

Dark energy and cosmological solutions in second-order string gravity

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Abstract. We study the cosmological evolution based upon a D -dimensional action in low-energy effective string theory in the presence of second-order curvature corrections and a modulus scalar field (dilaton or compactification modulus). A barotropic perfect fluid coupled to the scalar field is also allowed. Phase space analysis and the stability of asymptotic solutions are performed for a number of models which include (i) fixed scalar field, (ii) linear dilaton in string frame, and (iii) logarithmic modulus in Einstein frame. We confront analytical solutions with observational constraints for deceleration parameter and show that Gauss-Bonnet gravity (with no matter fields) may not explain the current acceleration of the universe. We also study the future evolution of the universe using the GB parametrization and find that big rip singularities can be avoided even in the presence of a phantom fluid because of the balance between the fluid and curvature corrections. A non-minimal coupling between the fluid and the modulus field also opens up the interesting possibility to avoid big rip regardless of the details of the fluid equation of state.

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1. Introduction

During the past few years precision cosmology experiments have confirmed the big bang/inflationary standard scenario of a flat universe characterized by nearly scale-invariant, Gaussian primordial density perturbations. Two thirds of the total energy density, which is close to the critical one, is contributed by an exotic form of matter popularly known as dark energy (see [1]–[5] and references therein for an overview on the subject). Moreover, recent observations of extra-galactic supernovae treated as standardized candles indicate that the universe is presently undergoing a phase of accelerated expansion which receives an independent support from CMB and large-scale structure data. Within the framework of General Relativity, the cosmic speed-up can be fueled by an energy-momentum tensor with negative pressure – dark energy. The analysis of high redshift type Ia supernovae exhibits a marginal evidence of a possible super-acceleration [6]. The latter eventuality has triggered many speculations on the underlying nature of dark energy, among which the phantom hypothesis has been under active investigations (see Refs. [7]–[10] for early works on phantom). By definition, a phantom is a fluid with pressure and energy density satisfying an equation of state $p < -\rho$; if the fluid is represented by a scalar field, its kinetic energy has a negative sign (hence phantoms and ghosts are relatives). This kind of contribution leads to a finite-time future singularity dubbed big rip. We also note that there exist other types of sudden future singularities even in the absence of the phantom [11]–[13]. An updated list of references on these subjects, including first and latest proposals, big rip phenomenology, and criticism, can be found in [14].

The search for a viable semiclassical or quantum model for the dark energy component and its dynamical properties is one of the topics explored by “string cosmology”, which attempts to understand the cosmological implications of string effective actions. In particular, quantum corrections to the Einstein–Hilbert action naturally arise for the closed string, bosonic or supersymmetric, and modify the gravitational interaction in a way which might be testable in the near future. The Gauss–Bonnet (GB) combination of curvature invariants with fixed dilaton is of special relevance: (i) it represents the unique 5D leading-order extension of the Einstein–Hilbert action that leads to second order gravitational field equations, and (ii) the highest derivative occurs in equations linearly, thereby ensuring a unique solution; finally (iii) with the GB parametrization, graviton interactions are ghost-free and spacetime perturbations are wave-like. However, in four dimensions, the GB term with fixed dilaton is topological and does not contribute to the equations of motions. Then, the only way to modify the cosmological evolution is to consider GB term in the bulk in a braneworld setup, so that the induced 4D Friedmann equation on the brane depends on the bulk back-reaction. The GB braneworld and its observational consequences, both in the early and recent universe, are still under extensive study (see [14] for a list of references).

Second-order gravity terms can generate non-trivial effects in a four-dimensional

spacetime even outside the braneworld lore, provided that we relax some common assumptions: the most important ones are fixed dilaton and GB parametrization. In the first case, it is assumed that non-perturbative potentials arise and dilaton and other moduli freeze out with a minimum non-zero mass, so that to preserve the observational bounds for the gravitational interaction [15, 16]. However, a massless dilaton still can be compatible with observations [17, 18, 19] (see also [20, 21]).

If the dilaton varies in spacetime, gradient terms appear and the higher-order curvature invariants interact with a non-constant α' -coupling. Also, compactification of the 26 or 10-dimensional target space is encoded in residual modulus fields which in general will evolve in time. Both cases result in a dynamical field non-minimally coupled to gravity.

Although the Gauss–Bonnet combination of curvature invariants is ghost-free, we should bear in mind that in the high-energy limit the uniqueness of the GB term is not guaranteed and, from the point of view of string theory, the GB parametrization cannot be distinguished from the others (see references in [22]).

Dilaton and modulus cosmologies in higher-order gravity were considered, e.g., in [22]–[39]. As regards the recent universe, dark energy models with higher-order terms were inspected, e.g., in [40] in the case of fixed moduli, while a dynamical dilaton as a dark energy candidate was studied in [41, 42]. Recently Nojiri *et al.* [43] investigated a dark energy scenario in GB gravity coupled to a dynamical scalar field with non-negative potential.

The understanding of the impact of higher-order curvature terms on the macrocosmic evolution may open up the possibility to verify high-energy fundamental theories through cosmology. We address this issue in two ways. First, we find cosmological solutions in the absence of an extra fluid in the action and constrain them via the recent supernovae data [6]. It turns out that, while solutions with non-GB parametrizations can fit the experimental evidence, the GB solutions are in general not viable, unless one takes the perfect fluid into account or, in the case of finite-time future singularity solutions, one fine-tunes the parameters of the model. Second, we solve numerically the equations of motion with a perfect fluid in four dimensions and study the future behaviour of its energy density and the Hubble parameter. In particular we are interested in the case of a phantom fluid which leads to the growth of the energy density of the universe. Then higher-order curvature corrections become inevitably important around the big rip. In fact, we will show that both second-order curvature terms and a non-minimal coupling between the fluid and the modulus field deeply affect the cosmological evolution. We shall classify the situations in which big rip singularities are reached or avoided. We keep arbitrary coefficients in the second-order gravity action, considering then the usual Gauss–Bonnet parametrization as a special case. Our results are in agreement with past investigations in literature where comparison is possible. Also, this work extends previous analyses to a spacetime of arbitrary dimension (including the 4D case) with a (coupled) barotropic fluid being taken into account.

The paper is organized as follows. In section 2 we present a general action which involves curvature, scalar field, and barotropic fluid terms, and derive basic equations for this system.

In section 3 we take up the issue of de Sitter and inflationary solutions in the case of a constant scalar field. We then work out the phase space and stability analysis for a linearly changing dilaton in the string frame in section 4. Some technical material is confined to the Appendix.

Section 5 is devoted to the logarithmic modulus case with stabilized dilaton, which solves either exactly or asymptotically the equations of motion. The stability of these solutions is studied, together with their application as concrete models for the late time evolution of the universe.

The properties of second-order gravity in the context of (phantom) dark energy are discussed in section 6. We show that it is possible to avoid big rip singularity in the presence of the Gauss–Bonnet corrections with a dynamically changing modulus field. The coupling Q between the modulus and the fluid can also play important roles in determining the future fate of the universe. Conclusions are given in section 7.

2. General setup

In our notation, $g_{\mu\nu}$ is the D -dimensional metric with signature $(-, +, \dots, +)$ and Greek indices run from 0 to $d \equiv D - 1$. The Riemann and Ricci tensors are defined as $R_{\beta\mu\nu}^{\alpha} \equiv \partial_{\mu}\Gamma_{\beta\nu}^{\alpha} - \partial_{\nu}\Gamma_{\beta\mu}^{\alpha} + \Gamma_{\mu\sigma}^{\alpha}\Gamma_{\beta\nu}^{\sigma} - \Gamma_{\nu\sigma}^{\alpha}\Gamma_{\beta\mu}^{\sigma}$ and $R_{\mu\nu} \equiv R_{\mu\alpha\nu}^{\alpha}$, respectively, where $\Gamma_{\beta\nu}^{\alpha}$ is the Christoffel symbol. The gravitational coupling κ_D and the Regge slope α' are set to unity.

2.1. General action

The gravity+matter action we begin from is (e.g., [44, 45])

$$\mathcal{S} = \int d^D x \sqrt{-g} \left[\frac{1}{2} f(\phi, R) - \frac{1}{2} \omega(\phi) (\nabla\phi)^2 - V(\phi) + \xi(\phi) \mathcal{L}_c^{(\phi)} + \mathcal{L}_\rho^{(\phi)} \right], \quad (1)$$

where g is the determinant of the D -dimensional metric, ϕ is a scalar field corresponding either to the dilaton or to another modulus, and f is a generic function of the scalar field and the Ricci scalar R . ω , ξ and V are functions of ϕ . In this paper we do not consider the cosmological dynamics in the presence of the field potential V , but we include this term in deriving basic equations. $\mathcal{L}_\rho^{(\phi)}$ is the Lagrangian of a D -dimensional perfect fluid with energy density ρ and pressure p . Later on we shall assume that the barotropic index $w \equiv p/\rho$ is a constant. In general the fluid will be coupled to the scalar field ϕ .

Finally, α' -order quantum corrections are encoded in the term

$$\mathcal{L}_c^{(\phi)} = a_1 R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R^2 + a_4 (\nabla\phi)^4, \quad (2)$$

where a_i are coefficients depending on the string model one is considering. In section 2.3 we will specify the functions f , ω , V and ξ together with the coefficients a_i according to the scenarios of interest.

We assume that the target spacetime is described by a flat Friedmann-Robertson-Walker metric

$$ds^2 = -N^2(t)dt^2 + a^2(t) \sum_{i=1}^d (dx^i)^2, \quad (3)$$

where $N(t)$ is the lapse function ($N = 1$ in synchronous gauge) and $a(t)$ is the scale factor describing the physical size of the universe. The size of the causal connected region centered on the observer is given by the inverse of the Hubble parameter $H \equiv \dot{a}/(Na)$. In the following, dots and primes denote derivatives with respect to synchronous time t and ϕ , respectively.

2.2. Equations of motion

With the metric (3), the Riemann invariants read

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 2d \left[2 \left(H^2 + \frac{\dot{H}}{N} \right)^2 + (d-1)H^4 \right], \quad (4)$$

$$R_{\mu\nu}R^{\mu\nu} = d \left[d \left(H^2 + \frac{\dot{H}}{N} \right)^2 + \left(dH^2 + \frac{\dot{H}}{N} \right)^2 \right], \quad (5)$$

$$R = d \left[(d+1)H^2 + 2\frac{\dot{H}}{N} \right], \quad (6)$$

where we used

$$\frac{\ddot{a}}{N^2a} - \frac{\dot{N}}{N} \frac{H}{N} = H^2 + \frac{\dot{H}}{N}. \quad (7)$$

For simplicity we shall limit the discussion to a homogeneous scalar field $\phi(t)$. Then the spatial volume can be integrated out from the measure in equation (1), which we rewrite as

$$\mathcal{S} = \int dt N a^d [\mathcal{L}_g + \mathcal{L}_\rho^{(\phi)}]. \quad (8)$$

The Hamiltonian constraint $\delta\mathcal{S}/\delta N|_{N=1} = 0$ is

$$\left(\mathcal{L}_g + N \frac{\partial \mathcal{L}_g}{\partial N} - dHN \frac{\partial \mathcal{L}_g}{\partial \dot{N}} - N \frac{d}{dt} \frac{\partial \mathcal{L}_g}{\partial \dot{N}} - \rho \right) \Big|_{N=1} = 0, \quad (9)$$

giving the Friedmann equation

$$d(d-1)FH^2 = RF - f - 2dH\dot{F} + 2(\rho + \rho_\phi + \xi\rho_c), \quad (10)$$

where

$$\rho_\phi = \frac{1}{2}\omega\dot{\phi}^2 + V, \quad (11)$$

$$\rho_c = 3a_4\dot{\phi}^4 - d[4c_1\Xi H^3 + (d-3)c_1H^4 + c_2(2\Xi H\dot{H} + 2dH^2\dot{H} - \dot{H}^2 + 2H\ddot{H})]. \quad (12)$$

Here $F \equiv \partial f / \partial R$, $\Xi \equiv \dot{\xi} / \xi$, and

$$c_1 \equiv 2a_1 + da_2 + d(d+1)a_3, \quad (13a)$$

$$c_2 \equiv 4a_1 + (d+1)a_2 + 4da_3. \quad (13b)$$

Note that all three energy densities, ρ , ρ_ϕ and ρ_c , depend on the field ϕ . In four dimensions ($d = 3$), the coefficients (13a)–(13b) read $c_1 = 2a_1 + 3a_2 + 12a_3$ and $c_2 = 4(a_1 + a_2 + 3a_3)$, while the H^4 term in ρ_c vanishes. At low energy it was shown that the unique higher-order gravitational Lagrangian giving a theory without ghosts is the Gauss–Bonnet one ($a_1 = a_3 = 1$, $a_2 = -4$, $a_4 = -1$). In this case, c_2 vanishes identically while $c_1 = 2 + d(d-3)$. With fixed dilaton coupling ($\Xi = 0$) equation (10) reduces to the standard Friedmann equation in four dimensions, in agreement with the fact that the GB term is topological when $d = 3$. In three dimensions ($d = 2$), the GB higher-derivative contribution vanishes identically except for the $\dot{\phi}^4$ term.

