Secular instability in quasi-viscous disc accretion

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ABSTRACT

A first-order correction in the α-viscosity parameter of Shakura & Sunyaev has been introduced in the standard inviscid and thin accretion disc. A linearised time-dependent perturbative study of the stationary solutions of this "quasi-viscous" disc leads to the development of a secular instability on large spatial scales. This qualitative feature is equally manifest for two different types of perturbative treatment — a standing wave on subsonic scales, as well as a radially propagating wave. Stability of the flow is restored when viscosity disappears.

Subject headings: accretion, accretion disks — hydrodynamics — instabilities — methods: analytical

1. Introduction

The compressible, inviscid and thin disc flow has by now become an established model in accretion studies (Abramowicz & Zurek 1981; Fukue 1987; Chakrabarti 1989; Nakayama & Fukue 1989; Chakrabarti 1990; Kafatos & Yang 1994; Yang & Kafatos 1995; Pariev 1996; Molteni et al. 1996; Lu et al. 1997; Das 2002; Das et al. 2003; Ray 2003a; Barai et al. 2004; Das 2004; Abraham et al. 2006; Das et al. 2006; Chaudhury et al. 2006; Goswami et al. 2007). This is a particularly expedient and simple physical system to study, especially as regards the rotating flow in the innermost regions of the disc, in the vicinity of a black hole. Steady global solutions of inviscid axisymmetric accretion on to a black hole have been meticulously studied over the years, and at present there exists an extensive body of literature devoted to the subject, with especial emphasis on the transonic nature of solutions, the multitransonic character of the flow, formation of shocks, and the stability of global solutions under time-dependent linearised perturbations.

Having stressed the usefulness of the inviscid model among researchers in accretion astrophysics, it must also be recognised that this model has its own limitations. It is easy to understand that while the presence of angular momentum leads to the formation of an accretion disc in the first place, a physical mechanism must also be found for the outward transport of angular momentum, which should then make possible the inward drift of the accreting matter into the potential well of the accretor. Viscosity has been known all along to be a just such a physical means to effect infall, although the exact presciption for viscosity in an accretion disc is still a matter of much debate (Frank et al. 2002). What is well appreciated, however, is that the viscous prescription should be compatible with an enhanced outward transport of angular momentum. The very well-known α parametrisation of Shakura & Sunyaev (1973) is based on this principle.

And so it transpires that on global scales — especially on the very largest scales of the disc — the inviscid model will encounter difficulties in the face of the fact that without an effective outward transport of angular momentum, the accretion process cannot be sustained globally. To address this adverse issue, what is being introduced in this paper is the "quasi-viscous" disc model. This model involves prescribing a very small first-order viscous correction in the α-viscosity parameter of Shakura & Sunyaev (1973), about the zeroth-order inviscid solution. In doing this, a viscous generalisation of the inviscid flow can be logically extended to capture the important physical properties of accretion discs on large length scales, without compromising on the fundamentally simple and elegant features of the inviscid model. This is the single most appealing aspect of the quasi-viscous disc model vis-a-vis many other standard models of axisymmetric flows which involve viscosity (Shakura & Sunyaev 1973; Liang & Thomson 1980; Pringle 1981; Matsumoto et al. 1984; Muchotrzeb-Czerny 1986; Abramowicz et al. 1988; Narayan & Yi 1994; Chakrabarti 1996a,b,c; Chen et al. 1997; Peitz & Appl 1997; Frank et al. 2002; Afshordi & Paczyński 2003; Umurhan et al. 2006).

While many previous works have also taken up the question of the stability of viscous thin disc accretion (Lightman & Eardley 1974; Shakura & Sunyaev 1976; Livio & Shaviv 1977; Kato 1978; Umurhan & Shaviv 2005), the specific objective of the present paper is to study the stability of stationary quasi-viscous inflow solutions under the influence of a time-dependent and linearised radial perturbation. It has already been shown in some earlier works that inviscid solutions are stable under a perturbation of this kind (Ray 2003a; Chaudhury et al. 2006). What has been found through this particular work is that with the merest presence of viscosity (i.e. to a first order in α, which itself is much less than unity) about the stationary inviscid solutions, instabilities develop exponentially on large length scales. This has disturbing implications, because all physically meaningful inflow solutions will have to pass through these length scales, connecting the outer boundary of the flow with the surface of the accretor (or the event horizon, if the accretor is a black hole).

The perturbative treatment has been executed in two separate ways — as a standing wave and as a high-frequency travelling wave, and in both cases the perturbation displays growth behaviour. It need not always be true that standing and travelling waves will simultaneously exhibit the same qualitative properties as far as stability is concerned. Many instances in fluid dynamics bear this out. In the case of binary fluids, standing waves indicate instability as opposed to travelling waves (Cross & Hohenberg 1993; Bhattacharya & Bhattacharjee 2005), while the whole physical picture is quite the opposite for the fluid dynamical problem of the hydraulic jump (Bohr et al. 1993; Ray & Bhattacharjee 2004; Singha et al. 2005). Contrary to all this the axisymmetric stationary quasi-viscous flow is greatly disturbed both by a standing wave and by a travelling wave. This provides convincing evidence of its unstable character, and it is very much in consonance with similar conclusions drawn from some earlier studies. For high-frequency radial perturbations Chen & Taam (1993) have found that inertial-acoustic modes are locally unstable, with a greater degree of growth for the outward travelling modes than the inward ones. On the other hand, Kato et al. (1988) have revealed a growth in the amplitude of a non-propagating perturbation at the critical point, which, however, stabilises in the inviscid regime.

