kruskal coordinate patch for each of them—no single patch of usual Kruskal-type coordinates can cover both the horizons. For example, to remove the singularity near the black-hole horizon r_- , introduce the coordinates

$$U_- = -\frac{1}{\kappa_-} e^{-\kappa_- u}; \quad V_- = \frac{1}{\kappa_-} e^{\kappa_- v} \tag{17}$$

so that the metric becomes

$$ds^2 = \frac{2M}{r} \left(1 - \frac{r}{r_+}\right)^{1+\frac{\kappa_-}{\kappa_+}} \left(1 + \frac{r}{r_- + r_+}\right)^{2-\frac{\kappa_-}{\kappa_+}} dU_- dV_- \tag{18}$$

Although this is non-singular near the black hole horizon r_- , it is clear that the singularity at the other horizon r_+ still prevails. Thus the coordinates (U_-, V_-) are not defined for the region $r > r_+$. Near the cosmological horizon r_+ , we can introduce another set of Kruskal coordinates

$$U_+ = \frac{1}{\kappa_+} e^{\kappa_+ u}; \quad V_+ = -\frac{1}{\kappa_+} e^{-\kappa_+ v} \quad (19)$$

so that the metric becomes

$$ds^2 = \frac{2M}{r} \left(\frac{r}{r_-} - 1 \right)^{1 + \frac{\kappa_+}{\kappa_-}} \left(1 + \frac{r}{r_- + r_+} \right)^{2 - \frac{\kappa_+}{\kappa_-}} dU_+ dV_+ \quad (20)$$

This metric, in turn, is singular at $r = r_-$ and hence these coordinates cannot be extended to the region $r < r_-$. In the region of overlap ($r_- < r < r_+$), where both the coordinate patches are well defined, they are related to each other by

$$(-\kappa_- U_-)^{-\frac{1}{\kappa_-}} = (\kappa_+ U_+)^{\frac{1}{\kappa_+}}, \quad (\kappa_- V_-)^{-\frac{1}{\kappa_-}} = (-\kappa_+ V_+)^{\frac{1}{\kappa_+}} \quad (21)$$

These considerations already show that one will expect non-trivial differences between the case of single horizon and multi-horizon scenarios. We shall now extend the concepts introduced for a single horizon to SDS metric and see whether there is any global temperature associated with the spacetime.

3.1 Expectation values of the stress tensor

The calculations performed for the expectation values of the stress tensor in the single horizon case can be extended to the SDS spacetime by identifying the corresponding vacuum states. There is no ambiguity in the Boulware vacuum, defined in terms of the modes $(4\pi\omega)^{-1/2} [e^{-i\omega u}, e^{-i\omega v}]$. The calculations and the expressions are identical to the single horizon case. Since the (u, v) coordinate system is badly behaved at both the horizons, the expectation values for an inertial observer will diverge at both the horizons.

The usual Hartle–Hawking vacuum is defined in terms of the Kruskal coordinates. Since we have two different patches of Kruskal coordinates, it is possible to define two separate Hartle–Hawking vacua. Let us call them as

$$\begin{aligned} H_- &: (4\pi\omega)^{-1/2} [e^{-i\omega U_-}, e^{-i\omega V_-}] \\ H_+ &: (4\pi\omega)^{-1/2} [e^{-i\omega U_+}, e^{-i\omega V_+}]. \end{aligned} \quad (22)$$

For these vacua, one can trivially extend the calculations for the single horizon case. As expected, for an inertial observer in the H_- vacuum, the expectation values will be finite at the black hole horizon r_- , but will diverge at the cosmological horizon r_+ . Similar conclusions will hold for H_+ too. It also turns out that, as expected, the

H_- vacuum corresponds to a thermal bath at a temperature $\kappa_-/2\pi$, while the H_+ vacuum gives a thermal bath at a temperature $\kappa_+/2\pi$.

Similarly, there will now be two different sets of Unruh vacua too. Let us call them as

$$\begin{aligned} U_- &: (4\pi\omega)^{-1/2}[e^{-i\omega U_-}, e^{-i\omega v}] \\ U_+ &: (4\pi\omega)^{-1/2}[e^{-i\omega u}, e^{-i\omega V_+}]. \end{aligned} \tag{23}$$

The only non-trivial result one obtains from the standard Unruh vacuum is the presence of a thermal flux. Following straightforward calculations, one can show in this case

$$\langle U_- | T_t^r | U_- \rangle = \frac{\kappa_-^2}{48\pi}, \quad \langle U_+ | T_t^r | U_+ \rangle = -\frac{\kappa_+^2}{48\pi} \tag{24}$$

This indicates that each vacuum represents a flux with a temperature associated with the corresponding horizon.

All the results mentioned above are direct generalizations of the corresponding results in the case of single horizon. There is, however, one non-trivial new result that arises in the SDS spacetime. We can define another non-trivial vacuum state for a spacetime with two horizons which has no analogue for the single horizon spacetimes. It is very much like the standard Unruh vacuum, and is defined as

$$U_{-+} : (4\pi\omega)^{-1/2}[e^{-i\omega U_-}, e^{-i\omega V_+}]. \tag{25}$$

This has the same outgoing modes as the standard U_- vacuum, while the ingoing modes are like the U_+ . The flux obtained from this vacuum state has the expression

$$\langle U_{-+} | T_{xt} | U_{-+} \rangle = \frac{\kappa_-^2 - \kappa_+^2}{48\pi} \tag{26}$$

This indicates the presence of two oppositely driven thermal fluxes with different temperatures—one flowing from the black hole horizon with temperature $\kappa_-/2\pi$, while other flowing from the cosmological horizon with temperature $\kappa_+/2\pi$. Just as the standard Unruh vacuum is useful for studying collapse of matter in Schwarzschild spacetime [62], it has been argued that the U_{-+} is appropriate for describing collapse of matter in a spacetime with cosmological horizon [11–13].