The continuity equation for the dark energy fluid contains a source term given by the coupling between this fluid and the string scalar field. We choose the covariant coupling considered in [46, 47, 48]: $\delta \mathcal{S}_\rho / \delta \phi = -\sqrt{-g} Q(\phi) \rho$, where $\mathcal{S}_\rho = \int d^D x \sqrt{-g} \mathcal{L}_\rho^{(\phi)}$ and $Q(\phi)$ is an unknown function which we shall set to a constant later. In synchronous gauge we have

$$\dot{\rho} + dH\rho(1+w) = Q\dot{\phi}\rho, \quad (14)$$

while the equation of motion for the field ϕ is

$$\omega(\ddot{\phi} + dH\dot{\phi}) + V' - \xi' \mathcal{L}_c^{(\phi)} + 4a_4 \xi \dot{\phi}^2 (3\ddot{\phi} + dH\dot{\phi} + \Xi\dot{\phi}) + \left(\dot{\omega}\dot{\phi} - \omega' \frac{\dot{\phi}^2}{2} - \frac{f'}{2} \right) = -Q\rho, \quad (15)$$

where the Lagrangian of the quantum correction is written as

$$\mathcal{L}_c^{(\phi)} = d \left[(d+1)c_1 H^4 + 4c_1 H^2 \dot{H} + c_2 \dot{H}^2 \right] + a_4 \dot{\phi}^4. \quad (16)$$

Equations (10)–(15) are the master equations of the physical system under study.

2.3. Dilaton and modulus scenarios

In order to reproduce general relativity at low energy, we impose that the leading term is given by the Einstein–Hilbert linear Lagrangian, i.e., $f(\phi, R) = F(\phi)R$.

The massless dilaton field arises in the string loop expansion of the low-energy effective theory, its vacuum expectation value being directly related to the string coupling (e.g., [50]). It is possible to consider the physical properties of the dilaton in two different frames (see, e.g., [51, 52, 53]). In the “string frame” (subscript S), the dilatonic action corresponds to

$$F = -\omega = e^{-\phi}, \quad (17a)$$

$$V = 0, \quad (17b)$$

$$\xi = \frac{\lambda}{2} e^{-\phi}, \quad (17c)$$

where $\lambda = 1/4, 1/8$ for the bosonic and heterotic string, respectively, whereas $\lambda = 0$ in the Type II superstring. In fact, the above expression for ξ is the tree-level term in the full contribution of n -loop corrections, given by

$$\xi = \frac{1}{2}\lambda \sum_{n=0} C_n e^{(n-1)\phi}, \quad (18)$$

where $C_0 = 1$. In this work we shall take only the tree-level term (17c) into account. Note that our function $\xi(\phi)$ is dubbed $-\frac{1}{2}\alpha'\lambda\xi(\phi)$ in [45].

One can transform to the ‘‘Einstein frame’’ ($f = R$, subscript E) by making a conformal transformation

$$g_{\mu\nu}^{(S)} = F^{\frac{2}{D-2}} g_{\mu\nu}^{(E)} = e^{\frac{2\phi}{D-2}} g_{\mu\nu}^{(E)}, \quad (19)$$

so that $F_E = \omega_E = 1$, while the term $\mathcal{L}_c^{(\phi)}$ is modified accordingly (note that $\xi_E = \xi_S$ in four dimensions). Although the analysis can be performed in any frame, the equivalence principle is preserved only in the Einstein frame (in the string frame, the dilaton coupling to gravity results in a varying gravitational ‘‘constant’’). One can shift from the string to the Einstein frame by noting that

$$H_E = e^{\frac{\phi}{D-2}} \left(H_S - \frac{\dot{\phi}}{D-2} \right), \quad (20)$$

$$\dot{H}_E = e^{\frac{2\phi}{D-2}} \left[\dot{H}_S + \frac{\dot{\phi}H_S}{D-2} - \frac{\ddot{\phi}}{D-2} - \left(\frac{\dot{\phi}}{D-2} \right)^2 \right], \quad (21)$$

where dots in the left (right) hand side denote derivatives with respect to the Einstein (string) time coordinate. Two frames coincide when the dilaton is fixed.

In the string frame, the equations of motion read:

$$d(d-1)H^2 = (2dH - \dot{\phi})\dot{\phi} + 2e^\phi \rho + \lambda \rho_c, \quad (22)$$

$$-Qe^\phi \rho = \ddot{\phi} + dH\dot{\phi} + \frac{\lambda}{2} \mathcal{L}_c^{(\phi)} + 2a_4 \lambda \dot{\phi}^2 (3\ddot{\phi} + dH\dot{\phi} - \dot{\phi}^2) + \frac{1}{2} (\dot{\phi}^2 + R), \quad (23)$$

where one has $\Xi = -\dot{\phi}$ in the definition of ρ_c .

In general, other moduli appear whenever a submanifold of the target spacetime is compactified with compactification radii described by the expectation values of the moduli themselves. In the case of a single modulus (one common characteristic length) and heterotic string ($\lambda = 1/8$), the four-dimensional action corresponds to [23]

$$F = 1, \quad (24a)$$

$$\omega = 3/2, \quad (24b)$$

$$a_4 = 0, \quad (24c)$$

$$\xi = -\frac{\delta}{16} \ln[2e^\phi \eta^4(ie^\phi)], \quad (24d)$$

where η is the Dedekind function and δ is a constant proportional to the 4D trace anomaly and depending on the number of chiral, vector, and spin-3/2 massless supermultiplets of the $N = 2$ sector of the theory. In general it can be either positive

or negative, but it is positive for the theories in which not too many vector bosons are present. Again the scalar field corresponds to a flat direction in the space of nonequivalent vacua and $V = 0$. At large ϕ the last equation can be approximated as

$$\xi \approx \xi_0 \cosh \phi, \quad (25)$$

$$\xi_0 \equiv \frac{\pi\delta}{24}, \quad (26)$$

which we shall use instead of the exact expression. In fact it was shown in Ref. [31] that this approximation gives results very close to those of the exact case.

In any realistic situation both the dilaton and the modulus can appear in the action. However, we can consider the effect of each of them separately for the following reason. As already mentioned, a varying dilaton is strongly constrained by gravitational experiments. Although there is still a possibility to allow for a non-trivial dynamical contribution, one can assume to freeze it out at the minimum of its non-perturbative potential. Once the dilaton is fixed, gravity is not coupled to the surviving modulus field. Also, while in the modulus case one considers the heterotic string already compactified to $d = 3$, in the dilaton case we should start from the 26- or 10-dimensional action and then compactify down to four dimensions. To set $d = 3$ into the action is equivalent to consider a stabilized modulus (or moduli) while keeping the dilaton as a flat direction of the theory.

3. de Sitter and inflationary solutions: fixed scalar field

3.1. Preliminary remarks and geometrical inflation

The search for de Sitter (dS) solutions of the higher-derivative equations of motion [54] is important at least for two reasons. First, they can give useful insight as regards sensible implementations of string-motivated scenarios in the early-universe inflationary context [24, 25]. Secondly, stable dS solutions may avoid the infall of the late universe in eventual finite-time singularities. In order to take into account the case of inflation, we shall consider small but non-vanishing time derivatives of the Hubble parameter. We introduce the slow-roll (SR) parameter

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad (27)$$

while leaving the other SR parameters specified implicitly in the second-order relation $\ddot{H} = O(\epsilon^2)$. Since the evolution equation for ϵ is at second-order in the SR parameters themselves, these are constant if small enough (slow-roll approximation). Pure dS solutions correspond to $\epsilon = \dots = 0$, where dots stand for other parameters of the SR tower. The Friedmann equation (10) can be truncated at lowest order in the SR parameters by neglecting $O(\epsilon^2)$ contributions. § Let us study the case of a constant scalar field ($\dot{\phi} = 0$) in order to present a preliminary example.

§ Typical inflationary scenarios are achieved via the introduction of a dynamical scalar field. If we consider ρ in the action (1) as the energy density of the inflaton, then it would be trivial to find quasi

In general, (quasi) de Sitter solutions can exist for a non-linear $f(R)$ as in the Starobinsky model [55]. In the case of linear gravity, we can fix $F = 1 = \xi$ without loss of generality. If the scalar field ϕ vanishes or is constant with vanishing potential ($\rho_\phi = 0 = \Xi$), the Friedmann equation becomes a differential Riccati-type equation in H ,

$$4c_2 d\dot{H} + 2(d-3)c_1 H^2 + (d-1) = 0, \quad (28)$$

where we have factored out the constraint giving the trivial Minkowski spacetime $H = 0$. Clearly there are no dS solutions for $d = 1$ and $d = 3$. At low energies the only invariant Lagrangian which is second order in the Riemann tensor and contains no ghosts is the Gauss–Bonnet invariant, which coincides with the Euler characteristic of the target manifold in four dimensions. As $c_2 = 0$, not even inflationary solutions would exist in the GB case. Nevertheless it is worth studying the case of $c_2 \neq 0$ for the reasons explained in the introduction. Then the 4D inflationary solution is

$$H(t) = -\frac{t}{6c_2}, \quad (29)$$

with $c_2 < 0$. When $1 \neq d \neq 3$, there are both dS and inflationary solutions. The former is given by [22] $H^2 = -(d-1)/[2(d-3)c_1]$. If $d > 3$, then c_1 must be negative, which excludes the GB case. The quasi dS solution for equation (28) with $c_2 \neq 0$ is

$$H(t) = \sqrt{\frac{d-1}{2(3-d)\xi_0 c_1}} \tanh \left[\sqrt{\frac{(d-1)(3-d)c_1}{8d^2 \xi_0 c_2^2}} t \right], \quad (30)$$

which is obtained by fixing the integration constant so that $H(0) = 0$ and by assuming $c_1, c_2 < 0$ ($H(t)$ is positive and monotonic in this case). Changing the sign of c_1 results in a periodic tangent behaviour. The Hubble rate (30) approaches a constant in the limit $t \rightarrow \infty$. To summarize, higher-derivative terms can induce an inflationary phase of pure geometrical nature. This unusual situation^{||} is anyway avoided in any dimension when considering a Gauss–Bonnet coupling. This might be another reason why to fix the coefficients c_i to their GB value.

A stability analysis can be performed as follows. Let H_c be a solution of equation (28) and $H = H_c + \delta H$ be another solution given by the first plus a small time-dependent perturbation $\delta H \neq 0$. If the perturbation decays in time, then the solutions are stable. Dropping $\mathcal{O}(\delta H^2)$ terms in equation (28), one gets

$$\delta H(t) = a(t)^{(3-d)c_1/(dc_2)}, \quad (31)$$

where we have used the definition of the Hubble parameter. Therefore the dS solution for $d > 3$ exists and is stable if $c_1, c_2 < 0$.[¶] Since the inflationary solutions can be de Sitter solutions. Here our aim is to consider the pure gravitational action with $\rho = 0$ and see what kind of cosmological evolution is obtained in the presence of curvature corrections.

^{||} Curvature-driven accelerating models were already considered in, e.g., [55]–[62] and references therein. See also [63] for a case which can be compatible with solar system experiments.