This kind of instability — one that manifests itself only if some dissipative mechanism (viscous dissipation in the case of the quasi-viscous rotational flow) is operative — is called *secular instability* (Chandrasekhar 1987). It should be very much instructive here to furnish a parallel instance of the destabilising influence of viscous dissipation in a system undergoing rotation : that of the effect of viscous dissipation in a Maclaurin spheroid (Chandrasekhar 1987). In studying ellipsoidal figures of equilibrium, Chandrasekhar (1987) has discussed that a secular instability develops in a Maclaurin spheroid, when the stresses derive from an ordinary viscosity which is defined in terms of a coefficient of kinematic viscosity (as the α parametrisation is for an accretion disc), and when the effects arising from viscous dissipation are considered as small perturbations on the inviscid flow, to be taken into account in the first order only. It is exactly in this spirit that the "quasi-viscous" approximation has been prescribed for the thin accretion disc, although, unlike a Maclaurin spheroid, an astrophysical accretion disc is an open system.

Curiously enough, the geometry of the fluid flow also seems to be having a bearing on the issue of stability. The same kind of study, as has been done here with viscosity in a rotational flow, had also been done earlier for a viscous spherically symmetric accreting system. In that treatment (Ray 2003b) viscosity was found to have a stabilising influence on the system, causing a viscosity-dependent decay in the amplitude of a linearised standing-wave perturbation. This is quite in keeping with the understanding that the respective roles of viscosity are at variance with each other in the two distinctly separate cases of spherically symmetric flows and thin disc flows. While viscosity contributes to the resistance against infall in the former case, in the latter it aids the infall process.

Finally, an important aspect of the time-dependent perturbative analysis that may be emphasised is that although the flow has been considered to be driven by the Newtonian potential, none of the physical conclusions of this work will be qualified in any serious way upon using any of the pseudo-Newtonian potentials (Paczyński & Wiita 1980; Nowak & Wagoner 1991; Artemova et al. 1996), which are regularly invoked in accretion-related literature to describe rotational flows on to a Schwarzschild black hole, even while preserving the Newtonian construct of space and time. This shall be especially true of the flow on large scales, where all pseudo-Newtonian potentials converge to the Newtonian limit, and, therefore, the conclusions of this perturbative treatment will also have a similar bearing on pseudo-Schwarzschild flows.

2. The quasi-viscous axisymmetric flow

For the thin disc, under the condition of hydrostatic equilibrium along the vertical direction (Matsumoto et al. 1984; Frank et al. 2002), two of the relevant flow variables are the drift velocity, v, and the surface density, Σ. In the thin-disc approximation the latter has been defined by vertically integrating the volume density, ρ, over the disc thickness, $H(r)$. This gives $\Sigma \cong \rho H$, and in terms of Σ, the continuity equation is set down as

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma v r\right) = 0. \tag{1}$$

For a flow driven by the Newtonian potential, $V(r) = -GMr^{-1}$, assumption of the hydrostatic equilibrium in the vertical direction will give the condition $H = r(c_{\rm s}/v_{\rm K})$, in which the local speed of sound, $c_{\rm s}$, and the local Keplerian velocity, $v_{\rm K}$, are, respectively, defined as $c_{\rm s}^2 = \gamma K \rho^{\gamma-1}$ and $v_{\rm K}^2 = GMr^{-1}$, with the constants γ and K deriving from the application of a polytropic equation of state, $P = K\rho^{\gamma}$, in terms of which, the speed of sound may given as $c_{\rm s}^2 = \partial P/\partial \rho$. Written explicitly, the disc height is, therefore, expressed as

$$H = \left(\frac{\gamma K}{GM}\right)^{1/2} \rho^{(\gamma-1)/2} r^{3/2}, \tag{2}$$

and with the use of this result, the continuity equation could then be recast as

$$\frac{\partial}{\partial t}\left[\rho^{(\gamma+1)/2}\right] + \frac{1}{r^{5/2}}\frac{\partial}{\partial r}\left[\rho^{(\gamma+1)/2} v r^{5/2}\right] = 0. \tag{3}$$

The condition for the balance of specific angular momentum in the flow is given by (Frank et al. 2002),

$$\frac{\partial}{\partial t}\left(\Sigma r^2 \Omega\right) + \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\Sigma v r\right) r^2 \Omega\right] = \frac{1}{2\pi r}\left(\frac{\partial \mathcal{G}}{\partial r}\right), \tag{4}$$

where Ω is the local angular velocity of the flow, while the torque is given as

$$\mathcal{G} = 2\pi r \nu \Sigma r^2 \left(\frac{\partial \Omega}{\partial r}\right), \tag{5}$$

with ν being the kinematic viscosity associated with the flow. With the use of the continuity equation, as equation (1) gives it, and going by the Shakura & Sunyaev (1973) prescription for the kinematic viscosity, $\nu = \alpha c_{\mathrm{s}} H$, it would be easy to reduce equation (4) to the form (Frank et al. 2002; Narayan & Yi 1994)

$$\frac{1}{v}\frac{\partial}{\partial t}\left(r^2\Omega\right) + \frac{\partial}{\partial r}\left(r^2\Omega\right) = \frac{1}{\rho v r H}\frac{\partial}{\partial r}\left[\frac{\alpha \rho H c_{\mathrm{s}}^2 r^3}{\Omega_{\mathrm{K}}}\left(\frac{\partial \Omega}{\partial r}\right)\right], \tag{6}$$

with Ω_{K} being defined from $v_{\mathrm{K}} = r\Omega_{\mathrm{K}}$.