It turns out from the above analysis that there is no notion of a global temperature which is characteristic of the entire spacetime. We shall now see that the same conclusion can be drawn by studying the response of particle detectors in different vacua.

3.2 Detector response

We have already defined five vacuum states for the SDS metric in the previous section. The calculation for the detector response functions in this spacetime is quite

straightforward. We just mention the final results:

$$\begin{aligned}
 \frac{\mathcal{F}_B(E)}{\text{unit proper time}} &= 0, \\
 \frac{\mathcal{F}_{H_-}(E)}{\text{unit proper time}} &\propto \frac{1}{E \left(e^{2\pi\sqrt{f(\bar{r})}E/\kappa_-} - 1 \right)} \\
 \frac{\mathcal{F}_{H_+}(E)}{\text{unit proper time}} &\propto \frac{1}{E \left(e^{2\pi\sqrt{f(\bar{r})}E/\kappa_+} - 1 \right)} \\
 \frac{\mathcal{F}_{U_-}(E)}{\text{unit proper time}} &\propto \frac{1}{E \left(e^{2\pi\sqrt{f(\bar{r})}E/\kappa_-} - 1 \right)} \\
 \frac{\mathcal{F}_{U_+}(E)}{\text{unit proper time}} &\propto \frac{1}{E \left(e^{2\pi\sqrt{f(\bar{r})}E/\kappa_+} - 1 \right)}
 \end{aligned} \tag{27}$$

The main conclusions drawn in the previous section go through unchanged in each of these cases. The Boulware vacuum shows no flux, while the Hartle–Hawking and Unruh vacua exhibit two different temperatures depending on the state that is chosen.

The new feature arises in the U_{-+} state. Here we find that:

$$\frac{\mathcal{F}_{U_{-+}}(E)}{\text{unit proper time}} \propto \frac{1}{E} \left[\frac{1}{(e^{2\pi\sqrt{f(\bar{r})}E/\kappa_-} - 1)} + \frac{1}{(e^{2\pi\sqrt{f(\bar{r})}E/\kappa_+} - 1)} \right] \tag{28}$$

That is, the detector in the U_{-+} vacuum register *simultaneously* two different thermal fluxes of temperatures $\kappa_-/2\pi$ and $\kappa_+/2\pi$. Since the superposition of two Planckian spectra cannot be mapped to a Planck spectrum of single temperature, there is no indication of a global temperature associated with the spacetime. [At the Rayleigh limit the Planck spectrum is proportional to the temperature and, in this limit, the effective temperature is simply the sum of the two temperatures. At the other extreme of high frequencies, the higher temperature will dominate. So clearly, the notion of effective temperature is frequency dependent.]

3.3 Euclidean time and periodicity

All the above results were tuned to the static coordinate system $[r^*, t]$ and the *two* Kruskal coordinate systems obtained from them. Since all these coordinate patches are singular, all of them are conceptually comparable to Rindler coordinate system in flat spacetime and we need to find the analogue of global Minkowski coordinate system. In this aspect, the SDS differs drastically from either Schwarzschild or De Sitter, in either of which, the Kruskal coordinates cover the global manifold and are non-singular. One might, therefore, argue that since we have not used a global coordinate system which can cover both the horizons, it is understandable that we do not have a global temperature. In the next section, we take up this issue in detail.

4 Analysis in a global coordinate chart

The first indication, that a global notion of temperature might exist, arises when we study the periodicity of the Greens function, which is analytic over the whole space-time, in the Euclidean time. The most direct manner in which this issue can be settled is by introducing a global coordinate patch which can cover the whole manifold, though it is also possible to reach the same conclusions by considering the single-valuedness for the two Kruskal extensions defined in Eqs. (17) and (19) [46].

In the case of single horizon, we have seen that the Kruskal coordinates are adequate for this purpose. On the contrary, for a spacetime with multiple horizons, one has to look for something else. It turns out that there is a *family* of coordinate systems which are globally well defined, covers the full manifold and are non-singular. We take a particular one of them in this paper. The details of the global coordinates (\bar{T}, \bar{R}) —which are quite complicated algebraically—are discussed in Appendix B. The only relation we need here is the dependences of the global Euclidean time coordinate $\bar{T}_E = i\bar{T}$ and \bar{R} on the original t_E [which is the analogue of Eq. (14)]. The relation for \bar{T}_E is given by

$$\begin{aligned} \bar{T}_E = & \frac{1}{\kappa_-} \frac{\tan\left(e^{\kappa_- r^*} \sin \kappa_- t_E\right) \operatorname{sech}^2\left(e^{\kappa_- r^*} \cos \kappa_- t_E\right)}{1 + \tanh^2\left(e^{\kappa_- r^*} \cos \kappa_- t_E\right) \tan^2\left(e^{\kappa_- r^*} \sin \kappa_- t_E\right)} \\ & + \frac{1}{\kappa_+} \frac{\tan\left(e^{-\kappa_+ r^*} \sin \kappa_+ t_E\right) \operatorname{sech}^2\left(e^{-\kappa_+ r^*} \cos \kappa_+ t_E\right)}{1 + \tanh^2\left(e^{-\kappa_+ r^*} \cos \kappa_+ t_E\right) \tan^2\left(e^{-\kappa_+ r^*} \sin \kappa_+ t_E\right)} \end{aligned} \tag{29}$$

Note that the first term periodic when t_E has a period $\beta = 2\pi n_- / \kappa_-$, while the second term is periodic when $\beta = 2\pi n_+ / \kappa_+$. This implies that \bar{T}_E is periodic only when $\kappa_+ / \kappa_- = n_+ / n_-$. (The same conclusion can be drawn by considering the expression for \bar{R} .) This, in turn, implies that the Euclidean Greens function is periodic in t_E only when the ratio κ_+ / κ_- is rational. In that case, the Feynman propagator describes a thermal state having a temperature

$$\beta^{-1} = \frac{\kappa_-}{2\pi n_-} = \frac{\kappa_+}{2\pi n_+} . \tag{30}$$

In case the ratio κ_+ / κ_- is not rational, the Euclidean Greens function \bar{T}_E is not a periodic function of t_E and the propagator does not characterize a thermal state any more and hence there is no notion of a globally defined temperature.