[¶] This result, confirmed by the following numerical analysis, is in disagreement with [22], where it was claimed that dS solutions are unstable regardless of the parametrization. In particular, we were not able to reproduce their equations (33)–(36).

regarded as perturbations of the dS ones, intuitively these are stable when dS solutions are; indeed the inflationary solution (30) is stable.

In the 4D case corresponding to equation (29), the perturbation is constant and therefore the solution is not stable, as could be guessed by the fact that there is no dS background to perturb.

We caution the reader that from the point of view of quantum gravity, models with a non-GB parametrization are unstable because of the presence of ghost fields. In this respect, whenever we mention the stability issue for such models we are implicitly referring to the classical stability against linear perturbations of a classical solution within a phenomenological effective theory. The details of the latter are still far from being established, as we shall discuss in the concluding section.

3.2. Phase space analysis

One may wonder if the slow-roll approximation we used in the previous subsection is really justified. A priori the \ddot{H} term may play a relevant role in the analysis, since the stability in that direction was not considered. In the next sections we clarify this point and generalize to the case of a dynamical scalar field. We shall consider the cases $c_2 \neq 0$ and $c_2 = 0$ (GB scenario) separately.

By introducing the variables

$$x \equiv H, \quad y \equiv \dot{H}, \quad u \equiv \phi, \quad v \equiv \dot{\phi}, \quad z \equiv \rho, \quad (32)$$

we obtain the phase space evolution equations

$$\dot{x} = y, \quad (33)$$

$$\dot{y} = -\frac{(d-1)Fx}{4\xi c_2} - \frac{\dot{F}}{2\xi c_2} + \frac{2z + \omega v^2 + 2V}{4d\xi c_2 x} + \frac{3a_4 v^4}{2dc_2 x} - \frac{4c_1 \Xi x^3 + (d-3)c_1 x^4 + c_2(2\Xi xy + 2dx^2 y - y^2)}{2c_2 x}, \quad (34)$$

$$\dot{u} = v, \quad (35)$$

$$\dot{v} = \frac{-d\omega xv - V' + \xi' \mathcal{L}_c^{(\phi)} - 4a_4 \xi v^3(dx + \Xi) - \dot{\omega}v + (\omega'v^2 + F'R)/2 - Qz}{\omega + 12a_4 \xi v^2}, \quad (36)$$

$$\dot{z} = [-dx(1+w) + Qv]z, \quad (37)$$

where

$$\mathcal{L}_c^{(\phi)} = d[(d+1)c_1 x^4 + 4c_1 x^2 y + c_2 y^2] + a_4 v^4, \quad (38)$$

$$R = d[(d+1)x^2 + 2y]. \quad (39)$$

In what follows we do not take into account the scalar field potential, which is not present in a perturbative string framework. In addition the coupling Q is assumed to be constant.

In the GB case ($c_2 = 0$) differentiating the Friedmann equation (10) with respect to time gives the following equation

$$2d[\dot{F} + (d-1)Fx + 12c_1 \dot{\xi} x^2 + 4(d-3)c_1 \xi x^3] \dot{x} = (RF - f)^\bullet - 2d\ddot{F}x - d(d-1)\dot{F}x^2$$

$$+2\dot{z} + \dot{\omega}v^2 + 2\omega v\dot{v} + 2vV' + 6a_4v^3(\dot{\xi}v + 4\xi\dot{v}) - 2d(d-3)c_1\dot{\xi}x^4 - 8dc_1\ddot{\xi}x^3. \quad (40)$$

Let us search for a dS solution which is characterized by $\dot{x} = 0$, $\dot{y} = 0$ with a vanishing fluid ($\dot{z} = z = 0$). We find that

$$2\xi d(d-3)c_1x^4 + 8\xi dc_1\Xi x^3 + d(d-1)Fx^2 + 2d\dot{F}x - \omega v^2 - 6a_4\xi v^4 = 0. \quad (41)$$

When the scalar field ϕ is fixed with vanishing potential ($\xi = 1 = F$), equations (10) and (14) reduce to

$$d(d-1)H^2 = 2\rho - 2d \left[(d-3)c_1H^4 + c_2(2dH^2\dot{H} - \dot{H}^2 + 2H\ddot{H}) \right], \quad (42)$$

$$\dot{\rho} + dH(1+w)\rho = 0. \quad (43)$$

Defining the vector $\tilde{X} \equiv (x, y, z)^t$, the above equations of motion can be written as

$$\dot{x} = y, \quad (44)$$

$$\dot{y} = \frac{(3-d)c_1}{2c_2}x^3 - dxy + \frac{y^2}{2x} - \frac{d-1}{4c_2}x + \frac{z}{2dc_2x}, \quad (45)$$

$$\dot{z} = -d(1+w)xz, \quad (46)$$

where we assumed that the perfect fluid has a constant barotropic index w . In the following we treat the case $w = -1$ (cosmological constant) separately. Also, we consider a space with $d > 3$ in this section.

3.3. Perfect fluid with $w \neq -1$

The dS fixed point $\dot{\tilde{X}}_c = 0$ for the previous set of equations is

$$x_c = \sqrt{\frac{(d-1)}{2(3-d)c_1}}, \quad y_c = 0, \quad z_c = 0, \quad (47)$$

where we have considered only the expanding case. When $d > 3$ we require the condition $c_1 < 0$ for the existence of the dS solution.

Linearizing the equations of motion under a perturbation $\delta\tilde{X} = \tilde{X} - \tilde{X}_c$ with respect to a solution \tilde{X}_c , one obtains

$$\delta\dot{x} = \delta y, \quad (48)$$

$$\delta\dot{y} = \left[\frac{3(3-d)c_1}{2c_2}x_c^2 - dy_c - \frac{y_c^2}{2x_c^2} - \frac{(d-1)}{4c_2} - \frac{z_c}{2dc_2x_c^2} \right] \delta x + \left(\frac{y_c}{x_c} - dx_c \right) \delta y + \frac{1}{2dc_2x_c} \delta z, \quad (49)$$

$$\delta\dot{z} = -d(1+w)z_c\delta x - d(1+w)x_c\delta z. \quad (50)$$

This can be rewritten in compact notation as $\delta\dot{\tilde{X}} = \tilde{M}\delta\tilde{X}$, where \tilde{M} is a 3×3 matrix with elements m_{ij} and eigenvalues γ defined by the Jordan constraint

$$\gamma^3 - (m_{33} + m_{22})\gamma^2 + (m_{22}m_{33} - m_{21})\gamma + (m_{21}m_{33} - m_{31}m_{23}) = 0. \quad (51)$$

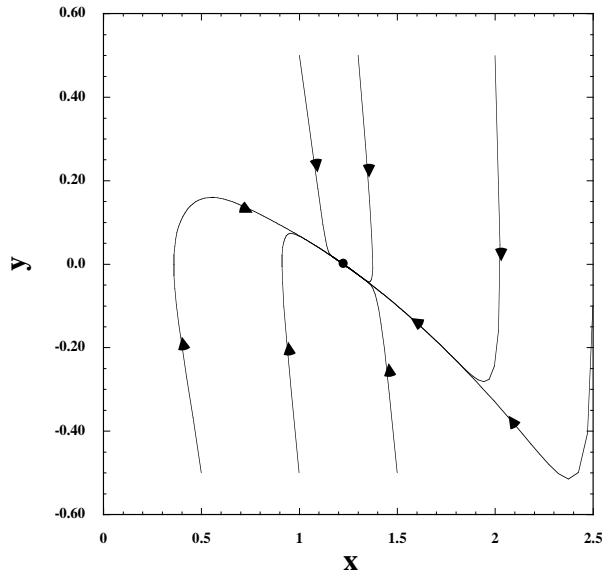


Figure 1. The phase space plot for $d = 4$, $\xi = 1$, $F = 1$, $w = -0.9$, $c_1 = -1$ and $c_2 = -1$ when the scalar field ϕ is fixed. In this case the de Sitter fixed point $(x_c, y_c, z_c) = (1.22, 0, 0)$ is a stable attractor.

For the dS fixed point given by equation (47) one has $m_{31} = -d(1+w)z_c = 0$. In this case equation (51) is factored out as $\gamma = m_{33}$ and $\gamma^2 - m_{22}\gamma - m_{21} = 0$. Therefore we find that the eigenvalues are

$$\gamma_1 = -d(1+w)x_c, \quad \gamma_{2,3} = \frac{1}{2} \left[-dx_c \pm \sqrt{d^2x_c^2 + \frac{2(d-1)}{c_2}} \right]. \quad (52)$$

This implies that the dS solution is unstable for a phantom fluid, since γ_1 is positive for $w < -1$. When $c_2 < 0$ the real part of both γ_2 and γ_3 is negative. Therefore the dS expanding solution is stable for $w > -1$ and $c_2 < 0$, as is illustrated in figure 1. When c_2 is positive we have $\gamma_1 > 0$ and $\gamma_2 < 0$, implying that the dS solution is a saddle point. In summary, there is a dS solution for $c_1 < 0$ and $d > 3$ which is a stable attractor for $c_2 < 0$ and $w > -1$. Note that parameters with this sign choice do not avoid either ghost-free graviton scattering amplitudes or naked singularity structures [64].

3.4. Cosmological constant ($w = -1$)

In this subsection we shall discuss the situation in which a cosmological constant Λ is present instead of a barotropic fluid. This corresponds to dropping equation (46) and replacing ρ for Λ . Then we find that the dS solution satisfies

$$\Lambda = d(d-3)c_1x_c^4 + \frac{d(d-1)x_c^2}{2}, \quad y_c = 0. \quad (53)$$

There exists one solution for this equation when $(d-3)c_1$ is positive, whereas two solutions exist for $(d-3)c_1 < 0$ as long as the cosmological constant satisfies $0 < \Lambda <$

$d(d-1)^2/[16(3-d)c_1]$. In the latter case each solution $x = x_c$ belongs to the range

$$0 < x_c < x_M \equiv \sqrt{\frac{(d-1)}{4(3-d)c_1}}, \quad \text{and} \quad x_M < x_c < \sqrt{\frac{(d-1)}{2(3-d)c_1}}. \quad (54)$$

One can compute the eigenvalues of the 2×2 matrix for perturbations δx and δy about the dS fixed points given by equation (53). They are

$$\gamma_{1,2} = \frac{1}{2} \left(-dx_c \pm \sqrt{d^2 x_c^2 + 4m_{21}} \right), \quad m_{21} = -\frac{1}{2c_2} [4(d-3)c_1 x_c^2 + (d-1)]. \quad (55)$$

If $(d-3)c_1$ is positive, we find $m_{21} < 0$ for $c_2 > 0$, which means that both γ_1 and γ_2 are negative. Then this case corresponds to a stable attractor. Meanwhile the dS solution is unstable for $(d-3)c_1 > 0$ and $c_2 < 0$, since one has $\gamma_1 > 0$ and $\gamma_2 < 0$.

When $(d-3)c_1 < 0$ there exist two dS fixed points given by equation (54). For the first critical point in equation (54), one has $m_{21} < 0$ for $c_2 > 0$ and $m_{21} > 0$ otherwise. Therefore the dS solution is stable for $c_2 > 0$ and a saddle point otherwise. Similarly we find that the second critical point in equation (54) is stable for $c_2 < 0$ and a saddle point otherwise.

From the above argument it is clear that the stability of the dS solutions crucially depends upon the signs of c_1 and c_2 .

4. Linear dilaton case

From now on we shall study the cosmological evolution in the presence of a dynamically varying scalar field. For the dilatonic action in the string frame, the condition (41) for the existence of the dS solution yields

$$\lambda d(d-3)c_1 x^4 - 4d\lambda c_1 v x^3 + d(d-1)x^2 - 2dvx + v^2 - 3a_4 \lambda v^4 = 0. \quad (56)$$

This suggests that dS solutions may exist when $v = \dot{\phi}$ is a non-zero constant.

Using $\dot{v} = 0$ in equation (36), one gets

$$-\lambda d(d+1)c_1 x^4 - 4d\lambda a_4 v^3 x - d(d+1)x^2 + 2dvx - v^2 + 3a_4 \lambda v^4 = 0. \quad (57)$$

From the above two equations we find that the dS solution satisfies

$$2\lambda c_1 x^3 + 2\lambda(c_1 v x^2 + a_4 v^3) + x = 0. \quad (58)$$

We note that the Minkowski solution ($x = 0$ and $v = 0$) also satisfies equations (56) and (57).