Going back to equation (3), a new variable is defined as $f = \rho^{(\gamma+1)/2} v r^{5/2}$, whose steady value, as it is very easy to see from equation (3), can be closely identified with the constant matter flux rate. In terms of this new variable, equation (3) can be modified as

$$\frac{\partial}{\partial t}\left[\rho^{(\gamma+1)/2}\right] + \frac{1}{r^{5/2}}\frac{\partial f}{\partial r} = 0, \tag{7}$$

while equation (6) can be rendered as

$$\frac{1}{v}\frac{\partial}{\partial t}\left(r^2\Omega\right) + \frac{\partial}{\partial r}\left(r^2\Omega\right) = \alpha\left(\frac{\gamma K}{GM}\right)\frac{1}{f}\frac{\partial}{\partial r}\left[f\left(\frac{f^2\Omega_{\mathrm{K}}}{\rho^2 v^3}\right)\frac{\partial \Omega}{\partial r}\right]. \tag{8}$$

The inviscid disc model is given by the requirement that $r^2\Omega = \lambda$, in which λ is the constant specific angular momentum. The quasi-viscous disc that is being proposed here will introduce a first-order correction in terms involving α, the Shakura & Sunyaev (1973) viscosity parameter, about the constant angular momentum solution. Mathematically this will be represented by the prescription of an effective specific angular momentum,

$$\lambda_{\mathrm{eff}}(r) = r^2\Omega = \lambda + \alpha r^2\tilde{\Omega}, \tag{9}$$

with the form of $\tilde{\Omega}$ having to be determined from equation (8), under the stipulation that the dimensionless α-viscosity parameter is much smaller than unity. This smallness of the quasi-viscous correction induces only very small changes on the constant angular momentum background, and, therefore, neglecting all orders of α higher than the first, and ignoring any explicit time-variation of the viscous correction term, the latter being a standard method adopted also for Keplerian flows (Lightman & Eardley 1974; Shakura & Sunyaev 1976; Pringle 1981; Frank et al. 2002), the dependence of $\tilde{\Omega}$ on v and ρ is obtained as

$$\tilde{\Omega} = -\frac{2\lambda}{r^2}\left(\frac{\gamma K}{GM}\right)\left[\frac{f^2\Omega_{\mathrm{K}}}{\rho^2 v^3 r^3} + \int \frac{f^2\Omega_{\mathrm{K}}}{\rho^2 v^3 r^3}\left(\frac{1}{f}\frac{\partial f}{\partial r}\right)\mathrm{d}r\right]. \tag{10}$$

The effect of the absence and the presence of a small viscous correction to the inviscid background flow has been schematically shown in Figs. 1 & 2, respectively.

The stationary solution of equation (7) can be easily obtained as a first integral, and with the help of equation (2), it can be set down as

$$2\pi\left(\frac{\gamma K}{GM}\right)^{1/2}\rho^{(\gamma+1)/2} v r^{5/2} = -\dot{m}, \tag{11}$$

with $\dot{m}$ being the conserved matter inflow rate. The negative sign arises because for inflows, v goes with a negative sign. Further, under stationary conditions, equation (9) can be written in a modified form as

$$\lambda_{\mathrm{eff}}(r) = \lambda - 2\alpha\lambda\left(\frac{c_{\mathrm{s}}^2}{v v_{\mathrm{K}}}\right). \tag{12}$$

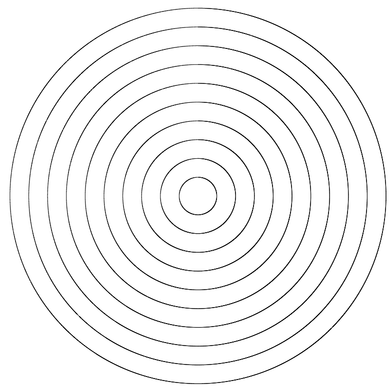

Fig. 1.— A schematic representation of the axisymmetric inviscid disc. Viewed from the top (along the vertical axis of the disc) the flow will be seen to form closed circular paths.

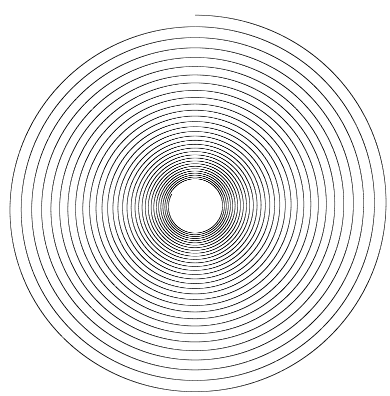

Fig. 2.— A schematic representation to trace the spiralling behaviour of the physical flow trajectories upon the introduction of a small viscous correction to the background inviscid flow.

From equation (11), with ρ approaching a constant asymptotic value on large length scales, the drift velocity, v, can be seen to go asymptotically as $r^{-5/2}$. Bearing in mind that for inflows, $v < 0$, the asymptotic dependence of the effective angular momentum can be shown to be

$$\lambda_{\rm eff}(r) \sim \lambda + 2\alpha\lambda \left(\frac{r}{r_{\rm s}}\right)^3 , \tag{13}$$

with $r_{\rm s}$ being a scale of length, which, to an order-of-magnitude, is given by $r_{\rm s}^3 \sim GM\dot{m}[c_{\rm s}^3(\infty)\rho(\infty)]^{-1}$. This asymptotic behaviour is entirely to be expected, because the physical role of viscosity is to transport angular momentum to large length scales of the accretion disc.

Lastly, the equation for radial momentum balance in the flow will also have to be modified under the condition of quasi-viscous dissipation. This has to be done according to the scheme outlined in equation (9) by which, the centrifugal term, $\lambda_{\rm eff}^2(r)/r^3$, of the radial momentum balance equation, will have to be corrected upto a first order in α. This will finally lead to the result

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} + V'(r) - \frac{\lambda^2}{r^3} - 2\alpha\frac{\lambda}{r^3}\left(r^2\tilde{\Omega}\right) = 0, \tag{14}$$

with $\tilde{\Omega}$ being given by equation (10), and P being expressed as a function of ρ with the help of a polytropic equation of state, as has been mentioned earlier. The steady solution of equation (14) is given as

$$v\frac{{\rm d}v}{{\rm d}r} + \frac{1}{\rho}\frac{{\rm d}P}{{\rm d}r} + V'(r) - \frac{\lambda^2}{r^3} + 4\alpha\frac{\lambda^2}{r^3}\left(\frac{c_{\rm s}^2}{vv_{\rm K}}\right) = 0, \tag{15}$$

whose first integral cannot be obtained analytically because of the α-dependent term. In the inviscid limit, though, the integral is easily obtained. This case will be governed by conserved conditions, and its solutions have been well-known in accretion literature (Chakrabarti 1989; Das 2002; Das et al. 2003). They will either be open solutions passing through saddle points or closed paths about centre-type points. The slightest presence of viscous dissipation, however, will radically alter the nature of solutions seen in the inviscid limit, and it may be easily understood that solutions forming closed paths about centre-type points, will, under conditions of small-viscous correction, be of the spiralling kind (Liang & Thomson 1980; Matsumoto et al. 1984; Afshordi & Paczyński 2003). This state of affairs is appreciated very easily by the analogy of the simple harmonic oscillator. In the undamped state the phase trajectories of the oscillator will, very much like the solutions of the inviscid flow, be either closed paths about centre-type points or open paths through saddle points. With the presence of even very weak damping the closed paths change into spiralling solutions.