When we have $\kappa_+ / \kappa_- = n_+ / n_-$, and that there is a global temperature $\kappa_- / (2\pi n_-) = \kappa_+ / (2\pi n_+)$, one could also verify that the usual limits exist when $M \rightarrow 0$ or $H \rightarrow 0$. When the black hole mass vanishes ($M \rightarrow 0$), we have $\kappa_- \rightarrow \infty$ and the period of the term containing κ_- will go to zero. This essentially means that the relevant term will oscillate rapidly, and in the limit of its frequency going to infinity, it will complete infinite number of periods by the time the other term completes one period. In that case, the period of \bar{T}_E , and hence the temperature of the spacetime, will be determined by the period of the term containing κ_+ , and the temperature will

be simply be the horizon temperature $\kappa_+/2\pi$. In the other limit when $H \rightarrow 0$, we have $\kappa_+ \rightarrow 0$. As κ_+ vanishes, the term containing κ_+ will stop oscillating and will essentially be a constant. In this case the period will be determined by κ_- and we will get back the black hole temperature. The difference in the manner in which the limits are obtained has nothing to do with our analysis of the problem. It merely reflects the following well-known, but curious, feature: when $H \rightarrow 0$ in the pure De Sitter metric, the spacetime becomes flat and the surface gravity [and temperature] vanishes. But when $M \rightarrow 0$ in the Schwarzschild metric, the spacetime becomes flat *but the surface gravity and the temperature diverge*.

The most interesting situation arises in the case of $\kappa_+/\kappa_- = n_+/n_-$. In such a situation, there does exist a notion of a global temperature which, however, turns out to be coordinate dependent. This is most easily seen from the analysis in Sect. 3. If we choose the H_\pm or U_\pm , U_{-+} vacuum states, the stress tensor expectation values or the detector response will still lead to the results in Eq. (28). There are two temperatures and fluxes corresponding to two temperatures, except that one temperature is rational multiple of the other. On the other hand, quantum field theory in the global coordinate chart will describe a system with an effective temperature $\beta^{-1} = \kappa_-(2\pi n_-) = \kappa_+(2\pi n_+)$. Thus we see that both the claims in the literature (“the SDS has two temperatures” and “SDS has an effective single temperature”) are correct but applies to different coordinate charts.

At first sight this might seem somewhat peculiar to the SDS case. However, we would like to emphasize that it is not, which can be understood from the following line of argument: it is quite obvious that the horizons we are usually interested in come with associated temperatures which can be related to the slope of the metric function near the horizon. The point to note is that this effect depends on the coordinate system used. For example, in the flat spacetime, the Rindler chart has one horizon and one temperature but the same is not true in the Minkowski chart. The number of different temperatures, N_t , depends on the coordinate chart used. In the case of the flat spacetime, Minkowski chart has $N_t = 0$ while Rindler chart has $N_t = 1$. We have shown that the same concept applies in the case of SDS spacetime too. In the SDS spacetime, there is a coordinate chart in which $N_t = 2$, i.e., the spacetime has two different temperatures in this coordinate chart (Sect. 3 of this paper). What we have shown here is that when the ratio of surface gravities is rational, it is possible to find another coordinate chart for SDS in which $N_t = 1$, i.e., a single global temperature exists (Sect. 4). Thus the $N_t = (0, 1)$ situation of flat spacetime generalizes to $N_t = (1, 2)$ in the case of SDS. In this sense, the SDS spacetime exhibits properties very similar to the flat spacetime.

5 Conclusions

In this section, we summarise the main conclusions of our analysis and discuss some of the implications. We find that:

- There exists a global coordinate chart for the SDS spacetime which covers the entire manifold. Our analysis in this global coordinate chart implies that there exists a global notion of the temperature provided the the surface gravities satisfy the condition $\kappa_+/\kappa_- = n_+/n_-$. In this case, the metric can be described as having a temperature $\beta^{-1} = \kappa_-(2\pi n_-) = \kappa_+(2\pi n_+)$.

- More conventional coordinate charts lead to the interpretation of SDS spacetime having two different temperatures, i.e., the spacetime behaves like a non-equilibrium system characterized by two different temperatures.

It is thus clear that the SDS spacetime possesses a dual thermal interpretation (provided, of course, the ratio of the surface gravities is a rational number) in a manner similar to flat spacetime in Minkowski and Rindler coordinate charts. This result is new (as far as the authors are aware of) in the following sense: In the Minkowski/Rindler coordinate charts we have an example of global temperature being zero in one coordinate system and nonzero in the other. In the SDS, the global coordinate chart can have a nonzero single temperature (when the ratio of surface gravities is rational) while in other coordinate charts, there are two temperatures.

At this point, it might be worth commenting on the ratio of the surface gravities with some previous results. The result that a precise relation between the surface gravities of the two horizons must hold in order to have a single temperature has been used in the Euclidean quantum gravity approach [14,65–67]. In such a framework this condition must be imposed in order to work with a non-singular solution of the (Euclidean) Einstein equations to remove the conical singularities at the horizons which indeed corresponds to fix the periodicity of the Euclidean time.

