QUANTUM FLUCTUATIONS IN THE SCHWARZSCHILD SPACE-TIME

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T. Padmanabhan and J.V. Narlikar

Tata Institute of Fundamental Research
Theoretical Astrophysics Group
Bombay 400 005/INDIA

Recently the method of path integrals was used to analyse the effect of conformal fluctuations in a Friedman universe and in the case of a collapsing dust ball /1-3/. The results indicated that the fluctuations diverge near the classical singularity allowing nonzero probability for quantum transitions to nonsingular metrics. Further, if one considers the 'effective metric' which incorporates the quantum fluctuations, the corresponding space-time is non-singular.

We consider here the effect of conformal fluctuations in the Schwarzschild and Reissner-Nordstrom solutions. These are the classical static solutions and we find that the conformal fluctuations are severely damped near the event horizon of the classical solution. We also find that transitions between solutions of Schwarzschild or Reissner-Nordstrom type with different mass or charge values are forbidden, when only those degrees of freedom are treated as quantum fluctuations. This result, however, does not remain true when other fluctuations are also considered. In particular when coupled with 'conformal' fluctuations, these transitions between different M values can occur.

The results arise mathematically as follows: consider first the question of conformal fluctuations in the Schwarzschild metric. Here one considers a metric of the form,

$$ds^2 = \Omega^2(t)\left[\left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2(d\theta^2+\sin^2\theta\, d\phi^2)\right]. \quad (1)$$

The quantum mechanical probability amplitude for the metric to make a transition from $\Omega(t_1) = \Omega_1$ to $\Omega(t_2) = \Omega_2$ is formally given by the propagator

$$K\,[\Omega_2 t_2;\ \Omega_1 t_1\,] = \int \exp\frac{iS}{\hbar}\,\mathcal{D}\Omega(t) \quad (2)$$

where S is the action functional for the theory. There is a mathematical difficulty here because the conventional Einsteinian action (units $c = G = 1$)

$$S = \frac{1}{16\pi}\int_V R\sqrt{-g}\ d^4x \quad (3)$$

involves the second derivatives, while the path integral formalism requires only first derivatives. The difficulty can be resolved by adding appropriate surface terms /2/. Or, equivalently one can integrate by parts the second derivatives and omit the surface terms. We shall always assume this to have been done in what follows. Then it turns out that the functional in eq. (2) is of quadratic form and the expressions can be integrated to give, with $\hbar = 1$,

$$K[\Omega_2, t_2; \Omega_1, t_1]$$

$$= \left[\frac{f(R_1,R_2)}{i\pi(t_2-t_1)}\right]^{\frac{1}{2}} \exp\left[-\frac{if(R_1,R_2)(\Omega_2-\Omega_1)^2}{t_2-t_1}\right] \quad (4)$$

where

$$f(R_1, R_2) = \frac{1}{2}\left((R_2-2M)^3 - (R_1-2M)^3\right) + \frac{3M}{2}\left((R_2-2M)^2 - (R_1-2M)^2\right) + 6M^2(R_2-R_1) + 12M^3 \ln\frac{R_2-2M}{R_1-2M}. \quad (5)$$

Here $2M < R_1 < R_2$ and we confine our attention to the fluctuations in the region bounded between R_1 and R_2. By the definition of the propagator we have the equation,

$$\psi(\Omega_2,t_2) = \int K(\Omega_2\ t_2\ ;\Omega_1\ t_1)\ \psi\ (\Omega_1 t)\ d\Omega_1 \quad (6)$$

for the "wave function(al)" for the conformal factor Ω. If one assumes the initial state to be a Gaussian wave packet with a width $\Delta(t_1)$, then it is easy to see that the width increases to the value,

$$\Delta(t_2) = \Delta(t_1)\left[1 + \frac{(t_2-t_1)^2}{4\Delta_o^2\ f^2}\right]^{\frac{1}{2}},\quad \Delta_o = \Delta(t_1). \quad (7)$$

From eq (5) it is clear that $f \to \infty$ as $R_1 \to 2M$. In other words when the region of consideration is close to the classical event horizon then the fluctuations are severely attenuated. (In a way one may consider this limit as analogous to the limit $\hbar \to 0$ from quantum mechanics to classical mechanics.) This result is not unexpected since the event horizon is conformally invariant.

The Schwarzschild metric, is specified by a single dynamical information viz the mass. Hence it is tempting to make this a quantum variable and consider fluctuations in it. In other words, one takes the metric to be of the form,

$$ds^2 = \left(1 - \frac{2M(t)}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M(t)}{r}\right)} - r^2(d\theta^2 + \sin^2\theta\ d\phi^2) \quad (8)$$

and asks for the probability of a transition from M_1 to M_2. It turns out, however, that such transitions are forbidden. The scalar curvature reduces to the expression,

$$R = - \frac{d^2}{dt^2} \left(\frac{1}{1 - \frac{2M(t)}{r}} \right) \qquad (9)$$

which identically vanishes when the surface terms are used. It can be shown that this result is extendible to Reissner-Nordstrom metric. This shows that the classical static solution is stable against simplest type of quantum fluctuations.

This conclusion does not hold true when one considers more complicated forms of fluctuations. For example one can consider the combination of the two fluctuations - of $\Omega(t)$ and $M(t)$, and take the metric to be of the form,

$$ds^2 = \Omega^2(t) \left[\left(1 - \frac{2M(t)}{r}\right) dt^2 - \frac{ds^2}{\left(1 - \frac{2M(t)}{r}\right)} - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right]. \qquad (10)$$

In this case, it turns out that, the kernel is expressible as,

$$K[X_2, Y_2, t_2; X_1, Y_1, t_1] = F(t_2 - t_1) \exp \frac{iV}{10(t_2 - t_1)} \left((X_2 - X_1)^2 - (Y_2 - Y_1)^2 \right) \qquad (11)$$

where $F(\tau) = (10 \pi i\tau)^{-1}$; $\tau = t_2 - t_1$, and where X and Y are defined through the following relations,

$$\Omega = (X - Y)^{\frac{1}{5}} \qquad Q = (X + Y)(X - Y)^{\frac{3}{5}} \qquad (12)$$

with

$$Q(M) = \frac{3}{(R_2^3 - R_1^3)} \int_{R_1}^{R_2} \frac{r^2 \, dr}{\left(1 - \frac{2M}{r}\right)} \qquad (13)$$

If one considers an initial wave function which is Gaussian, then analysis similar to those of previous sections can be performed [see /4/ for details]. Especially one can show that the root mean square fluctuation in the mass is given by,

$$< M^2 > = M_o^2 + \left(\frac{4\pi c^2 R_1^2}{9GVM_o} \right)^2 (t_2 - t_1)^2 .$$

Thus even though direct transitions from vacuum to $M \neq 0$ are not allowed a conformal factor allows spontaneous mass production.

A similar analysis of conformal fluctuation was performed in the case of Bianchi models. Here it seems that (because of anisotropy) the fluctuations near the classical singularity do not diverge fast enough to quench the singularity.

References :

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