After having understood the qualitative nature of the stationary flows in the quasi-viscous accretion disc, as given by equations (11), (12) and (15), it will now become possible to carry out a real-time linear stability analysis about the stationary solutions, with the help of equations (3) and (14).

3. An equation for a time-dependent perturbation on stationary solutions

About the stationary solutions of the flow variables, v and ρ, a time-dependent perturbation is introduced according to the scheme, $v(r,t) = v_0(r) + v'(r,t)$, $\rho(r,t) = \rho_0(r) + \rho'(r,t)$ and $f(r,t) = f_0(r) + f'(r,t)$, in all of which, the subscript "0" implies stationary values, with f_0 especially, as can be seen from equation (3), being a constant. This constant, as it is immediately evident from a look at equation (11), is very much connected to the matter flow rate, and, therefore, the perturbation f' is to be seen as a disturbance on the steady, constant background accretion rate. For spherically symmetric flows, this Eulerian perturbation scheme has been applied by Petterson et al. (1980) and Theuns & David (1992), while for inviscid axisymmetric flows, the same method has been used equally effectively by Ray (2003a,b) and Chaudhury et al. (2006).

The definition of f will lead to a linearised dependence among f', v' and ρ' as

$$\frac{f'}{f_0} = \left(\frac{\gamma+1}{2}\right)\frac{\rho'}{\rho_0} + \frac{v'}{v_0}, \tag{16}$$

while from equation (3), an exclusive dependence of ρ' on f' will be obtained as

$$\frac{\partial \rho'}{\partial t} + \beta^2 \frac{v_0 \rho_0}{f_0}\left(\frac{\partial f'}{\partial r}\right) = 0 \tag{17}$$

with $\beta^2 = 2(\gamma+1)^{-1}$. Combining equations (16) and (17) will render the velocity fluctuations as

$$\frac{\partial v'}{\partial t} = \frac{v_0}{f_0}\left(\frac{\partial f'}{\partial t} + v_0 \frac{\partial f'}{\partial r}\right) \tag{18}$$

which, upon a further partial differentiation with respect to time, will give

$$\frac{\partial^2 v'}{\partial t^2} = \frac{v_0}{f_0}\left[\frac{\partial^2 f'}{\partial t^2} + v_0 \frac{\partial}{\partial r}\left(\frac{\partial f'}{\partial t}\right)\right]. \tag{19}$$

From equation (14) the linearised fluctuating part could be extracted as

$$\frac{\partial v'}{\partial t} + \frac{\partial}{\partial r}\left(v_0 v' + c_{\mathrm{s0}}^2 \frac{\rho'}{\rho_0}\right) + 4\alpha\lambda^2 \frac{\sigma}{r^3}\left[2\frac{f'}{f_0} - 2\frac{\rho'}{\rho_0} - 3\frac{v'}{v_0} + \frac{1}{\sigma}\int \sigma \frac{\partial}{\partial r}\left(\frac{f'}{f_0}\right) \mathrm{d}r\right] = 0, \tag{20}$$

in which $\sigma = c_{\mathrm{s0}}^2/(v_0 v_{\mathrm{K}})$ and c_{s0} is the local speed of sound in the steady state. Differentiating equation (20) partially with respect to t, and making use of equations (17), (18) and (19) to substitute for all the first and second-order derivatives of v' and ρ', will deliver the result

$$\frac{\partial^2 f'}{\partial t^2} + 2\frac{\partial}{\partial r}\left(v_0 \frac{\partial f'}{\partial t}\right) + \frac{1}{v_0}\frac{\partial}{\partial r}\left[v_0\left(v_0^2 - \beta^2 c_{\mathrm{s0}}^2\right)\frac{\partial f'}{\partial r}\right] - 4\alpha\lambda^2 \frac{\sigma}{v_0 r^3}\left[\frac{\partial f'}{\partial t} + \left(\frac{3\gamma - 1}{\gamma+1}\right) v_0 \frac{\partial f'}{\partial r} - \frac{1}{\sigma}\int \sigma \frac{\partial}{\partial r}\left(\frac{\partial f'}{\partial t}\right) \mathrm{d}r\right] = 0 \tag{21}$$

entirely in terms of f'. This is the equation of motion for a perturbation imposed on the constant mass flux rate, f_0, and it shall be important to note here that the choice of a driving potential, Newtonian or pseudo-Newtonian, has no explicit bearing on the form of the equation.

4. Linear stability analysis of stationary solutions

With a linearised equation of motion for the perturbation having been derived, a solution of the form $f'(r,t) = g_\omega(r)\exp(-\mathrm{i}\omega t)$ is applied to it. From equation (21), this will give

$$\omega^2 g_\omega + 2\mathrm{i}\omega \frac{\mathrm{d}}{\mathrm{d}r}\left(v_0 g_\omega\right) - \frac{1}{v_0}\frac{\mathrm{d}}{\mathrm{d}r}\left[v_0\left(v_0^2 - \beta^2 c_{\mathrm{s0}}^2\right)\frac{\mathrm{d}g_\omega}{\mathrm{d}r}\right] + 4\alpha\lambda^2 \frac{\sigma}{v_0 r^3}\left[-\mathrm{i}\omega g_\omega + \left(\frac{3\gamma-1}{\gamma+1}\right) v_0 \frac{\mathrm{d}g_\omega}{\mathrm{d}r} + \frac{\mathrm{i}\omega}{\sigma}\int \sigma \left(\frac{\mathrm{d}g_\omega}{\mathrm{d}r}\right)\mathrm{d}r\right] = 0. \tag{22}$$

The perturbation may now be treated as a standing wave spatially confined between two suitably chosen boundaries, as well as a radially propagating high-frequency wave. These two distinct cases will be taken up separately in what follows, to see how stationary solutions are affected by the perturbation.