The rational numbers are infinitely dense in the space of real numbers and in practical sense the condition on the surface gravities may *not* impose any condition on M and H at all—however, the very fact that demanding equilibrium in a spacetime imposes a restriction on the surface gravities is an interesting point just as a matter of principle. It would be interesting to examine the possibility of whether any semiclassical calculations can lead to such quantization of surface gravities. It has been suggested from two different classes of semiclassical gravity calculations that the areas of the horizons might be quantized [in units of (Planck length)²] (for reviews, see [4,5,68]). In that case we have $r_+^2/r_-^2 = N_+/N_-$, where N_{\pm} are (relatively prime) positive integers. The condition on the ratio of surface gravities is then

$$\frac{\kappa_+}{\kappa_-} = \frac{N_- + 2\sqrt{N_+N_-}}{N_+ + 2\sqrt{N_+N_-}} \tag{31}$$

The ratio κ_+/κ_- can still be an irrational number, hence the quantization of areas does *not* necessarily imply the existence of thermal equilibrium. It is clear from the above expression that κ_+/κ_- will be rational only when $\sqrt{N_+/N_-}$ is rational, i.e., $N_+/N_- = N_1^2/N_2^2$, where N_1 and N_2 form another set of relatively prime integers. Hence, for the existence of a global temperature one requires that $r_+/r_- = N_1/N_2$. This means that the quantization of areas is not sufficient, there must be further restrictions on the horizon radii. *In particular, it is adequate if the radii of horizons are quantized in the units of Planck length which, of course, is consistent with the notion of area quantization.* In such case, the condition on MH will be

$$MH = \frac{N_2}{2} \frac{N_1N_2(N_1 + N_2)}{(N_1^2 + N_1N_2 + N_2^2)^{3/2}} \tag{32}$$

The implications of this result are under investigation.

The situation is more unclear when the ratio of surface gravities is not a rational number. The global coordinate system, defined in Appendix B, still exists covering the whole manifold but the metric in this coordinate system does not lead to any thermal interpretation. The other, singular coordinate charts, of course, lead to the conventional view of two different temperatures for the SDS. A non-singular Euclidean quantum field theory does not exist in this case. It is, however, unclear whether one should *demand* the existence of the non-singular Euclidean field theory while working on a given curved spacetime. After all, an arbitrary, time dependent background spacetime may not even have a Euclidean continuation, let alone a non-singular one. But if we demand that, the only cases that are allowed to exist are those in which non singular Euclidean extension is possible, then we obtain certain bizarre conclusions. For example, if the universe has a cosmological constant, then it will approach the De Sitter solution asymptotically and any black hole that forms in it in the asymptotic future must have a mass which satisfies the quantization condition in Eq. (32).

Finally, we mention that the thermal behaviour of horizons is closely related to the quasi-normal modes (QNM) as pointed out in recent analyses of QNMs using Born approximation [69, 70]. This investigation shows that the asymptotic QNMs of the SDS spacetime arises essentially from those of the Schwarzschild metric. In the study performed in this paper, however, both the horizons contribute in equal footing. There are calculations, based on other techniques, which indicate that the QNM's have dependence on the surface gravity of the cosmological horizon κ_+ [71]—however, they seem to be independent of κ_+ for large asymptotic frequencies. It will be, therefore, interesting to analyse the semiclassical wave modes in the global coordinate system and compare them with the results in the singular coordinate charts. This, and related issues, are under study.

Appendix A: Horizons, surface gravities and their relations in the SDS spacetime

In this appendix, let us discuss the basic properties of the SDS metric. The SDS spacetime, described by (16), has two horizons at r_- and r_+ (with $r_- < r_+$):

$$r_- = \frac{2}{\sqrt{3}H} \sin\left(\frac{\theta}{3}\right), \quad r_+ = \frac{2}{\sqrt{3}H} \sin\left(\frac{\theta + 2\pi}{3}\right) \quad (\text{A1})$$

where

$$\sin\theta = 3^{3/2}MH; \quad 0 < \theta < \frac{\pi}{2} \quad (\text{A2})$$

and we have assumed $0 < MH < 3^{-3/2}$. As discussed, the horizon at r_- is called the black hole horizon, while that at r_+ is called the cosmological horizon. As $MH \rightarrow 0$, one obtains the two known limits $r_- \rightarrow 2M$ and $r_+ \rightarrow H^{-1}$. The surface gravity for

the two horizons are given by

$$\begin{aligned} \kappa_- &= \frac{H^2}{2r_-}(r_+ - r_-)(r_+ + 2r_-) \\ \kappa_+ &= \frac{H^2}{2r_+}(r_+ - r_-)(r_- + 2r_+) \end{aligned} \tag{A3}$$

As $MH \rightarrow 0$, one gets the usual limits $\kappa_- \rightarrow (4M)^{-1}$ and $\kappa_+ \rightarrow H$.

The metric can be written in the conformal form by introducing the ‘tortoise coordinate’

$$\begin{aligned} r^* &= \int dr \left(1 - \frac{2M}{r} - H^2 r^2\right)^{-1} \\ &= \frac{1}{2\kappa_-} \ln \left| \frac{r}{r_-} - 1 \right| - \frac{1}{2\kappa_+} \ln \left| 1 - \frac{r}{r_+} \right| - \frac{1}{2} \left(\frac{1}{\kappa_-} - \frac{1}{\kappa_+} \right) \ln \left| \frac{r}{r_- + r_+} + 1 \right| \end{aligned} \tag{A4}$$

The null coordinates can be defined as $u = t - r^*$, $v = t + r^*$.