When $d = 3$ equations (56) and (58) give

$$x^2 = \frac{-v^2(9\lambda a_4 v^2 + 1)}{6(1 + 2\lambda c_1 v^2)}. \quad (59)$$

For the bosonic correction with a GB term one has $\lambda = 1/4$, $a_4 = -1$ and $c_1 = 2$. Then by combining equations (58) and (59), we obtain the following dS solution with a linearly changing dilaton:

$$x_c = 0.62, \quad v_c = 1.40. \quad (60)$$

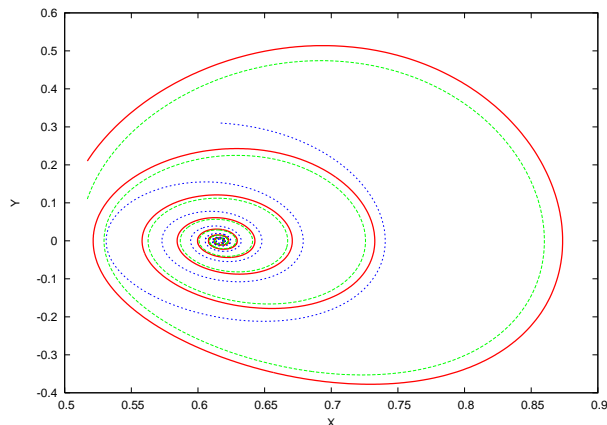


Figure 2. The phase portrait (plot of y versus x) of the system in the linear dilaton case with $d = 3$, $c_1 = 2$, $\lambda = 1/4$, $a_4 = -1$ and $c_2 = -1$. Trajectories starting anywhere in the phase space converge at $(x_c, y_c, z_c) = (0.62, 0, 0)$ for a constant value of $v = 1.40$. The fixed point corresponds to a stable de Sitter solution.

This agrees with the result in Ref. [45]. It is clear from equation (59) that the dS solution does not exist for $a_4 = 0$ for the GB coupling ($c_1 = 2$). If we allow for a negative value of c_1 , it is possible to have a dS solution even when $a_4 = 0$. Note that these results do not depend on c_2 .

The issue of the stability of such solutions is presented in more details in the Appendix (including the case of $c_2 = 0$). A rather counter-intuitive result is the following. Although the dS solution is insensitive to the actual value of c_2 , its stability depends upon the sign of c_2 . More precisely, the solution is stable against linear perturbations for $a_4 < 0$, $c_1 > 0$, $c_2 \geq 0$, whereas for negative c_2 one of the eigenvalues has a positive real part and the solution is unstable. We have checked other situations for $d = 3$, which can be summarized as follows (we considered only real solutions with $x_c > 0$):

- $c_1 > 0$, $a_4 \geq 0$: There is no real solution.
- $c_1 > 0$, $a_4 < 0$: Real solutions ($a_4 < -(9\lambda v^2)^{-1}$) are stable only for $c_2 \geq 0$.
- $c_1 < 0$, $a_4 > 0$: Real solutions ($c_1 < -(2\lambda v^2)^{-1}$) are stable only for $c_2 \geq 0$.
- $c_1 < 0$, $a_4 \leq 0$: Real solutions are stable only for $c_2 \leq 0$. See figure 2.

We should stress that the dS solutions we found appear in the presence of the ξ -dependent terms. In the low-energy regime where the effect of higher curvature terms is neglected, we obtain a logarithmic evolution of the dilaton corresponding to the pre/post-big-bang solution in pre-big-bang cosmology [52, 53]. Later we shall briefly discuss the case of a logarithmic dilaton in the string frame.

4.1. Impact of the barotropic fluid

In the case of vanishing coupling Q , the above stability analysis holds as long as we assume a standard equation of state characterized by $w > -1$. In the presence of a phantom fluid (diverging z_c) the eigenvalue $\gamma_2 = -d(1+w)x_c$ is always positive for $x_c > 0$ and all solutions are unstable (see the Appendix). If $Q \neq 0$ the dS solutions are stable when

$$Q \leq d(1+w)\frac{x_c}{v_c}, \quad (61)$$

where we assumed $v_c > 0$. Therefore the phantom fluid with positive Q is always unstable.

4.2. Cosmological constant ($w = -1$)

When $w = -1$ and $z = \Lambda = \text{const}$, the left-hand side of equation (56) has an extra term $-2\Lambda e^u$. This term vanishes or diverges depending on the sign of u . In order to find a non-trivial dS solution with cosmological constant, one has to assume that the dilaton field u approaches 0 asymptotically, $v \rightarrow 0$, $u \rightarrow 0$ as $t \rightarrow \infty$. In this regime, equation (57) gives

$$x_c^2 = \frac{\Lambda}{d} = -\frac{1}{\lambda c_1}, \quad (62)$$

which is positive for $c_1 < 0$. The value of the cosmological constant is fixed by the choice of the coefficients. We ran our numerical code for various values of $c_1 < 0$ and c_2 and checked that the dS solution in four dimensions is unstable against linear perturbations. Note that this behaviour does not coincide with what has been found for a fixed dilaton in section 3.4. This is because the dynamical equations are affected by the presence of the scalar field even before reaching the asymptotical regime.

A quick inspection of the dynamical equations shows that in the Einstein frame there are no dS solutions, except under the same assumption adopted for the cosmological constant case ($u \rightarrow 0$ asymptotically).

5. Logarithmic scalar field and the late time cosmological evolution

In the first part of this section we shall consider the cosmological dynamics in the presence of the compactification modulus under the assumption that the dilaton is fixed. This means that the analysis is identical both in the Einstein and string frame. In section 5.5 the logarithmic dilaton in the string frame will be studied as well. In order to keep our discussion as general as possible, we shall not fix the dimension d of the target space except at the end, where the four-dimensional case $d = 3$ will be inspected among the others. The equations of motion for the modulus action corresponding to Eqs. (24a)-(24d) read

$$\dot{x} = y, \quad (63)$$

$$\dot{y} = \frac{2z + 3v^2/2 - d(d-1)x^2 - 8dc_1\dot{\xi}x^3 - 2d(d-3)c_1\xi x^4 - 2dc_2\xi y(2\Xi x + 2dx^2 - y)}{4dc_2\xi x}, \quad (64)$$

$$\dot{v} = \frac{2d}{3}\xi'[(d+1)c_1x^4 + 4c_1x^2y + c_2y^2] - dxv - \frac{2Q}{3}z, \quad (65)$$

$$\dot{z} = [-dx(1+w) + Qv]z. \quad (66)$$

While only derivatives of ξ appear in the equations of motion for $d = 3$ and $c_2 = 0$ (GB case), there are non-vanishing contributions of ξ itself for general coefficients c_i . When $c_2 = 0$, the equations of motion for x and v read, from equations (40) and (36),

$$\dot{x} = \frac{\dot{z} + 3v\dot{v}/2 - dc_1x^3[4\ddot{\xi} + (d-3)\dot{\xi}x]}{dx[(d-1) + 12c_1\dot{\xi}x + 4(d-3)c_1\xi x^2]}, \quad (67)$$

$$\frac{\dot{v}}{v^2} = \frac{2d}{3}c_1\xi' \frac{x^2}{v^2}[(d+1)x^2 + 4\dot{x}] - d\frac{x}{v} - \frac{2Q}{3}\frac{z}{v^2}, \quad (68)$$

while the Friedmann equation is

$$d(d-1)\frac{x^2}{v^2} - \frac{3}{2} - 2\frac{z}{v^2} + 2dc_1\xi \frac{x^3}{v^2}[4\Xi + (d-3)x] = 0. \quad (69)$$

When finding solutions one can use the equations for $c_2 \neq 0$ and then set $c_2 = 0$ at the end, although equations (67)–(69) are required for numerical purposes. In addition c_1 can be set equal to 1 and absorbed in the definition of ξ_0 , so that the coefficient c_2 is the only free parameter of the higher-order Lagrangian.

Following the approach of [23, 29], we search for future asymptotic solution of the form

$$x \sim \omega_1 t^\beta, \quad y \sim \beta \omega_1 t^{\beta-1}, \quad (70a)$$

$$u \sim u_0 + \omega_2 \ln t, \quad v \sim \frac{\omega_2}{t}, \quad (70b)$$

$$\xi \sim \frac{1}{2}\xi_0 e^{u_0 t^{\omega_2}}, \quad (70c)$$

$$z \sim z_0 t^{Q\omega_2} \exp\left[-\frac{d(1+w)\omega_1}{\beta+1} t^{\beta+1}\right], \quad \beta \neq -1, \quad (70d)$$

$$z \sim z_0 t^\alpha, \quad \beta = -1, \quad (70e)$$

where the barotropic index w is constant and

$$\alpha \equiv Q\omega_2 - d(1+w)\omega_1. \quad (71)$$

We define

$$\tilde{\delta} \equiv \frac{1}{2}c_1\xi_0 e^{u_0}, \quad (72)$$

so that in any claim involving the sign of $\tilde{\delta}$ (δ) a positive c_1 coefficient is understood. In order to find a solution in the limit $t \rightarrow +\infty$, one has to match the exponents of t to get algebraic equations in the parameters β, ω_i, c_2 and $\tilde{\delta}$.

We shall consider the following four cases:

- (i) A low-curvature regime in which ξ terms are subdominant at late times.

- (ii) An intermediate regime where some terms in the equations of motion, either coupled to ξ or not, are damped.
- (iii) A high-curvature regime in which ξ terms dominate.
- (iv) A solution of the form (70a)–(70e) for the *full* equations of motion.⁺

Our results will be in agreement with [23] (and [29], taking into account our extra $\sqrt{2/3}$ factor in the normalization of v) for $d = 3$ and $z = 0$.

In order to keep contact with observations, we shall compare the obtained values of ω_1 of the geometrical ($z = 0$) modulus solutions with the estimate of the deceleration parameter from supernovae data [6], which does not depend on the Einstein equations governing the background evolution [65] (while the relation between the deceleration parameter and the dark energy barotropic index does). Solutions in agreement with observations might be used to describe the recent evolution of the universe. We shall not consider the logarithmic dilaton solution since it does not belong to the runaway dilaton models of [17, 18] and may be in conflict with gravity experiments, although we have not checked this explicitly.

As we will see later, one has $\beta = -1$ for all except one case. Then ω_1 is given in terms of the deceleration parameter $q \equiv \epsilon - 1$:

$$\omega_1 = \frac{1}{1 + q}. \quad (73)$$

The first thing to note is that equation (70a) can describe only cosmologies with $q > -1$, that is, the effective phantom regime is avoided.

At late times (redshift $z \sim 0$), the observational data give the constraint $-1.0 \lesssim q_0 \lesssim -0.5$ [6], where $q_0 = q(z)|_{z \sim 0}$. This results in the condition for ω_1

$$\omega_1^{(0)} \gtrsim 2, \quad (74)$$

if the obtained solution describes the present universe.

Of course this constraint is loosened if we use the 3σ observational bounds, but we need to exclude the region $q < -1$ for the above argument. For $-1.3 \lesssim q_0 \lesssim -0.2$ (after exclusion of the extremal negative region), one has $\omega_1^{(0)} \gtrsim 1.2$.

In order to allow a super-accelerating phase ($q < -1$), we can shift the origin of time so that $t \rightarrow t - t_s$. For positive increasing time $t < t_s$, the solution

$$x = \frac{\omega_1}{t - t_s}, \quad (75)$$

⁺ Low- and high-curvature regimes may be viewed even as “weak coupling” and “strong coupling” models, respectively. These correspond to (late-time) stages at which higher-order terms become either subdominant or dominant because of the relative magnitude of the theory’s coefficients, evaluated at some moment of the cosmological evolution. Quotation marks are in order since actually the curvature is not related to the modulus expectation value u ; however, in the Friedmann and modulus equations the higher-order term $\xi\rho_c$ dominates both when $\xi \gg 1$ and $\rho_c \gg 1$, and the two descriptions are equivalent (with $a_4 = 0$). The varying coupling picture may be confusing in our context, since curvature terms may dominate because of their asymptotical time dependence, for natural values of their couplings, while the weak/strong coupling approximation is assumed to hold in a time interval Δt where the dynamical variables have the above time dependence.

is expanding when $\omega_1 < 0$. The constraints on the parameters can be found by considering the previous asymptotic behaviour for $t \rightarrow 0$, that is near the singularity at t_s . Then the previously discarded 3σ region $-1.3 \lesssim q_0 \lesssim -1$ corresponds to

$$\omega_1^{(0)} \lesssim -57.8. \quad (76)$$

In order to satisfy this bound, however, the parameters of the solutions will require rather unnatural adjustments.