4.1. Standing waves

It can be easily appreciated that with viscous dissipation present in the accreting system, multiply-valued solutions in the phase portrait are a distinct possibility (Liang & Thomson 1980; Abramowicz & Kato 1989; Afshordi & Paczyński 2003). This, however, is physically not feasible for a fluid flow, and, therefore, for all solutions which are multiply-valued about a critical point, it should be necessary to have the inner branch of a solution fitted to its outer branch via a shock, which is a standard practice in general fluid dynamical studies (Bohr et al. 1993). The outer branch of this discontinuous solution will connect the shock with the outer boundary of the disc itself, and in the phase portrait of the flow, there will exist an entire family of these outer solutions which arguably shall be subsonic. It would be worthwhile at this point to remember that for this kind of a thin disc system, the local speed of acoustic propagation would go as βc_{s0}, and subsonic solutions would have their bulk flow velocity, v_0, at less than this value. Over the entire subsonic range, at two chosen points, the perturbation may be spatially constrained by requiring it to be a standing wave, which dies out at the two chosen boundaries. The outer point could suitably be chosen to be at the outer boundary of the flow itself, where, by virtue of the boundary condition on the steady flow, the perturbation would naturally decay out. The inner boundary of the standing wave is to be chosen infinitesimally close to the shock front, through which the flow will be discontinuous, and the perturbation will be made to die out in its neighbourhood. Between these two points, for solutions which are entirely subsonic, it should be necessary to multiply equation (22) by $v_0 g_\omega$, and then carry out an integration by parts. The requirement that all integrated "surface terms" vanish at the two boundaries of the standing wave, will give a quadratic dispersion relation in ω, which will read as

$$\mathcal{A}\omega^2 + \mathcal{B}\omega + \mathcal{C} = 0, \tag{23}$$

in which the coefficients of each successive term will be given by

$$\mathcal{A} = \int \left(v_0 g_\omega^2\right) \,\mathrm{d}r, \qquad \mathcal{B} = -4\mathrm{i}\alpha\lambda^2 \int \left[\frac{g_\omega}{r^3} \int g_\omega \left(\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right) \,\mathrm{d}r\right] \mathrm{d}r$$

and

$$\mathcal{C} = \int v_0 \left(v_0^2 - \beta^2 c_{\mathrm{s0}}^2\right) \left(\frac{\mathrm{d}g_\omega}{\mathrm{d}r}\right)^2 \mathrm{d}r + 2\alpha\lambda^2 (3\gamma - 1) \int \frac{\beta^2 \sigma g_\omega v_0}{r^3} \left(\frac{\mathrm{d}g_\omega}{\mathrm{d}r}\right) \mathrm{d}r.$$

It is possible now to find a solution for ω, but of greater immediate significance is the fact that this solution will yield an α-dependent real part of the temporal component of the perturbation, which will go as

$$\Re\left(-\mathrm{i}\omega\right) = 2\alpha \left[\int \left(v_0 g_\omega^2\right) \xi(r) \,\mathrm{d}r\right] \left[\int \left(v_0 g_\omega^2\right) \,\mathrm{d}r\right]^{-1} \sim \alpha\xi(r), \tag{24}$$

with $\xi(r)$ itself being expressed as

$$\xi(r) = \frac{\lambda^2}{v_0 g_\omega r^3} \int g_\omega \left(\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right) \mathrm{d}r \sim \frac{\lambda^2 c_{\mathrm{s0}}^2}{v_0^2 v_{\mathrm{K}} r^3} = \frac{\lambda^2}{\mathcal{M}^2 v_{\mathrm{K}} r^3}, \tag{25}$$

in which the Mach number, $\mathcal{M} = v_0/c_{\mathrm{s0}}$. It is globally true that $v_{\mathrm{K}} \sim r^{-1/2}$. In this situation, under the assumption that the Mach number has a power-law dependence on the radial distance, given as $\mathcal{M}^2 \sim r^{-\varepsilon}$, it is possible to write down,

$$\frac{\mathrm{d}(\ln \xi)}{\mathrm{d}(\ln r)} = \varepsilon - \frac{5}{2}. \tag{26}$$

On large length scales, with c_{s0} approaching a constant ambient value, and with $v_0 \sim r^{-5/2}$ from the continuity condition, it is easily deduced that $\xi(r) \sim r^{5/2}$. This indicates that on large length scales, the amplitude of the spatially constrained standing-wave perturbation grows in time, i.e. the subsonic solutions on which this perturbation has been imposed, display unstable behaviour. It is easy to see that when α vanishes, all stationary solutions restricted by the condition $v_0 < \beta c_{s0}$, are stable under a standing-wave perturbation, which displays a purely oscillatory behaviour with no growth in amplitude. For the inviscid disc (with $\alpha = 0$) this has been shown by Ray (2003a). Regarding this point, it will be instructive to mention again that Kato et al. (1988) have pointed to the existence of a non-propagating growing perturbation localised at the critical point, but interestingly enough they have also found that this instability disappears in inviscid transonic flows.