One can, in principle, write the parameters M and H in terms of the surface gravities κ_- and κ_+ of the two horizons. However, it turns out that this cannot be written in a closed form. On the other hand, it is possible to write the combination MH in terms of the ratio κ_+/κ_- . One can get from Eq. (A3)

$$k = \frac{1 + 2x}{x(x + 2)} \tag{A5}$$

where we have defined

$$x \equiv \frac{r_+}{r_-} \tag{A6}$$

and

$$k \equiv \frac{\kappa_+}{\kappa_-} \tag{A7}$$

This can be inverted to obtain

$$x = \frac{1 - k + \sqrt{1 - k + k^2}}{k} \tag{A8}$$

Note that $k = 1$ only if $x = 1$, i.e., the surface gravities of the two horizons are equal only when the two horizons coincide. Also, since $x \geq 1$, one gets $k \leq 1$. This implies that the temperature corresponding to the cosmological horizon is always less than that of the black hole horizon.

Now, use Eq. (A1) to write

$$x = \frac{\sin[(\theta + 2\pi)/3]}{\sin(\theta/3)} \tag{A9}$$

The terms containing θ can be directly related to the combination MH using Eq. (A2), so that

$$MH = \frac{1}{2} \frac{x(1+x)}{(x^2+x+1)^{3/2}} \tag{A10}$$

and finally

$$MH = \frac{k}{2} \frac{2 - 2k + k^2 + (2 - k)\sqrt{1 - k + k^2}}{[2(1 - k + k^2) + (2 - k)\sqrt{1 - k + k^2}]^{3/2}} \tag{A11}$$

Appendix B: Global coordinates for the SDS spacetime

This appendix discusses the existence and properties of a global coordinate system in the SDS spacetime. As discussed in the main text, unlike the case of a metric with single horizon, one cannot cover the entire SDS manifold with the conventional Kruskal-like patches. However, it turns out that there exists a class of coordinate systems which are analytic over the whole space. Let us study one explicit example for this. The global coordinates (\bar{U}, \bar{V}) can be defined in each region of the Penrose diagram (see Fig. 1) in terms of the Kruskal coordinates defined in the corresponding region. For completeness, we give the detailed expressions for the global coordinates in each region:

Region I: $U_- < 0, V_- > 0, U_+ > 0, V_+ < 0$

$$\begin{aligned} \bar{U} &= \frac{1}{\kappa_-} \tanh \kappa_- U_- + \frac{1}{\kappa_+} \tanh \kappa_+ U_+ \\ &= \frac{1}{\kappa_-} \tanh \kappa_- U_- + \frac{1}{\kappa_+} \tanh(-\kappa_- U_-)^{-\frac{\kappa_+}{\kappa_-}} \\ &= -\frac{1}{\kappa_-} \tanh(\kappa_+ U_+)^{-\frac{\kappa_-}{\kappa_+}} + \frac{1}{\kappa_+} \tanh \kappa_+ U_+ \end{aligned}$$

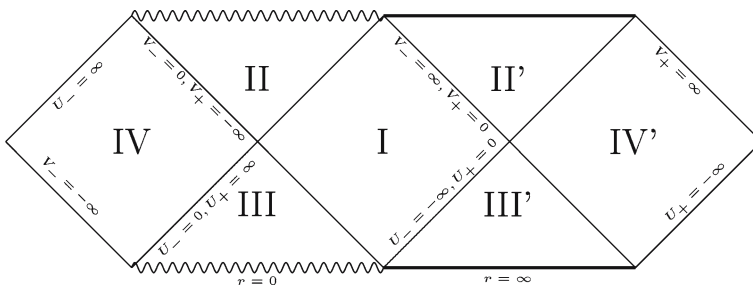


Fig. 1 Part of the Penrose diagram for the Schwarzschild–De Sitter spacetime. The pictured group of blocks keeps repeating infinitely on the *left* and on the *right*

$$\begin{aligned}
 \bar{V} &= \frac{1}{\kappa_-} \tanh \kappa_- V_- + \frac{1}{\kappa_+} \tanh \kappa_+ V_+ \\
 &= \frac{1}{\kappa_-} \tanh \kappa_- V_- - \frac{1}{\kappa_+} \tanh(\kappa_- V_-)^{-\frac{\kappa_+}{\kappa_-}} \\
 &= \frac{1}{\kappa_-} \tanh(-\kappa_+ V_+)^{-\frac{\kappa_-}{\kappa_+}} + \frac{1}{\kappa_+} \tanh \kappa_+ V_+
 \end{aligned}
 \tag{B1}$$

Region II: $U_- > 0, V_- > 0$

$$\begin{aligned}
 \bar{U} &= \frac{1}{\kappa_-} \tanh \kappa_- U_- - \frac{1}{\kappa_+} \tanh(\kappa_- U_-)^{-\frac{\kappa_+}{\kappa_-}} + \frac{2}{\kappa_+} \\
 \bar{V} &= \frac{1}{\kappa_-} \tanh \kappa_- V_- - \frac{1}{\kappa_+} \tanh(\kappa_- V_-)^{-\frac{\kappa_+}{\kappa_-}}
 \end{aligned}
 \tag{B2}$$

Region III: $U_- < 0, V_- < 0$

$$\begin{aligned}
 \bar{U} &= \frac{1}{\kappa_-} \tanh \kappa_- U_- + \frac{1}{\kappa_+} \tanh(-\kappa_- U_-)^{-\frac{\kappa_+}{\kappa_-}} \\
 \bar{V} &= \frac{1}{\kappa_-} \tanh \kappa_- V_- + \frac{1}{\kappa_+} \tanh(-\kappa_- V_-)^{-\frac{\kappa_+}{\kappa_-}} - \frac{2}{\kappa_+}
 \end{aligned}
 \tag{B3}$$