We shall not discuss the stability of these solutions since they will hold at times $t \sim t_s$, that is, only near the sudden future singularity.

5.1. Structure of the solutions and behaviour of the perfect fluid

Before proceeding, let us explain the general structure of the solutions. Each of the equations of motion, under the substitution (70a)–(70e), is of the form

$$\sum_i \left(g_i^+ t^{\alpha_i^+} + g_i^- t^{-\alpha_i^-} \right) = 0, \quad (77)$$

where $g_i^\pm = g_i^\pm(\beta, \omega_1, \omega_2, c_1, c_2, \xi_0, \dots)$ are the functions of the parameters of the theory, and $\alpha_i^\pm = \alpha_i^\pm(\beta, \omega_1, \omega_2, Q, w)$ are non-negative exponents. In the limit $t \rightarrow \infty$, an asymptotic solution with

$$\alpha_i^+ = 0, \quad \alpha_i^- > 0, \quad (78)$$

is $\sum_i g_i^+ = 0$. If the g_i^+ coefficients do not depend on ξ_0 , then the solution represents a low-curvature regime. If $\partial g_j^+ / \partial \xi_0 \neq 0$ for some j , then we will say that the solution describes an intermediate regime, when some (but not all the) higher-order terms survive. If $\partial g_i^+ / \partial \xi_0 \neq 0$ for all i , only the higher-order terms contribute and the solution is in a high-curvature regime.

In the limit $t \rightarrow 0$, the solution (78) becomes $\sum_i g_i^- = 0$, with eventually (in fact, only in the intermediate regime) some subdominant $g_i^- t^{-\alpha_i^-}$ terms dying away. Therefore all the above cases are interpreted in the complementary way, and low (high) curvature solutions become high (low) curvature.

The solution satisfying $\alpha_i^+ = \alpha_i^- = 0$ is valid at all times and the only distinction between solutions with or without future singularity is the sign of ω_1 .

As regards the contribution of the barotropic fluid, when $\beta = -1$ we can parametrize its time dependence in equation (77) as

$$g_z(t) = z_0 t^{\alpha - \alpha_*}, \quad (79)$$

where α is defined in equation (71) and α_* is a constant determined by the kind of solution (low/high curvature, exact, etc.) one is searching for.

We can adopt two perspectives. In the first one, we do not fine-tune the fluid parameters and impose that $g_z(t)$ vanishes asymptotically (purely geometrical solution). In this case, $z \rightarrow 0$ asymptotically in the equations of motion and $\alpha_* - \alpha \in \{\alpha_i^-\}$. From equation (78) it follows that $\alpha < \alpha_*$. In the second one, the expansion is driven by the

perfect fluid and $g_z = \text{const}$, that is, $\alpha - \alpha_* \in \{\alpha_i^+\}$, and $\alpha = \alpha_*$. We summarize these constraints in the damping/matching condition

$$\alpha \leq \alpha_*. \quad (80)$$

When the inequality holds, the fluid decays away in the limit $t \rightarrow +\infty$ and is consistently discarded.

When equation (80) is an equality, the fluid contributes to the asymptotical equations together with other geometrical terms and affects the dynamics of the system. However, this does not mean that the energy density $z(t)$ increases in time, since α can be negative. In this case the barotropic fluid can be regarded as a model for dark energy with

$$\epsilon = \frac{1}{\omega_1} = \frac{d(1 + w_{\text{eff}})}{2}, \quad (81)$$

$$w_{\text{eff}} \equiv \frac{2w - Q\omega_2}{2 + Q\omega_2}. \quad (82)$$

Then constraints on q would result in bounds for w_{eff} . In the asymptotic limit $t \rightarrow +\infty$, solutions exist (and do not burst out) only for the effective barotropic index $w_{\text{eff}} > -1$. However, in this section we shall constrain only the geometrical solutions with a scalar field ϕ , postponing a more complete discussion on dark energy to the next section.

To keep the notation clear, we shall write explicitly the time dependence of fluid terms and discuss separately the damping and matching cases. In the last section we shall compare numerically evolved dark energy scenarios to the analytical solutions below. In some cases the fluid will be dominating or decaying and either the matching or the damping condition will hold at the infinite future, with fixed constant parameters ω_1 and ω_2 . At intermediate times, where avoidance or incoming of a big rip become manifest, there is enough freedom in varying w and Q to span a wide range of dark energy phenomena, since $dg_z/dt \neq 0$ in general.

5.2. Low-curvature and intermediate regimes

The solution in a low-curvature regime corresponds to the one where ξ -dependent terms are neglected. The asymptotic future solution we consider is ($d \neq 1$)

$$\beta = -1, \quad \omega_2 < 2, \quad (83)$$

together with the constraints

$$\omega_1 = \frac{1}{d} - \frac{2Qz_0t^{\alpha+2}}{3d\omega_2}, \quad (84)$$

$$3\omega_2^2 = 2d(d-1)\omega_1^2 - 4z_0t^{\alpha+2}, \quad (85)$$

$$\alpha \leq -2, \quad (86)$$

so that the energy density $z(t)$ decreases in time. ξ terms decay away regardless of the value of c_2 . According to equation (86), if $\omega_2 > 0$ and the barotropic fluid is phantom-like ($w < -1$), then Q should be negative and of high enough magnitude in an expanding

universe ($\omega_1 > 0$).^{*} The reader can check other situations. In the damping case $\alpha < -2$, the solution is $\omega_1 = 1/d$ and $\omega_2 = \pm\sqrt{2(d-1)/(3d)}$. This case (for $d = 3$) was labelled A_∞ in Ref. [23].

In the marginal case $\alpha = -2$, the scale factor expands for $Q/\omega_2 < 3/(2z_0)$. This requires either an ordinary fluid with $Q\omega_2 > -2$ or a phantom fluid with $Q\omega_2 < -2$. When $Q = 0$ one has $w = 1$. In addition equation (85) states that $\omega_1^2 > 2z_0/[d(d-1)]$. We have numerically checked that this solution exists for suitable values of Q .

Another solution, valid in an intermediate regime and only for the $c_2 = 0$ and $\omega_1 > 0$ case, is

$$\beta = -2, \quad \omega_2 = 5, \quad Q \leq -2/5, \quad \omega_1^3 = \frac{1}{16d\tilde{\delta}} (15 - 2Qz_0t^{5Q+2}), \quad (87)$$

for a non-vanishing fluid. We require the condition $\tilde{\delta} > 0$ in order to have an expanding scale factor $a(t) \sim a_0 \exp(-\omega_1/t)$. Actually this goes from 0 to a_0 , that is, it reaches Minkowski spacetime asymptotically. If the fluid decays, then one recovers the C_∞ solution of Ref. [23] with $d = 3$ and $z = 0$. Interestingly, the coupling Q cannot vanish when $z \neq 0$.

For both the low-curvature and intermediate regimes, there exist no solutions corresponding to (70a)–(70e) in the cosmological constant case ($w = -1$), since the Λ/v^2 term diverges asymptotically.

We performed numerical simulations for $c_2 = 0$, $z = 0$, and $\omega_2 > 0$ with general initial conditions determined by the Hamiltonian constraint. With either positive or negative values of $\tilde{\delta}$, we found that in the asymptotic future the solutions tend to approach the general-relativistic (GR) solution (83) rather than the intermediate solution (87). One example is shown in figure 3, corresponding to the case without barotropic fluid and $d = 3$. The asymptotic solution is actually described by constant values of ω_1 and ω_2 characterized by equations (84) and (85), i.e., $\omega_1 = 1/3$ and $\omega_2 = 2/3$. The $\delta < 0$ case is consistent with the result in Ref. [23], where non-singular cosmological solutions were constructed by joining the future GR solution (83) with the past intermediate solution (87). In fact we numerically found that the asymptotic past solutions are well described by equation (87) for negative δ . One can either choose a positive or negative sign of $\dot{\phi}$ in order to obtain non-singular cosmological solutions. When $\delta > 0$, we found that the past solutions are not described by equation (87), which is associated with the fact that singularity-free solutions do not exist in this case [23].

It is easy to check the stability of low-curvature solutions without fluid analytically. Using equations (67)–(69) for $z = 0$, we obtain, after dropping ξ -dependent terms,

$$\dot{x} \approx -dx^2, \quad (88)$$

which gives

$$\dot{\omega}_1 = \frac{\omega_1(1 - d\omega_1)}{t}. \quad (89)$$

^{*} Constraints on the sign of Q will depend on the sign of ω_2 . Assuming $v < 0$ would simply flip $\text{sgn}(Q)$ in these considerations.

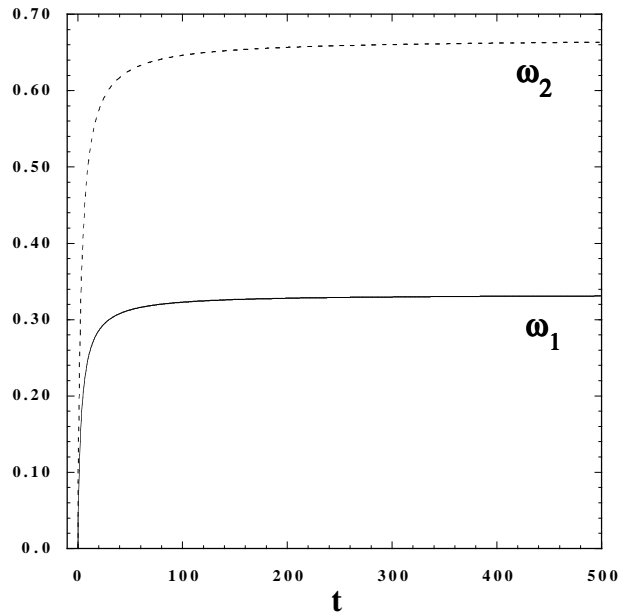


Figure 3. Evolution of ω_1 and ω_2 for $c_2 = 0$, $d = 3$, $z = 0$, and $\xi_0 = 2$ without a barotropic fluid. We find that the system approaches the low-curvature solution (83) characterized by constant ω_1 and ω_2 ($\omega_1 = 1/3$ and $\omega_2 = 2/3$).

Considering perturbations $\delta\omega_1(t)$ around the critical point $\omega_1^{(0)} = 1/d$, we obtain

$$\delta\omega_1 \sim \frac{1}{t^{2d\omega_1^{(0)}-1}} = \frac{1}{t}. \quad (90)$$

Since v and x are proportional to each other, the perturbations in ω_2 decay as $\delta\omega_1$, thereby showing the stability of the GR solution. In the presence of a fluid with $Q = 0$, it can be shown that the solution is stable again, provided that $w > -1$.

The solution with a sudden future singularity at t_s corresponds to

$$\beta = -1, \quad \omega_2 > 2; \quad (91)$$

and an eventual cosmological constant term damps away. The parameter constraints are equations (84) and (85). In the marginal case

$$\alpha = -2, \quad (92)$$

the scale factor expands for $Q > 3\omega_2/(2z_0) > 3/z_0$. This requires a phantom fluid with $w + 1 = (2 + Q\omega_2)/(d\omega_1) < 0$. We have numerically checked that this solution exists for suitable values of Q .

5.3. High curvature regime

When higher-order terms become important we can consider the asymptotic future solution

$$\beta = -1, \quad \omega_2 > 2, \quad \alpha \leq \omega_2 - 4, \quad (93)$$

together with constraint equations

$$\tilde{\delta}d\omega_1^2[(d+1)\omega_1^2 - 4\omega_1 + c_2] - Qz_0t^{\alpha-\omega_2+4} = 0, \quad (94)$$

$$d\tilde{\delta}\omega_1^2[(3-d)\omega_1^2 - 4\omega_2\omega_1 + c_2(2d\omega_1 + 2\omega_2 - 3)] + z_0t^{\alpha-\omega_2+4} = 0. \quad (95)$$

In the GB case ($c_2 = 0$) the solution corresponding to a decaying fluid ($\alpha < \omega_2 - 4$) is

$$\omega_1 = \frac{4}{d+1}, \quad \omega_2 = \frac{3-d}{d+1}, \quad (96)$$

which contradicts the condition (93) in any dimension. Therefore in the GB scenario with $\omega_2 \neq 4$ only the marginal case $\alpha = \omega_2 - 4$ is allowed.