4.2. Radially propagating waves

The perturbation is now made to behave in the manner of a radially travelling wave, whose wavelength is suitably constrained to be small, i.e. it is to be smaller than any characteristic length scale in the system. A perturbative treatment of this nature has been carried out before on spherically symmetric flows (Petterson et al. 1980) and on axisymmetric flows (Ray 2003a; Chaudhury et al. 2006). In both these cases the radius of the accretor was chosen as the characteristic length scale in question, and the wavelength of the perturbation was required to be much smaller than this length scale. In this study of an axisymmetric flow driven by a Newtonian potential, the radius of the accreting star, $r_\star$, could be a choice for such a length scale. As a result, the frequency, ω, of the waves should be large.

An algebraic rearrangement of terms in equation (22) will lead to an integro-differential equation of the form

$$\mathcal{P}\frac{\mathrm{d}^2 g_\omega}{\mathrm{d}r^2} + \mathcal{Q}\frac{\mathrm{d}g_\omega}{\mathrm{d}r} - \mathcal{R}g_\omega + \mathcal{T}\int g_\omega \left(\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right)\,\mathrm{d}r = 0, \tag{27}$$

with its coefficients being given by

$$\mathcal{P} = v_0^2 - \beta^2 c_{s0}^2, \qquad \mathcal{Q} = 3v_0\frac{\mathrm{d}v_0}{\mathrm{d}r} - \frac{1}{v_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(v_0\beta^2 c_{s0}^2\right) - 2\mathrm{i}\omega v_0 - 2\alpha\lambda^2\left(3\gamma - 1\right)\frac{\beta^2\sigma}{r^3},$$

$$\mathcal{R} = 2\mathrm{i}\omega\frac{\mathrm{d}v_0}{\mathrm{d}r} + \omega^2 \qquad \text{and} \qquad \mathcal{T} = \frac{4\mathrm{i}\omega\alpha\lambda^2}{v_0 r^3}.$$

At this stage, bearing in mind the constraint that ω is large, the spatial part of the perturbation, $g_\omega(r)$, is prescribed as $g_\omega(r) = \exp(s)$, with the function s itself being represented as a power series of the form

$$s(r) = \sum_{n=-1}^{\infty}\frac{k_n(r)}{\omega^n}. \tag{28}$$

The integral term in equation (27), can, through some suitable algebraic substitutions, be recast as

$$\int g_\omega\left(\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right)\,\mathrm{d}r = \int \exp(s)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}s}\right)\,\mathrm{d}s = g_\omega(r)\mathcal{S},$$

with $\mathcal{S}$ itself being given by another power series as

$$\mathcal{S} = \sum_{m=1}^{\infty}\left(-1\right)^{m+1}\frac{\mathrm{d}^m\sigma}{\mathrm{d}s^m}. \tag{29}$$

Following this, all the terms in equation (27) can be expanded with the help of the power series for $g_\omega(r)$. Under the assumption (whose self-consistency will be justified soon) that to a leading order

$$\mathcal{S} \sim \frac{\mathrm{d}\sigma}{\mathrm{d}s} \simeq \frac{\mathrm{d}\sigma}{\mathrm{d}r}\left(\omega\frac{\mathrm{d}k_{-1}}{\mathrm{d}r}\right)^{-1},$$

the three successive highest-order terms (in a decreasing order) involving ω will be obtained as ω^2, ω and ω^0. The coefficients of each of these terms are to be collected first and then individually summed up. This is to be followed by setting each of these sums separately to zero, which will yield for ω^2, ω and ω^0, respectively, the conditions

$$\left(v_0^2 - \beta^2 c_{\rm s0}^2\right)\left(\frac{{\rm d}k_{-1}}{{\rm d}r}\right)^2 - 2{\rm i}v_0\frac{{\rm d}k_{-1}}{{\rm d}r} - 1 = 0 \tag{30}$$

$$\left(v_0^2 - \beta^2 c_{\rm s0}^2\right)\left(\frac{{\rm d}^2k_{-1}}{{\rm d}r^2} + 2\frac{{\rm d}k_{-1}}{{\rm d}r}\frac{{\rm d}k_0}{{\rm d}r}\right) + \left[3v_0\frac{{\rm d}v_0}{{\rm d}r} - \frac{1}{v_0}\frac{\rm d}{{\rm d}r}\left(v_0\beta^2c_{\rm s0}^2\right) - 2\alpha\lambda^2\left(3\gamma-1\right)\frac{\beta^2\sigma}{r^3}\right]\frac{{\rm d}k_{-1}}{{\rm d}r}$$
$$-2{\rm i}v_0\frac{{\rm d}k_0}{{\rm d}r} - 2{\rm i}\frac{{\rm d}v_0}{{\rm d}r} = 0 \tag{31}$$

and

$$\left(v_0^2 - \beta^2 c_{\rm s0}^2\right)\left[\frac{{\rm d}^2k_0}{{\rm d}r^2} + 2\frac{{\rm d}k_{-1}}{{\rm d}r}\frac{{\rm d}k_1}{{\rm d}r} + \left(\frac{{\rm d}k_0}{{\rm d}r}\right)^2\right] + \left[3v_0\frac{{\rm d}v_0}{{\rm d}r} - \frac{1}{v_0}\frac{\rm d}{{\rm d}r}\left(v_0\beta^2c_{\rm s0}^2\right) - 2\alpha\lambda^2\left(3\gamma-1\right)\frac{\beta^2\sigma}{r^3}\right]\frac{{\rm d}k_0}{{\rm d}r}$$
$$-2{\rm i}v_0\frac{{\rm d}k_1}{{\rm d}r} + 4{\rm i}\alpha\lambda^2\frac{1}{v_0r^3}\frac{{\rm d}\sigma}{{\rm d}r}\left(\frac{{\rm d}k_{-1}}{{\rm d}r}\right)^{-1} = 0. \tag{32}$$

Out of these, the first two, i.e. equations (30) and (31), will deliver the solutions

$$k_{-1} = \int \frac{\rm i}{v_0 \pm \beta c_{\rm s0}}\,{\rm d}r \tag{33}$$

and

$$k_0 = -\frac{1}{2}\ln\left(v_0\beta c_{\rm s0}\right) \pm \alpha\lambda^2\left(3\gamma-1\right)\int\frac{\beta c_{\rm s0}\left(v_0 \pm \beta c_{\rm s0}\right)}{v_0 v_{\rm K} r^3\left(v_0^2 - \beta^2 c_{\rm s0}^2\right)}\,{\rm d}r, \tag{34}$$

respectively.