Region II': $U_+ > 0, V_+ > 0$

$$\begin{aligned}
 \bar{U} &= -\frac{1}{\kappa_-} \tanh(\kappa_+ U_+)^{-\frac{\kappa_-}{\kappa_+}} + \frac{1}{\kappa_+} \tanh \kappa_+ U_+ \\
 \bar{V} &= -\frac{1}{\kappa_-} \tanh(\kappa_+ V_+)^{-\frac{\kappa_-}{\kappa_+}} + \frac{1}{\kappa_+} \tanh \kappa_+ V_+ + \frac{2}{\kappa_-}
 \end{aligned}
 \tag{B4}$$

Region III': $U_+ < 0, V_+ < 0$

$$\begin{aligned}
 \bar{U} &= \frac{1}{\kappa_-} \tanh(-\kappa_+ U_+)^{-\frac{\kappa_-}{\kappa_+}} + \frac{1}{\kappa_+} \tanh \kappa_+ U_+ - \frac{2}{\kappa_-} \\
 \bar{V} &= \frac{1}{\kappa_-} \tanh(-\kappa_+ V_+)^{-\frac{\kappa_-}{\kappa_+}} + \frac{1}{\kappa_+} \tanh \kappa_+ V_+
 \end{aligned}
 \tag{B5}$$

Region IV: $U_- > 0, V_- < 0$

$$\begin{aligned}
 \bar{U} &= \frac{1}{\kappa_-} \tanh \kappa_- U_- - \frac{1}{\kappa_+} \tanh(\kappa_- U_-)^{-\frac{\kappa_+}{\kappa_-}} + \frac{2}{\kappa_+} \\
 \bar{V} &= \frac{1}{\kappa_-} \tanh \kappa_- V_- + \frac{1}{\kappa_+} \tanh(-\kappa_- V_-)^{-\frac{\kappa_+}{\kappa_-}} - \frac{2}{\kappa_+}
 \end{aligned}
 \tag{B6}$$

Region IV': $U_+ < 0, V_+ > 0$

$$\begin{aligned} \bar{U} &= \frac{1}{\kappa_-} \tanh(-\kappa_+ U_+)^{-\frac{\kappa_-}{\kappa_+}} + \frac{1}{\kappa_+} \tanh \kappa_+ U_+ - \frac{2}{\kappa_-} \\ \bar{V} &= -\frac{1}{\kappa_-} \tanh(\kappa_+ V_+)^{-\frac{\kappa_-}{\kappa_+}} + \frac{1}{\kappa_+} \tanh \kappa_+ V_+ + \frac{2}{\kappa_-} \end{aligned} \tag{B7}$$

One can see that the relations are quite similar in various regions. It is clear from the expressions that the global coordinates (\bar{U}, \bar{V}) reduce to the Kruskal patch (U_-, V_-) near the black hole horizon, and to (U_+, V_+) near the cosmological horizon. In fact, any coordinate patch with such a property will be regular at both the horizons and can act as a global coordinate system.

We shall now discuss the various properties of this coordinate system. To avoid unnecessary complications, from now on, we shall concentrate on two or three particular regions, namely, I and II (and II' if required). The relations for all the other regions can be trivially extended. Note that the boundary between I and II denotes the future black hole horizon ($r = r_-, U_- = 0$) while that between I and II' denotes the future cosmological horizon ($r = r_+, V_+ = 0$).

B.1 Continuity

It is possible to show that \bar{U} and \bar{V} are continuous over the whole spacetime by checking their values at the boundaries. For example, consider the boundary between I and II, where $U_- = 0$. The expressions for \bar{V} are identical in these two regions, and hence the continuity of \bar{V} is obvious. The value of \bar{U} near the boundary in region I is

$$\lim_{U_- \rightarrow 0^-} \bar{U} = \frac{1}{\kappa_+} + U_- - \frac{\kappa_-^2 U_-^3}{3} \tag{B8}$$

while that in region II is

$$\lim_{U_- \rightarrow 0^+} \bar{U} = \frac{1}{\kappa_+} + U_- - \frac{\kappa_-^2 U_-^3}{3} \tag{B9}$$

This shows that \bar{U} and its derivative are continuous across the boundary. Similarly, near the boundary between I and II', we have $V_+ = 0$. This time the expressions for \bar{U} are identical in these two regions, and hence the continuity of \bar{U} is obvious. The value of \bar{V} near the boundary in region I is

$$\lim_{V_+ \rightarrow 0^+} \bar{V} = \frac{1}{\kappa_-} + V_+ - \frac{\kappa_+^2 V_+^3}{3} \tag{B10}$$

while that in region II' is

$$\lim_{V_+ \rightarrow 0^-} \bar{V} = \frac{1}{\kappa_-} + V_+ - \frac{\kappa_+^2 V_+^3}{3} \tag{B11}$$

and the continuity follows. Similar proof can be given for all the other cases.

B.2 Explicit form of the metric

The metric can be written in terms of the global coordinates as

$$ds^2 = C(\bar{U}, \bar{V}) d\bar{U} d\bar{V} \tag{B12}$$

The quantity $C(\bar{U}, \bar{V})$ can be written explicitly in various regions. For example, in region I, use Eq. (18) to write it in terms of r and (U_-, V_-)

$$C(\bar{U}, \bar{V}) = \frac{2M}{r} \left(1 - \frac{r}{r_+}\right)^{1 + \frac{\kappa_-}{\kappa_+}} \left(1 + \frac{r}{r_- + r_+}\right)^{2 - \frac{\kappa_-}{\kappa_+}} \left(\frac{d\bar{U}}{dU_-} \frac{d\bar{V}}{dV_-}\right)^{-1} \tag{B13}$$