When

$$\omega_2 = 4, \quad (97)$$

the cosmological constant term ($w = -1$) survives; an additional fluid term would be trivial since $\alpha = 0$. The constraints of this solution are

$$(d+1)\omega_1^2 - 4\omega_1 + c_2 = 0, \quad (98)$$

$$d\tilde{\delta}\omega_1^2[(3-d)\omega_1^2 - 16\omega_1 + c_2(2d\omega_1 + 5)] + \Lambda = 0. \quad (99)$$

In the GB case ($c_2 = 0$) we find

$$\omega_1 = \frac{4}{d+1}, \quad (100)$$

$$\Lambda = \tilde{\delta}d(5d+1) \left(\frac{4}{d+1} \right)^4. \quad (101)$$

The presence of a non-vanishing cosmological constant is mandatory for the consistency of this solution. For example one has $\Lambda = 48\tilde{\delta}$ for $d = 3$. It is remarkable that equation (97) is the only situation where a non-zero cosmological constant is not only allowed but even required in the GB case.

In order to check whether the solutions really approach the one given by equation (97), we run our numerical code for $d = 3$, $c_2 = 0$, and $\tilde{\delta} > 0$ with $\Lambda = 48\tilde{\delta}$. As illustrated in figure 4, the future asymptotic solutions are described by equations (97) and (100) with constant ω_1 ($=1$) and ω_2 ($=4$). We chose several different initial conditions and found that the solution (97) is a stable attractor. We also considered the case of a negative cosmological constant, but the solution (97) is found to be unstable.

When $c_2 \neq 0$, not all values and signs of the coefficient will give real expanding solutions. Since

$$\omega_1^\pm = \frac{2 \pm \sqrt{4 - (d+1)c_2}}{d+1}, \quad (102)$$

the reality condition reads $c_2 \leq 4/(d+1)$. If $c_2 < 0$, only the ω_1^+ solution is expanding, whereas both solutions are for $0 < c_2 \leq 4/(d+1)$. Again, the sign of the cosmological constant will depend upon δ .

Numerically, the solution of this system for $d = 3$ approaches the analytical value of ω_1^+ (and the associated ω_2) for positive Λ . The ω_1^- branch is found to be unstable

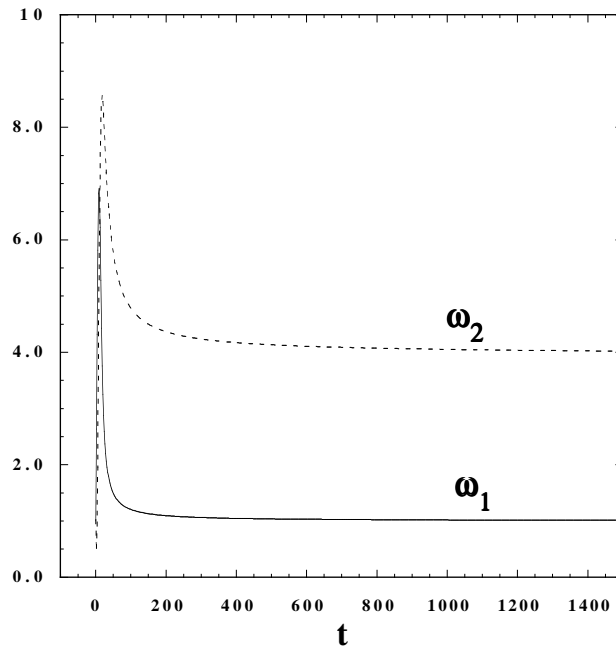


Figure 4. Variation of ω_1 and ω_2 for $c_2 = 0$, $d = 3$, and $\xi_0 = 2$ in the presence of a cosmological constant given by $\Lambda = 48\tilde{\delta}$. The asymptotic behavior of H and $\dot{\phi}$ is characterized by $\beta = -1$, $\omega_1 = 1$ and $\omega_2 = 4$ as estimated analytically.

regardless of the sign of the cosmological constant. Therefore the only stable solution is ω_1^+ with $\Lambda > 0$.

There are non-trivial positive solutions even with $\Lambda = 0$, corresponding to the critical case

$$\omega_1 = \frac{2}{d+1}, \quad c_2 = \frac{4}{d+1}. \quad (103)$$

We chose several different initial conditions and found no stable attractor. Since we did not check it for an extensive set of initial conditions, we cannot claim anything definite regarding the stability of this solution.

The sudden future solution corresponds to

$$\beta = -1, \quad \omega_2 < 2, \quad (104)$$

which describes a high-curvature regime where the ξ -dependent terms are dominant (a cosmological constant term is suppressed). The constraints on the parameters are equations (94), (95) and

$$\alpha \geq \omega_2 - 4, \quad (105)$$

where $z \rightarrow 0$ when the last equation is an inequality. In this case the solution (96) is contracting for $c_2 = 0$. Conversely, for $z = 0$ and $c_2 \neq 0$ the only expanding solution is ω_1^- given in equation (102) for $c_2 < 0$, together with

$$\omega_2 = 3 - d\omega_1^-. \quad (106)$$

5.4. Exact solution

An exact solution which is valid at all times is

$$\beta = -1, \quad \omega_2 = 2, \quad (107)$$

together with the constraints on ω_1 :

$$2\tilde{\delta}d\omega_1^2[(d+1)\omega_1^2 - 4\omega_1 + c_2] - 6d\omega_1 + 6 - Qz_0t^{\alpha+2} = 0, \quad (108)$$

$$2\tilde{\delta}d\omega_1^2[(d-3)\omega_1^2 + 8\omega_1 - c_2(2d\omega_1 + 1)] + d(d-1)\omega_1^2 - 6 - 2z_0t^{\alpha+2} = 0, \quad (109)$$

and equation (86). The cosmological constant term is forbidden in this case.

Let us consider the case with a vanishing fluid ($z \rightarrow 0$). After some manipulations, the above equations can be written as

$$4\tilde{\delta}\omega_1^2[(d-1)\omega_1 + 2 - dc_2] + (d-1)\omega_1 - 6 = 0, \quad (110)$$

$$2\tilde{\delta}\omega_1[d(d+1)\omega_1^2 - 2(d+1)\omega_1 + 4 - dc_2] - (5d+1) = 0. \quad (111)$$

When $c_2 = 0$, these equations completely fix the parameters of the theory. In four dimensions, for instance, the only real root of the above equations is

$$(\omega_1, \tilde{\delta}) = (0.21, 13.53), \quad (112)$$

in agreement with Ref. [23] (solution B_∞). Note that the original coupling δ is not fine-tuned, although $\tilde{\delta}$ is, since we have a freedom in choosing u_0 . There is no solution with a sudden future singularity.

Numerically we found that the exact solution (107) is not stable. As we already pointed out in section 5.2, the future stable attractor for $\Lambda = 0$ corresponds to the GR solution given by equation (83) for both positive and negative δ . The asymptotic past solution is described by equation (87), with $\delta < 0$.

In the case $c_2 \neq 0$, from equation (110) one can show that a solution of the system is given by ($d \neq 1$)

$$\omega_1 = \frac{6}{d-1}, \quad c_2 = \frac{8}{d}, \quad \tilde{\delta} = \frac{(d-1)^3}{48(d+2)}. \quad (113)$$

We checked numerically that this solution is unstable.

Other (real) solutions, with either $\omega_1 > 0$ or $\omega_1 < 0$, can be found by fixing the parameters known *a priori* from the theory, that is, c_2 and $\tilde{\delta}$.

5.5. Logarithmic dilaton

With little effort, we can check whether equations (70a)–(70e) are a solution of the dilatonic model in string frame for $t \rightarrow +\infty$. It is easy to find that the only possible case is $\beta = -1$, corresponding to a low-curvature regime where ξ -dependent terms are damped away. Then the model reduces to a scalar-tensor theory in Einstein gravity.

In the presence of a barotropic fluid, the matching/damping condition reads $\alpha \leq -2 - \omega_2$, i.e.,

$$(1+Q)\omega_2 - d(1+w)\omega_1 + 2 \leq 0, \quad (114)$$

while the constraints on the parameters read, with the above matching condition,

$$d\omega_1^2 - d\omega_1 + \omega_2 + (1+Q)e^{u_0}z_0 t^{\alpha+\omega_2+2} = 0, \quad (115)$$

$$d(d-1)\omega_1^2 - 2d\omega_1\omega_2 + \omega_2^2 - 2e^{u_0}z_0 t^{\alpha+\omega_2+2} = 0. \quad (116)$$

When $\alpha = -2 - \omega_2$, the solutions are

$$\omega_1^\pm = \pm \sqrt{\frac{1}{2d} \left[1 - 2(1+Q)e^{u_0}z_0 + \sqrt{1 + 4(1-Q)e^{u_0}z_0} \right]}, \quad (117)$$

$$\omega_2^\pm = \frac{1}{2} \left\{ -1 - \sqrt{1 + 4(1-Q)e^{u_0}z_0} \pm \sqrt{2d \left[1 - 2(1+Q)e^{u_0}z_0 + \sqrt{1 + 4(1-Q)e^{u_0}z_0} \right]} \right\}, \quad (118)$$

$$\tilde{\omega}_1^\pm = \pm \sqrt{\frac{1}{2d} \left[1 - 2(1+Q)e^{u_0}z_0 - \sqrt{1 + 4(1-Q)e^{u_0}z_0} \right]}, \quad (119)$$

$$\tilde{\omega}_2^\pm = \frac{1}{2} \left\{ -1 + \sqrt{1 + 4(1-Q)e^{u_0}z_0} \pm \sqrt{2d \left[1 - 2(1+Q)e^{u_0}z_0 - \sqrt{1 + 4(1-Q)e^{u_0}z_0} \right]} \right\}. \quad (120)$$

Without fluid ($Q = 0 = z_0$), the expanding solution reads

$$\omega_1^+ = \frac{1}{\sqrt{d}}, \quad (121)$$

$$\omega_2^+ = \sqrt{d} - 1, \quad (122)$$

in agreement with Ref. [52]. One has $\omega_1^+ \approx 0.58$ and $\omega_2^+ \approx 0.73$ for $d = 3$. This solution is found to be stable numerically.

In the presence of a cosmological constant [$z_0 t^\alpha = \Lambda$ and $Q = 0$ in equations (115) and (116)], we get $\omega_1 = -2/(1+d)$ and $\omega_2 = -2$, which describes a contracting universe.

The solution with a sudden future singularity at t_s and $\beta = -1$ corresponds to a high-curvature regime where ξ -dependent terms are dominant. In this case the constraints read

$$3\lambda a_4 \omega_2^4 - \lambda d \omega_1^2 [(d-3)c_1 \omega_1^2 - 4c_1 \omega_1 \omega_2 + c_2(2\omega_2 - 2d\omega_1 + 3)] + 2e^{u_0} z_0 t^{\alpha+\omega_2+4} = 0, \quad (123)$$

$$2\lambda a_4 \omega_2^3 (d\omega_1 - \omega_2 - 3) + \frac{\lambda}{2} \{ d \omega_1^2 [(d+1)c_1 \omega_1^2 - 4c_1 \omega_1 + c_2] + a_4 \omega_2^4 \} + Q e^{u_0} z_0 t^{\alpha+\omega_2+4} = 0. \quad (124)$$

When $\Lambda \neq z \neq 0$, the damping/matching condition for the fluid is

$$(1+Q)\omega_2 - d(1+w)\omega_1 + 4 \geq 0. \quad (125)$$

When the fluid vanishes or decays, we verified that there exist non-trivial solutions for suitable choices of parameters, both without and with a cosmological constant ($z = \Lambda$ and $\omega_2 = -4$ in the above equations). For instance, in four dimensions and in the GB case ($d = 3$, $c_1 = 2$, $c_2 = 0$, $a_4 = -1$), the only expanding solution is $\omega_1 = -2.87$ and $\Lambda = -188.88$. One can find other solutions by varying the parameters c_1, c_2, a_4, Λ .