The two foregoing expressions give the leading terms in the power series of $g_\omega(r)$. While dwelling on this matter, it will also be necessary to show that all successive terms of $s(r)$ will self-consistently follow the condition $\omega^{-n}|k_n(r)| \gg \omega^{-(n+1)}|k_{n+1}(r)|$, i.e. the power series given by $g_\omega(r)$ will converge very quickly with increasing n. In the inviscid limit, this requirement can be shown to be very much true, considering the behaviour of the first three terms in $k_n(r)$ from equations (33), (34) and (32). These terms can be shown to go asymptotically as $k_{-1} \sim r$, $k_0 \sim \ln r$ and $k_1 \sim r^{-1}$, given the condition that $v_0 \sim r^{-5/2}$ on large length scales, while $c_{\rm s0}$ approaches its constant ambient value. With the inclusion of viscosity as a physical effect, it can be seen from equations (33) and (34), respectively, that while k_{-1} remains unaffected, k_0 acquires an α-dependent term that goes asymptotically as r. This in itself is an indication of the extent to which viscosity might alter the inviscid conditions. However, since α has been chosen to be very much less than unity, and since the wavelength of the travelling waves is also very small, implying that $\omega \gg (v_0 \pm \beta c_{\rm s0})/r_\star$, the self-consistency requirement still holds. Therefore, as far as gaining a qualitative understanding of the effect of viscosity is concerned, it should be quite sufficient to truncate the power series expansion of $s(r)$ after considering the two leading terms only, and with the help of these two, an expression for the perturbation may be set down as

$$f^\prime(r,t) \simeq \frac{A_\pm}{\sqrt{\beta v_0 c_{\rm s0}}}\exp\left[\pm\alpha\lambda^2\left(3\gamma-1\right)\int\frac{\beta c_{\rm s0}\left(v_0\pm\beta c_{\rm s0}\right)}{v_0v_{\rm K}r^3\left(v_0^2-\beta^2c_{\rm s0}^2\right)}\,{\rm d}r\right]\exp\left(\int\frac{{\rm i}\omega}{v_0\pm\beta c_{\rm s0}}\,{\rm d}r\right)e^{-{\rm i}\omega t}, \tag{35}$$

which should be seen as a linear superposition of two waves with arbitrary constants A_+ and A_-. Both of these two waves move with a velocity $\beta c_{\rm s0}$ relative to the fluid, one against the bulk flow and the other along with it, while the bulk flow itself has a velocity v_0. It should be immediately evident that all questions pertaining to the growth or decay in the amplitude of the perturbation will be crucially decided by the real terms delivered from k_0. The viscosity-dependent term is especially crucial in this regard. For the choice of the lower sign in the real part of f' in equation (35), i.e. for the outgoing mode of the travelling wave solution, it can be seen that the presence of viscosity causes the amplitude of the perturbation to diverge exponentially on large length scales, where $c_{s0} \simeq c_s(\infty)$ and $v_0 \sim r^{-5/2}$, with $-v_0$ being positive for inflows. The inwardly travelling mode also displays similar behaviour, albeit to a quantitatively lesser degree. It is an easy exercise to see that stability in the system would be restored for the limit of $\alpha = 0$, and this particular issue has been discussed by Ray (2003a) and Chaudhury et al. (2006). The exponential growth behaviour of the amplitude of the perturbation, therefore, is exclusively linked to the presence of viscosity. Going back to a work of Chen & Taam (1993), it can be seen that the inertial-acoustic modes of short wavelength radial perturbations are locally unstable throughout the disc, with the outward travelling modes growing faster than the inward travelling modes in most regions of the disc, all of which is very much in keeping with what equation (35) indicates here.

With the help of equation (17) it should be easy to express the density fluctuations in terms of f' as

$$\frac{\rho'}{\rho_0} = \beta^2 \left(\frac{v_0}{\mathrm{i}\omega}\frac{\mathrm{d}s}{\mathrm{d}r}\right)\frac{f'}{f_0}, \tag{36}$$

and likewise, the velocity fluctuations may be set down from equation (16) as

$$\frac{v'}{v_0} = \left(1 - \frac{v_0}{\mathrm{i}\omega}\frac{\mathrm{d}s}{\mathrm{d}r}\right)\frac{f'}{f_0}. \tag{37}$$

In a unit volume of the fluid, the kinetic energy content is

$$\mathcal{E}_{\mathrm{kin}} = \frac{1}{2}\left(\rho_0 + \rho'\right)\left(v_0 + v'\right)^2, \tag{38}$$

while the potential energy per unit volume of the fluid is the sum of the gravitational energy, the rotational energy and the internal energy. For a quasi-viscous disc, to a first order in α, this sum is given by

$$\mathcal{E}_{\mathrm{pot}} = \left(\rho_0 + \rho'\right)\left[V(r) + \frac{\lambda_{\mathrm{eff}}^2}{2r^2}\right] + \rho_0\epsilon + \rho'\left[\frac{\partial}{\partial\rho_0}\left(\rho_0\epsilon\right)\right] + \frac{1}{2}{\rho'}^2\left[\frac{\partial^2}{\partial\rho_0^2}\left(\rho_0\epsilon\right)\right], \tag{39}$$

where ϵ is the internal energy per unit mass (Landau & Lifshitz 1987). In equation (39) the effective angular momentum for the quasi-viscous disc will have to be set up as a first-order correction about the inviscid conditions. Following this, a time-dependent perturbation has to be imposed about the stationary values of v and ρ. All first-order terms involving time-dependence in equations (38) and (39) will vanish on time-averaging. In this situation the leading contribution to the total energy in the perturbation comes from the second-order terms, which are all summed as