which gives

$$\begin{aligned} C(\bar{U}, \bar{V}) &= \frac{2M}{r} \left(1 - \frac{r}{r_+}\right)^{1 + \frac{\kappa_-}{\kappa_+}} \left(1 + \frac{r}{r_- + r_+}\right)^{2 - \frac{\kappa_-}{\kappa_+}} \\ &\times \left[\operatorname{sech}^2(\kappa_- U_-) + (-\kappa_- U_-)^{-\frac{\kappa_+}{\kappa_-} - 1} \operatorname{sech}^2(-\kappa_- U_-)^{-\frac{\kappa_+}{\kappa_-}} \right]^{-1} \\ &\times \left[\operatorname{sech}^2(\kappa_- V_-) + (\kappa_- V_-)^{-\frac{\kappa_+}{\kappa_-} - 1} \operatorname{sech}^2(\kappa_- V_-)^{-\frac{\kappa_+}{\kappa_-}} \right]^{-1} \end{aligned} \tag{B14}$$

[Similarly, one can start with Eq. (20) to write $C(\bar{U}, \bar{V})$ in terms of r and (U_+, V_+) .] Note that $C(\bar{U}, \bar{V})$ is positive definite in the region, as required. It might seem from the way it is written that it vanishes at the cosmological horizon $r = r_+$. However, a straightforward calculation shows that the conformal factor near $r = r_+$ is

$$C(\bar{U}, \bar{V}) \approx \frac{2M}{r} \left(1 + \frac{r}{r_- + r_+}\right)^{2 - \frac{\kappa_+}{\kappa_-}} \left(\frac{r}{r_-} - 1\right)^{\frac{\kappa_+}{\kappa_-} + 1} \tag{B15}$$

which is non-vanishing and non-singular. For completeness, let us determine its behaviour near the other horizon $r = r_-$

$$C(\bar{U}, \bar{V}) \approx \frac{2M}{r} \left(1 - \frac{r}{r_+}\right)^{1 + \frac{\kappa_-}{\kappa_+}} \left(1 + \frac{r}{r_- + r_+}\right)^{2 - \frac{\kappa_-}{\kappa_+}} \tag{B16}$$

Let us also write the explicit form of $C(\bar{U}, \bar{V})$ in region II. Note that only the coordinates (U_-, V_-) are defined in this region. Thus one can use Eq. (18) to get

$$\begin{aligned}
 C(\bar{U}, \bar{V}) &= \frac{2M}{r} \left(1 - \frac{r}{r_+}\right)^{1+\frac{\kappa_-}{\kappa_+}} \left(1 + \frac{r}{r_- + r_+}\right)^{2-\frac{\kappa_-}{\kappa_+}} \\
 &\times \left[\operatorname{sech}^2(\kappa_- U_-) + (\kappa_- U_-)^{-\frac{\kappa_+}{\kappa_-}-1} \operatorname{sech}^2(\kappa_- U_-)^{-\frac{\kappa_+}{\kappa_-}} \right]^{-1} \\
 &\times \left[\operatorname{sech}^2(\kappa_- V_-) + (\kappa_- V_-)^{-\frac{\kappa_+}{\kappa_-}-1} \operatorname{sech}^2(\kappa_- V_-)^{-\frac{\kappa_+}{\kappa_-}} \right]^{-1} \tag{B17}
 \end{aligned}$$

As required, $C(\bar{U}, \bar{V})$ is positive definite in the region too. Near the horizon $r = r_-$, one can perform a calculation similar to the case in region I, and show that

$$C(\bar{U}, \bar{V}) \approx \frac{2M}{r} \left(1 - \frac{r}{r_+}\right)^{1+\frac{\kappa_-}{\kappa_+}} \left(1 + \frac{r}{r_- + r_+}\right)^{2-\frac{\kappa_-}{\kappa_+}} \tag{B18}$$

which is identical to the corresponding expression in region I. Thus the metric is continuous and analytic across the horizon.

One can perform identical calculations for the other regions and find that the global coordinates (\bar{U}, \bar{V}) actually cover the whole space, and the metric is free from singularities at both the horizons.

B.3 The limiting cases

At this point, let us check the limiting case of $MH \rightarrow 0$ for the global coordinates. In region I, the global coordinates, written in terms of (t, r) coordinates are given by

$$\begin{aligned}
 \bar{U} &= -\frac{1}{\kappa_-} \tanh \left[e^{-\kappa_-(t-r^*)} \right] + \frac{1}{\kappa_+} \tanh \left[e^{\kappa_+(t-r^*)} \right] \\
 &= -\frac{1}{\kappa_-} \tanh \left[e^{-\kappa_- t} \sqrt{\frac{r}{r_-} - 1} \left(1 - \frac{r}{r_+}\right)^{-\frac{\kappa_-}{2\kappa_+}} \left(1 + \frac{r}{r_- + r_+}\right)^{\left(\frac{\kappa_-}{\kappa_+} - 1\right)/2} \right] \\
 &\quad + \frac{1}{\kappa_+} \tanh \left[e^{\kappa_+ t} \left(\frac{r}{r_-} - 1\right)^{-\frac{\kappa_+}{2\kappa_-}} \sqrt{1 - \frac{r}{r_+}} \left(1 + \frac{r}{r_- + r_+}\right)^{\left(\frac{\kappa_+}{\kappa_-} - 1\right)/2} \right] \tag{B19}
 \end{aligned}$$

and

$$\begin{aligned}
 \bar{V} &= \frac{1}{\kappa_-} \tanh \left[e^{\kappa_-(t+r^*)} \right] - \frac{1}{\kappa_+} \tanh \left[e^{-\kappa_+(t+r^*)} \right] \\
 &= \frac{1}{\kappa_-} \tanh \left[e^{\kappa_- t} \sqrt{\frac{r}{r_-} - 1} \left(1 - \frac{r}{r_+}\right)^{-\frac{\kappa_-}{2\kappa_+}} \left(1 + \frac{r}{r_- + r_+}\right)^{\left(\frac{\kappa_-}{\kappa_+} - 1\right)/2} \right] \\
 &\quad - \frac{1}{\kappa_+} \tanh \left[e^{-\kappa_+ t} \left(\frac{r}{r_-} - 1\right)^{-\frac{\kappa_+}{2\kappa_-}} \sqrt{1 - \frac{r}{r_+}} \left(1 + \frac{r}{r_- + r_+}\right)^{\left(\frac{\kappa_+}{\kappa_-} - 1\right)/2} \right] \tag{B20}
 \end{aligned}$$