5.6. Constraints from the current universe

We now compare the observational constraints on ω_1 for the recent evolution of the universe with the modulus solutions found in the previous section ($d = 3$). We consider the situation in which a perfect fluid is vanishing asymptotically. The results are summarized in Table 1 with the 68% CL bound. We discarded the intermediate regime or Minkowski solution.

Solutions $t \rightarrow \infty$	$\omega_1^{(0)}$ (68% CL)	Stability
Low curvature		Yes
High curvature, $c_2 = 0$		$\Lambda > 0$
High curvature, $c_2 \neq 0, \Lambda = 0$		
High curvature, $c_2 \neq 0, \Lambda \neq 0$	$\omega_1^+, c_2 \lesssim -8$	$\omega_1^+, \Lambda > 0$
Exact, $c_2 = 0$		
Exact, $c_2 \neq 0$, equation (113)	Yes	

Table 1. Constraints on modulus solutions in the asymptotic future for the Hubble parameter $H = \omega_1 t^{-1}$, vanishing fluid, and $d = 3$. Blank entries are excluded by experiments or numerical analysis.

We find that the logarithmic modulus solution with ghost-free GB parametrization and no extra fluid does not provide a viable cosmological evolution in the current late-time universe. However, we will see in the next section that the GB case in the presence of a cosmological fluid may display interesting features as regards the future evolution of the universe. At the 99% confidence level, the low redshift constraint on c_2 for the high curvature, $\Lambda \neq 0$ solution can be relaxed up to $c_2 \lesssim -1$. Therefore we have shown that there are models which can in principle explain the present acceleration without using the dark energy fluid.

The situation becomes more complicated in the presence of a barotropic fluid. The low-curvature solution can describe the very recent universe if Q is negative and non-vanishing. The other cases crucially depend upon the interplay between all the theoretical parameters.

As regards the models with a sudden future singularity, the bound equation (76) generally requires severe fine tunings for the parameters of the solutions. For instance one needs the condition $Qz_0 \gtrsim 10^3$ for the low-curvature solution with $\omega_2 = 4$ and $d = 3$. In the high-curvature case (104) with a vanishing fluid, we require that $|c_2| \gtrsim 10^4$. In the exact case ($c_2 \neq 0$), $|\tilde{\delta}| \lesssim 10^{-6}$. In the dilaton case with a cosmological constant, $|\Lambda| \gtrsim 10^7 - 10^9$.

However, we found a remarkable theoretical result. This kind of models with $c_2 = 0$ (Gauss–Bonnet case) always requires a non-vanishing contribution from the barotropic fluid for consistency. In the low-curvature case, even a positive coupling Q is necessary. In the dilaton case, a non-vanishing cosmological constant must be present if $z = 0$.

6. Gauss–Bonnet modulus gravity and the dark energy universe

In the previous section we compared cosmological solutions obtained in second-order gravity with the present evolution of the universe. In this section we shall adopt another perspective, namely, we will investigate the phenomenology of the full equations of motion as models of future evolution of the dark energy universe. At the infinite future, the properties of such models will agree with some of the above solutions.

In the typical phantom scenario ($w < -1$), the energy density of the fluid continues to grow and the Hubble rate eventually exhibits a divergence at finite time (big rip).

It is expected that the effect of higher-order curvature corrections becomes important when the energy density grows up to the Planck scale. In fact, it was shown in [13, 66, 67] that inclusion of quantum curvature corrections coming from conformal anomaly can moderate the future singularities.

Here we would like to consider the effect of $O(\alpha')$ string quantum corrections when the curvature of the universe increases in the presence of a phantom fluid. We shall concentrate on the modulus case with ξ given by equations (25) and (26). Our main interest is the cosmological evolution in four dimensions ($d = 3$) in the presence of a GB term ($c_1 = 2, c_2 = 0$) with $a_4 = 0$. We assume that the dilaton is stabilized, so that there are no long-range forces to take into account except gravity.

From the discussion in section 5, we found that the growth of the barotropic fluid is weaker than that of the Hubble rate when equation (86) is satisfied as an inequality. This condition is not achieved for a phantom fluid when the coupling Q between the fluid and the field ϕ is absent ($Q = 0$).

We ran our numerical code by varying initial conditions of H , ϕ and ρ in the full equations of motion. When $\delta < 0$, we numerically found that the solutions approach a big rip singularity for $Q = 0$ and $w < -1$ (see figure 5). Meanwhile the condition (86) can be satisfied for negative Q provided that ω_2 is positive. In figure 5 we show the evolution of H and ρ for $Q = -5$ and $w = -1.1$. In this case ρ decreases faster than t^{-2} , which means that the energy density of the fluid eventually becomes negligible relative to that of the modulus. Therefore the universe approaches the low-curvature solution given by equation (83) at late times, thereby showing the avoidance of big rip singularity even for $w < -1$. By substituting the asymptotic values $\omega_1 = 1/3$ and $\omega_2 = 2/3$ in equation (86), the condition for decaying fluid reads $Q < 3(w - 1)/2 = -3.15$. We checked that the big rip singularity can be avoided in a wide range of the parameter space for negative Q . These results do not change even for smaller values of δ such as $|\delta| = O(1)$ (corresponding to $|\xi_0| = 0.1$).

When Q is positive, the condition (86) is not fulfilled for $\omega_2 > 0$. However our numerical calculations show that $\dot{\phi}$ becomes negative even if $\dot{\phi} > 0$ initially. We found that the system approaches the low-energy regime characterised by $\omega_1 = 1/3$ and $\omega_2 = -2/3$. Since $\omega_2 < 0$, the big rip singularity can be avoided even for positive Q . In fact we numerically checked that the Hubble rate continues to decrease as long as the condition (86) is satisfied in the asymptotic regime.

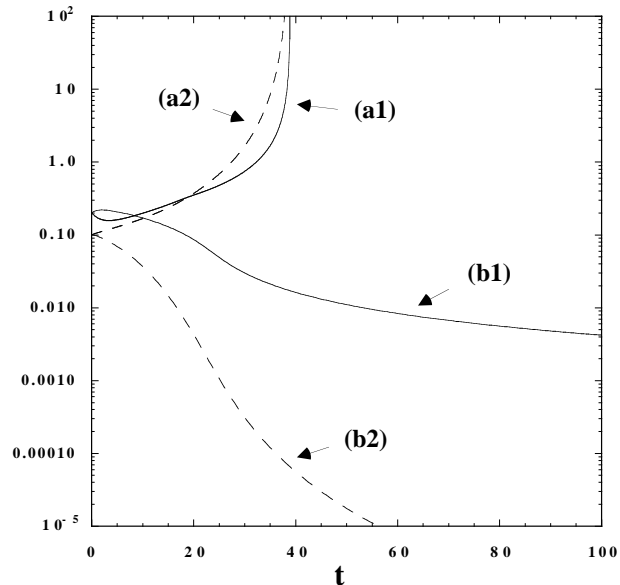


Figure 5. Evolution of H and ρ with $\xi_0 = -2$, $w = -1.1$ for (a) $Q = 0$ and (b) $Q = -5$. We choose initial conditions as $H_i = 0.2$, $\phi_i = 2.0$ and $\rho_i = 0.1$. The curves (a1) and (b1) represent the evolution of H for $Q = 0$ and $Q = -5$, respectively, while the curves (a2) and (b2) show the evolution of ρ for corresponding Q . We find that the big rip singularity is avoided when the negative coupling Q is present.

When $\delta > 0$, there is another interesting situation in which the Hubble rate decreases in spite of the increase of the energy density of the fluid. This corresponds to the solution in the high-curvature regime in which the growing energy density ρ can balance with the GB term ($\rho \approx 24H^3\xi$ in the Friedmann equation). One example is illustrated in figure 6; the big rip does not appear even when $w < -1$ and $Q = 0$. Thus the GB corrections coupled with a scalar field ϕ provides us several interesting possibilities to get rid of this singularity.

By examining the condition (80), one may think that the big rip singularity may appear even for $w > -1$ in the presence of the coupling Q . However we found that this is not the case. We show one example in figure 7, corresponding to $w = -0.8$ and $Q = +5$. It turns out that $\dot{\phi}$ becomes negative even when $\dot{\phi} > 0$ initially, as seen in the figure. We also checked that this property holds even for negative Q . This suggests that the system tends to evolve so that the condition

$$\alpha < 0 \quad \text{when} \quad Q \neq 0, \quad (126)$$

is satisfied. Then the presence of the coupling Q does not lead to the big rip singularity for a non-phantom fluid ($w > -1$).

7. Conclusions

In this paper we have studied cosmological solutions in the presence of second-order curvature corrections and their application to the dark energy universe. Our starting

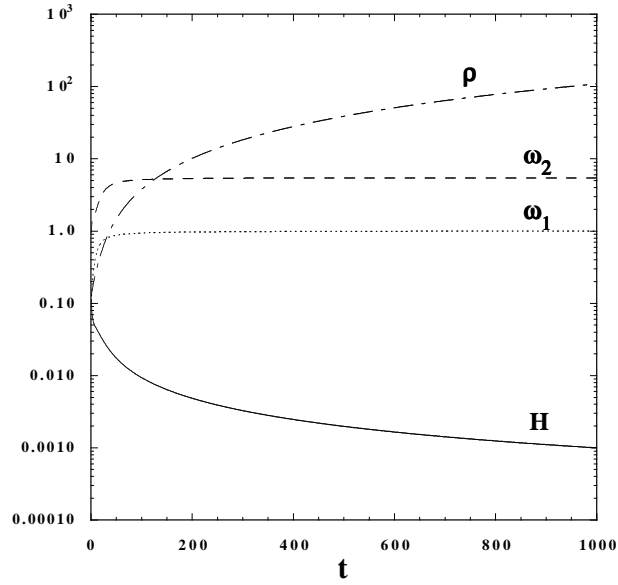


Figure 6. Variation of H , ρ , ω_1 and ω_2 with $\xi_0 = +2$, $w = -1.5$ and $Q = 0$. Initial conditions are chosen to be $H_i = 0.2$, $\phi_i = 2.0$ and $\rho_i = 0.1$. The Hubble rate continues to decrease while the energy density ρ increases.

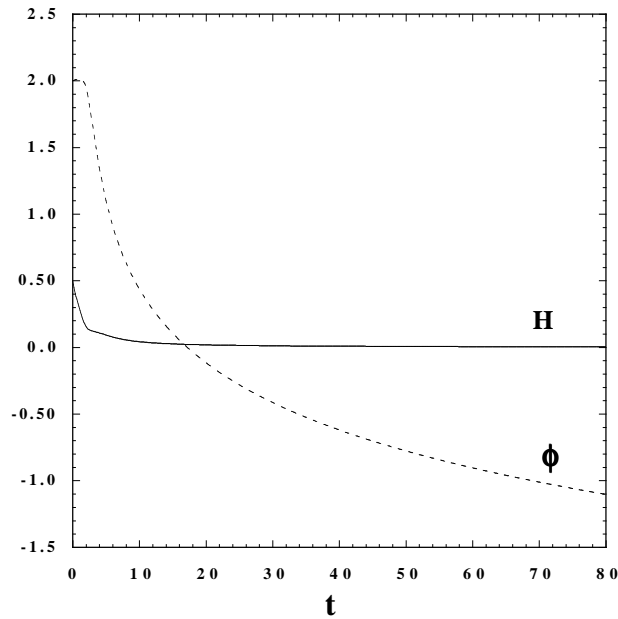


Figure 7. Evolution of H and ϕ with $\xi_0 = -2$, $w = -0.8$ and $Q = +5$. Initial conditions are chosen to be $H_i = 0.5$, $\phi_i = 2.0$ and $\rho_i = 0.3$. We find that both H and ϕ continue to decrease even in the presence of positive Q .

action (1) is based upon the low-energy effective string theory with a barotropic perfect fluid coupled to a scalar field ϕ . This is regarded as either the dilaton field characterising the strength of the string coupling, or a modulus field corresponding to the compactification radius of a dimensionally reduced theory. We adopted a general form of the second-order correction terms, given by equation (2), which includes the case of the Gauss–Bonnet curvature invariant. We have clarified the property of cosmological solutions in three classes of models: (i) fixed scalar field (section 3), (ii) dilaton field (sections 4 and 5), and (iii) modulus field (section 5).