$$\mathcal{E}_{\mathrm{pert}} = \frac{1}{2}\rho_0{v'}^2 + v_0\rho' v' + \frac{1}{2}{\rho'}^2\left[\frac{\partial^2}{\partial\rho_0^2}\left(\rho_0\epsilon\right)\right] - 2\alpha\lambda^2\frac{\rho_0\sigma}{r^2}\left[\left(\frac{\rho'}{\rho_0}\right)^2 + \left(\frac{f'}{f_0}\right)^2 + 6\left(\frac{v'}{v_0}\right)^2 - 2\frac{\rho' f'}{\rho_0 f_0} + 3\frac{v'\rho'}{v_0\rho_0} - 6\frac{f'v'}{f_0 v_0}\right.$$
$$\left. + \frac{1}{\sigma}\left(\frac{\rho'}{\rho_0}\right)\int\sigma\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f'}{f_0}\right)\mathrm{d}r + \frac{1}{\sigma}\int\sigma\left(\frac{f'}{f_0} - 2\frac{\rho'}{\rho_0} - 3\frac{v'}{v_0}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f'}{f_0}\right)\mathrm{d}r\right]. \tag{40}$$

In the preceding expression all terms involving ρ' and v' can be written in terms of f' with the help of equations (36) and (37), in both of which, to a leading order, $s \simeq \omega k_{-1}$. This is to be followed by a time-averaging over ${f'}^2$, which will contribute a factor of $1/2$. The total energy flux in the perturbation is obtained by multiplying $\mathcal{E}_{\mathrm{pert}}$ by the propagation velocity $(v_0 \pm \beta c_{s0})$ and then by integrating over the area of the cylindrical face of the accretion disc, which is $2\pi r H$. Under the thin-disc approximation, $H \ll r$, this will make it possible to derive an estimate for the energy flux as

$$\mathcal{F}(r) \simeq \frac{\pi \beta^2 A_{\pm}^2}{f_0}\sqrt{\frac{\gamma K}{GM}}\left[\pm 1 + \frac{1-\beta^2\left(2-\mu\right)}{2\beta\left(\mathcal{M}\pm\beta\right)}\right]\left(1-\frac{2\alpha\lambda^2\psi}{\beta^2\mathcal{M}^2 v_0 v_{\mathrm{K}} r^2}\right) \times \exp\left[\pm 2\alpha\lambda^2\left(3\gamma-1\right)\int\frac{\beta c_{\mathrm{s0}}\left(v_0\pm\beta c_{\mathrm{s0}}\right)}{v_0 v_{\mathrm{K}} r^3\left(v_0^2-\beta^2 c_{\mathrm{s0}}^2\right)}\,\mathrm{d}r\right], \qquad (41)$$

in which $\mathcal{M}$ is the Mach number, as defined earlier, while

$$\mu = \frac{\rho_0}{c_{\mathrm{s0}}^2}\left[\frac{\partial^2\left(\rho_0\epsilon\right)}{\partial\rho_0^2}\right],$$

and

$$\psi = 2\beta\left[\left(\beta^2-1\right)^2\mathcal{M}^2 \pm \mathcal{M}\beta\left(\beta^2-4\right)+\beta^2\right]\left[1\pm 2\beta\mathcal{M}+\beta^2\mu\right]^{-1}.$$

When $\mathcal{M} \longrightarrow 0$ on large length scales, ψ converges to a finite value. However, on these same length scales, what will *not* converge are the two terms involving α in equation (41). Under the asymptotic conditions on v_0 and c_{s0}, discussed earlier, one term will diverge exponentially as r, while another will have a power-law growth behaviour of r^6. The quasi-viscous disc will, therefore, be most pronouncedly unstable on large scales under the passage of a linearised radially propagating high-frequency perturbation. It is easy to check that under inviscid conditions, with $\alpha = 0$, and for an adiabatic perturbation with $\mu = 1$, the disc will immediately revert to stable behaviour (Ray 2003a).

5. Concluding remarks

The instability of the quasi-viscous disc raises some doubts, the primary one of these being on the possibility of a long-time evolution of the disc towards a stationary end. The quasi-viscous disc being a dissipative system, i.e. its energy being allowed to be drained away from this system, there cannot be any occasion to look for the selection of a particular solution, and a selection criterion thereof, on the basis of energy minimisation, as it can be done for an idealised inviscid flow, such as the Bondi solution in spherical symmetry (Bondi 1952; Garlick 1979; Ray & Bhattacharjee 2002). This, of course, shall also affect the flow rate, which, in this treatment, has been perturbed and has been found unstable. As a result of all this the whole system has been left without any well-defined criterion by which it could guide itself towards a steady state, transonic or otherwise.

One very important physical role of viscosity in an accretion disc is that it determines the distribution of matter in the disc. The manner in which viscosity redistributes an annulus of matter in a Keplerian flow around an accretor is very well known, with the inner region of this disc system drifting in because of dissipation, and consequently, through the conservation of angular momentum and its outward transport, making it necessary for the outer regions of the matter distribution to spread even further outwards (Pringle 1981; Frank et al. 2002). This state of affairs is qualitatively not altered in anyway for the quasi-viscous flow, except for the fact that with viscosity being very weak in this case, the outward transport of angular momentum can be conspicuous only on very large scales. It may rightly be conjectured that the instability that develops on the large subsonic scales of a quasi-viscous disc is intimately connected with the cumulative transfer of angular momentum on these very length scales. The accumulation of angular momentum in this region may create an abrupt centrifugal barrier against any further smooth inflow of matter. This adverse effect, on the other hand, could disappear if there should be some other means of transporting angular momentum from the inner regions of the disc. Astrophysical jets could readily afford such a means, insofar as jets actually cause a physical drift of angular momentum vertically away from the plane of the disc, instead of along it.

Acknowledgements

This research has made use of NASA's Astrophysics Data System. The authors express their indebtedness to Rajaram Nityananda and Paul J. Wiita for some helpful comments.

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