As $MH \rightarrow 0$, we have $r_- \rightarrow 2M, r_+ \rightarrow H^{-1}, \kappa_- \rightarrow (4M)^{-1}$ and $\kappa_+ \rightarrow H$. Then, in the limit, the expressions become

$$\begin{aligned} \bar{U} \approx & -4M \tanh \left[e^{-t/4M} \sqrt{\frac{r}{2M} - 1} \left(\frac{1 + Hr}{1 - Hr} \right)^{1/8MH} \right] \\ & + \frac{1}{H} \tanh \left[e^{Ht} \sqrt{\frac{1 - Hr}{1 + Hr}} \right] \end{aligned} \tag{B21}$$

Now, as $H \rightarrow 0$, we have

$$\bar{U} \approx -4M \tanh \left[e^{-t/4M} e^{r/4M} \sqrt{\frac{r}{2M} - 1} \right] \tag{B22}$$

where we have used $(1 + Hr)^{(1/8MH)} \approx \exp(r/8M)$ and $(1 - Hr)^{(-1/8MH)} \approx \exp(r/8M)$ as $H \rightarrow 0$. Thus the global coordinate reduces to tanh of the usual Kruskal coordinates in the Schwarzschild spacetime. Similarly, when $M \rightarrow 0$, we have

$$\bar{U} \approx \frac{1}{H} \tanh \left[e^{Ht} \sqrt{\frac{1 - Hr}{1 + Hr}} \right] \tag{B23}$$

which is the tanh of the usual Kruskal coordinates in the De Sitter spacetime. Similar calculations can be done for \bar{V} in region I, and then one can extend the calculations to all the other regions.

B.4 Imaginary time and periodicity

Let us introduce the coordinates (\bar{T}, \bar{R}) :

$$\bar{T} = \frac{\bar{U} + \bar{V}}{2}, \quad \bar{R} = \frac{\bar{V} - \bar{U}}{2} \tag{B24}$$

Now, analytically continue the \bar{T} coordinate to imaginary space and define the Euclidean time coordinates

$$\bar{T}_E = i\bar{T} \tag{B25}$$

One would like to express the set (\bar{R}, \bar{T}_E) in terms of (r, t_E) . Note that in region I, putting $t = -it_E$ in the expressions for \bar{U} and \bar{V} shows that $\bar{U} = -\bar{V}^\dagger$. Then $\bar{T}_E = i(\bar{U} + \bar{V})/2 = i(\bar{V} - \bar{V}^\dagger)/2 = -\Im(\bar{V})$ and similarly $\bar{R} = \Re(\bar{V})$.

A straightforward calculation shows that

$$\begin{aligned} \bar{T}_E &= \frac{1}{\kappa_-} \frac{\tan\left(e^{\kappa_- r^*} \sin \kappa_- t_E\right) \operatorname{sech}^2\left(e^{\kappa_- r^*} \cos \kappa_- t_E\right)}{1 + \tanh^2\left(e^{\kappa_- r^*} \cos \kappa_- t_E\right) \tan^2\left(e^{\kappa_- r^*} \sin \kappa_- t_E\right)} \\ &\quad + \frac{1}{\kappa_+} \frac{\tan\left(e^{-\kappa_+ r^*} \sin \kappa_+ t_E\right) \operatorname{sech}^2\left(e^{-\kappa_+ r^*} \cos \kappa_+ t_E\right)}{1 + \tanh^2\left(e^{-\kappa_+ r^*} \cos \kappa_+ t_E\right) \tan^2\left(e^{-\kappa_+ r^*} \sin \kappa_+ t_E\right)} \\ \bar{R} &= \frac{1}{\kappa_-} \frac{\sec^2\left(e^{\kappa_- r^*} \sin \kappa_- t_E\right) \tanh\left(e^{\kappa_- r^*} \cos \kappa_- t_E\right)}{1 + \tanh^2\left(e^{\kappa_- r^*} \cos \kappa_- t_E\right) \tan^2\left(e^{\kappa_- r^*} \sin \kappa_- t_E\right)} \\ &\quad - \frac{1}{\kappa_+} \frac{\sec^2\left(e^{-\kappa_+ r^*} \sin \kappa_+ t_E\right) \tanh^2\left(e^{-\kappa_+ r^*} \cos \kappa_+ t_E\right)}{1 + \tanh^2\left(e^{-\kappa_+ r^*} \cos \kappa_+ t_E\right) \tan^2\left(e^{-\kappa_+ r^*} \sin \kappa_+ t_E\right)} \end{aligned} \quad (\text{B26})$$

This shows that \bar{T}_E and \bar{R} is periodic only when $\kappa_+/\kappa_- = n_+/n_-$, and the period of t_E is given by $\beta = 2\pi n_+/\kappa_+ = 2\pi n_-/\kappa_-$.

One can show that the conformal factor $C(\bar{U}, \bar{V})$ is positive definite in region I when written in terms of the imaginary time (this follows from Eq. (B14) and from the fact that $U_- = -V_-^\dagger$ when written in terms of t_E).

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