When the field ϕ is fixed (model (i)), the existence of pure geometrical de Sitter solutions requires a spatial dimension $d \neq 3$, and that the parametrization of the second-order curvature correction is different from the GB one. Therefore these solutions are not realistic when applied to inflation or dark energy. Nevertheless we studied the classical stability of such solutions in the presence of a perfect fluid or a cosmological constant for completeness.

If the dilaton field is not fixed (model (ii)), it is possible to construct pure dS solutions in the string frame provided that the evolution of the dilaton is linear in time ($\dot{\phi} = \text{const}$). In the case of $d = 3$ spatial dimensions we obtained the dS solution given by Eq. (60) for the GB case, which was found to be stable. We also clarified the stability of other cases different from the GB parametrization. Moreover we showed that the existence of a phantom fluid generally makes the dS solution unstable.

When a modulus field is present with fixed dilaton (model (iii)), there exist cosmological solutions in which the field exhibits a logarithmic evolution, see equations (70a)–(70e). We showed the existence of low-curvature solutions (83) and Minkowski solutions (87), which can be joined each other if the coupling constant δ given in equation (24d) is negative. In addition we obtained an exact solution (107) for the modulus system, but this is found to be unstable. In the asymptotic future the solutions tend to approach the low-curvature one given by equation (83) rather than the others, irrespective of the sign of the modulus-to-curvature coupling δ . We also constructed high-curvature asymptotic solutions (93) in the presence of a growing perfect fluid or a cosmological constant. Finally, we placed constraints on the viability of modulus-driven solutions using the current observational data. It is interesting that the GB parametrisation is excluded in any of the above mentioned regimes when a barotropic fluid is vanishing; see table 1.

In section 6 we solved the full equations of motion to simulate a dark energy universe in the future. The numerical results were then compared to the solutions obtained in previous sections. We used the modulus coupling with the GB curvature correction in 3 spatial dimensions. In the presence of the coupling Q between the field ϕ and the phantom fluid ($w < -1$), it is possible to consider situations in which the energy density of the fluid decays in the future. In fact we have numerically found that the big rip singularity can be avoided for the coupling Q which satisfies the condition $Q\omega_2 - 3(1 + w)\omega_1 < -2$ asymptotically (see figure 5). This is actually achieved irrespective of the sign of Q and the asymptotic solutions are described by the low-

curvature one given by equation (83). When δ is positive, we also found that even for $Q = 0$ the Hubble parameter can decrease in spite of the growth of the energy density of the phantom fluid. Therefore the big rip can be avoided in this case as well, as illustrated in figure 6. When the equation of state of the fluid is non-phantom ($w > -1$), we found that the presence of the coupling Q *cannot* give rise to the big rip singularity, contrary to what one might expect when considering the dark energy continuity equation (see figure 7).

Another topic which is worth studying further is the behaviour of the asymptotic solutions at high redshift. For instance, when the deceleration parameter is parametrized as $q(z) = q_0 + q_1 z$,[‡] supernovae observations give constraints for both q_0 (redshift $z \sim 0$) and q_1 ($z \sim 0.5$). Here we limited the discussion to the low-redshift evolution, since the experimental uncertainties on high-redshift data, together with the correlated data analyses, are more delicate to deal with. This topic is still under investigation.

In the presence of a well-motivated underlying theory, one should specify the physical scale at which high-curvature terms with a non-GB parametrization arise and make sense, either at early times (high-energy inflation) or late times (dark energy around a big rip). This energy scale is related to the string coupling governing the perturbative higher-order corrections to the Einstein–Hilbert action. Although our main interest for the dark energy model of section 6 was only the Gauss–Bonnet case, we are still missing the precise theoretical setup in which a non-GB parametrization arise at high energies. This problem is still open and potentially rich of consequences.

Also, in order to get close to real-world cosmology, we implemented a matter contribution into the Lagrangian and interpreted it as a dark energy source. Although the dilatonic/modulus corrections are motivated by string theory (in particular with the GB parametrization), the origin of the matter term still lacks in theoretical background. A completely coherent picture of dark energy should clarify its origin from first principles. Nonetheless our results do not rely on a particular choice for the kind of fluid, except for the assumption of the continuity equation (14) and $w = \text{const}$. Field/string theory should also determine the coupling Q between the scalar field and matter. In our examples of big rip avoidance, the coupling Q is of order unity and has “natural” values anyway (in the sense specified by Dirac).

We hope that this kind of phenomenology would be able to originate from a more precise formulation of the underlying physics.

Acknowledgments

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[‡] A change in the parametrization can affect the bounds on q at large redshift; see [68] and references therein.

Appendix: Stability of dS solutions with varying dilaton

The equations of motion (33)–(37), linearized under a linear perturbation of the vector $X \equiv (x, y, u, v, z)^t$, read $\delta\dot{X} = M\delta X$, where the 5×5 matrix elements m_{ij} are

$$m_{11} = m_{13} = m_{14} = m_{15} = 0, \quad m_{12} = 1, \quad (\text{A.1})$$

$$m_{21} = -\frac{d-1}{2\lambda c_2} + \frac{4c_1 v x}{c_2} - dy - \frac{3(d-3)c_1 x^2}{2c_2} - \frac{ze^u}{d\lambda c_2 x^2} - \frac{3a_4 v^4}{2dc_2 x^2} - \frac{y^2}{2x^2} + \frac{v^2}{2d\lambda c_2 x^2},$$

$$m_{22} = v - dx + \frac{y}{x}, \quad m_{23} = zm_{25} = \frac{ze^u}{d\lambda c_2 x}, \quad (\text{A.2})$$

$$m_{24} = \frac{1}{\lambda c_2} + \frac{6a_4 v^3}{dc_2 x} + \frac{2c_1 x^2}{c_2} - \frac{v}{d\lambda c_2 x} + y, \quad (\text{A.3})$$

$$m_{31} = m_{32} = m_{33} = m_{35} = 0, \quad m_{34} = 1, \quad (\text{A.4})$$

$$m_{41} = \frac{d(d+1)x - dv + 2d(d+1)\lambda c_1 x^3 + 4d\lambda c_1 xy + 2d\lambda a_4 v^3}{1 - 6\lambda a_4 v^2}, \quad (\text{A.5})$$

$$m_{42} = \frac{d(1 + 2\lambda c_1 x^2 + \lambda c_2 y)}{1 - 6\lambda a_4 v^2}, \quad m_{43} = zm_{45} = \frac{Qze^u}{1 - 6\lambda a_4 v^2}, \quad (\text{A.6})$$

$$m_{44} = \frac{1}{1 - 6\lambda a_4 v^2} \left\{ v - dx - 6\lambda a_4 v^3 + 6d\lambda a_4 v^2 x - \frac{6\lambda a_4 v}{1 - 6\lambda a_4 v^2} [2dxv - 2dy - v^2 - d(d+1)\lambda c_1 x^4 - d(d+1)x^2 - (4c_1 x^2 + c_2 y)d\lambda y - 4d\lambda a_4 v^3 x + 3\lambda a_4 v^4 - Qze^u] \right\}, \quad (\text{A.7})$$

$$m_{51} = -d(1+w)z, \quad m_{52} = m_{53} = 0, \quad (\text{A.8})$$

$$m_{54} = Qz, \quad m_{55} = -d(1+w)x + Qv, \quad (\text{A.9})$$

where we have dropped the subscript c for the background quantities. For the dS solution with $Q = 0$, two eigenvalues of the matrix M are given by $\gamma_1 = 0$ and $\gamma_2 = m_{55}$. The other three can be extracted from

$$\gamma^3 - (m_{22} + m_{44})\gamma^2 + (m_{22}m_{44} - m_{24}m_{42} - m_{21})\gamma + (m_{21}m_{44} - m_{24}m_{41}) = 0. \quad (\text{A.10})$$

In the four-dimensional case ($d = 3$), by inserting $\lambda = 1/4$, $a_4 = -1$, $c_1 = 2$, and the values found in equation (60), we find that the real part of the eigenvalues reads

$$\text{Re}(\gamma_3) = 1.96, \quad \text{Re}(\gamma_4) = -2.40, \quad \text{Re}(\gamma_5) = -0.45, \quad (\text{A.11})$$

for $c_2 = 1$, while

$$\text{Re}(\gamma_3) = -0.45, \quad \text{Re}(\gamma_4) = \text{Re}(\gamma_5) = -0.22, \quad (\text{A.12})$$

for $c_2 = -1$. We have checked that the sign of the eigenvalues does not change for different absolute values of c_2 .

The GB case ($c_2 = 0$) must be treated separately. Together with the background equations (35)–(37), we consider the (differentiated) Friedmann equation

$$\dot{x} = \frac{B_2}{B_1}\dot{v} + (vz + \dot{z})\frac{e^u}{B_1}, \quad (\text{A.13})$$

where

$$B_1 \equiv d(d-1)x - dv + 2d(d-3)\lambda c_1 x^3 - 6d\lambda c_1 v x^2, \quad (\text{A.14})$$

$$B_2 \equiv dx - v + 6\lambda a_4 v^3 + 2d\lambda c_1 x^3. \quad (\text{A.15})$$

At the dS point the linear perturbations of the B_i coefficients are multiplied by vanishing factors. Defining the reduced vector $\tilde{X} \equiv (x, u, v, z)^t$, it can be shown that the eigenvalues of the 4×4 perturbation matrix \tilde{M} at the dS point are

$$\gamma_{1,3} = 0, \quad \gamma_3 = \tilde{m}_{55} = m_{55}, \quad (\text{A.16})$$

$$\gamma_4 = \tilde{m}_{11} + \tilde{m}_{44}, \quad (\text{A.17})$$

where

$$\tilde{m}_{11} = \frac{B_2}{B_1} \tilde{m}_{41} = \frac{B_2 m_{41}}{B_1 - B_2 m_{42}}, \quad (\text{A.18})$$

$$\tilde{m}_{14} = \frac{B_2}{B_1} \tilde{m}_{44} = \frac{B_2 m_{44}}{B_1 - B_2 m_{42}}. \quad (\text{A.19})$$

Note that $\tilde{m}_{11}\tilde{m}_{44} - \tilde{m}_{41}\tilde{m}_{14} = 0$. If $d = 3$ and $\lambda = 1/4$, the eigenvalue γ_4 is -0.45 for $a_4 = -1$ and $c_1 = 2$, and negative for other choices of parameters. Therefore the dS solution is always stable. Note that we may derive the same results by just multiplying the Jordan equation (A.10) times c_2 and then setting $c_2 = 0$. Since its coefficients are of the form $(\text{constant})_1 + (\text{constant})_2/c_2$, only the $(\text{constant})_2$ terms survive and the above equation becomes linear. We have checked this shortcut explicitly.

In the presence of a cosmological constant, $z = \Lambda = \text{const}$, there exist dS solutions only in the asymptotical limit $u, v \rightarrow 0$ as $t \rightarrow \infty$ as explained in the text. The only difference with respect to the previous expressions for $c_2 = 0$ is that they are evaluated at $v = 0$ and $\tilde{m}_{14} \rightarrow \tilde{m}_{14} + \Lambda/(B_1 - B_2 m_{42})$, $\tilde{m}_{44} \rightarrow \hat{m}_{44} = \tilde{m}_{44} + \Lambda m_{42}/(B_1 - B_2 m_{42})$. In this case the equation for the eigenvalue has a non-vanishing constant coefficient and the solutions are

$$\gamma_{3,4} = \frac{1}{2} \left[(\tilde{m}_{11} + \hat{m}_{44}) \pm \sqrt{(\tilde{m}_{11} + \hat{m}_{44})^2 + 4m_\Lambda} \right] \Big|_{v=0}, \quad (\text{A.20})$$

$$m_\Lambda \equiv \frac{\Lambda m_{41}}{B_1 - B_2 m_{42}}. \quad (\text{A.21})$$

For negative c_1 , one of the eigenvalues is always positive and the solution is unstable.

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