Rotation, closed timelike curves and a singularity theorem

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Abstract

Apart from topological pecularities, vorticity may lead to the occurrence of closed timelike curves. A sufficient condition for such occurrence is deduced. The paper then goes on to generalise a recently presented singularity theorem for open universes to include vorticity. A comparison with a special form of the Hawking-Penrose theorem on singularities is made and the relation between time like geodetic incompleteness and curvature singularity in case of open universes is examined.

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1. Introduction

Quite a number of theorems on the singularity in cosmological models exist in the literature. These are concerned both with the definition of singularity as well as the condition leading to its occurrence. The intuitive definition of singularity is an unacceptable behaviour of physical variables like their blow-up or abrupt discontinuity involving some breakdown of conservative principles. Of course such peculiarities would be reflected in the geometry of spacetime. However, it has been argued that such "singularities" may be removed out of sight by introducing coordinate systems whose domains do not include the singular regions. To take care of such situations a definition of singularity has emerged identifying singular solutions as those in which some null or timelike geodesic is incomplete. With this definition one may take a formulation of Hawking and Penrose as a standard singularity theorem. It states [1]

Spacetime is not timelike and null geodetically complete if (1) $R_{\alpha\beta}k^{\alpha}k^{\beta} \geq 0$ for every non-space like vector k^{α}; $R_{\alpha\beta}$ being the Ricci tensor. (2) Every non-space like geodesic contains a point at which $k_{[\alpha}R_{\beta]}\delta\gamma[\rho k_{\mu}]k^{\gamma}k^{\delta} \neq 0$ where k^{α} is the tangent vector to the geodesic. (3) There are no closed time like curves. (4) Ther exists at least one closed trapped surface.

With the field equations of general relativity, the first condition reduces to the strong energy condition (along with an attractive gravitation) whereas any violation of condition (3) would mean a breakdown of causality. Both these are fundamental elements of physics. Not so however are the other two. Indeed it seems difficult to reconcile the presence of the rather awkward and complicated condition (2) in the statement of a fundamental theorem. Regarding condition (4) any realistic physical model of the universe should have a variety of structures some of which will eventually undergo gravitational collapse leading to the formation of trapped surfaces. To eliminate trapped surfaces is to effectively leave aside the consideration of models having any semblance to reality.

In this background came the solution of Senovilla [2] The solution is free of any curvature or physical singularity. Of the four conditions of the Hawking Penrose theorem, only the condition regarding the trapped surface did not hold good in Senovilla's solution. In Senovilla's solution, the cosmic matter is a perfect fluid without rotation and if one works out the kinematic variables with reference to the velocity vector of matter, it turns out that the space time averages of all the kinematic scalars (that appear in the Raychaudhuri equation) as also the energy density vanish.

In a very thorough investigation, Chinea eta al [3] examined all non-space like geodesics in the Senovilla solution and came to the conclusion that they were all complete. (A contrary result was given by Joshi [4] that the Senovilla space time is non-space like geodetically incomplete but this seems to be wrong).

In a recent note [5], it was shown that the vanishing of the spacetime averages of the Raychaudhuri scalars taken over the entire space time is a general property of all nonsingular non-rotating solutions with the topology $R^3 \times R$. However, the vanishing of the space and time avarages separately as observed in Senovilla's solution is not a general property.

In the present paper, we examine the relation between "rotation" and the occurrence of closed timelike curves. This allows us to extend our theorem about the vanishing of space time avarages of scalars to rotating models as well provided there is no closed timelike curve. However, in the general case where the matter is not a perfect fluid, we use the timelike eigenvector of the Ricci tensor for constructing the kinematic variables and a singularity is identified with the blowing up of scalars built from the Ricci tensor and its covariant derivatives.

2. Vorticity and closed timelike curves

Quite generally we can write the metric in the form

$$ds^2 = g_{00}dx^{0^2} + 2g_{0i}dx^0dx^i + g_{ik}dx^idx^k \tag{1}$$

where the indices i, k run from 1 to 3 corresponding in general to space like coordinates and the index 0 refers to timelike coordinates. The domain of all the four coordinates extends from $-\infty to + \infty$. We adopt the convention $g_{00} > 0$. The signature condition for the metric (1) requires

$$det|g_{ik} - \frac{g_{0i}g_{ok}}{g_{00}}| < 0 \tag{2}$$

For a closed timelike curve with affine parameter λ, $\frac{dx^0}{d\lambda}$ must have some zeros and at these points the timelike condition of the curve requires

$$g_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda} > 0 \tag{3}$$

The inequalities (2) and (3) are consistent only if

$$g_{0i} \neq 0 \tag{4}$$

The relation (3) indicates that the three spaces spanned by the coordinates x^i must contain timelike lines and (4) shows that the unit timelike vectors v^α tangential to the x^0-lines are not hypersurface orthogonal. We then define the vorticity vector w^α corresponding to v^α :

$$\omega^\alpha = \frac{1}{2}\eta^{\alpha\beta\gamma\delta}v_\beta v_{\gamma;\delta} \tag{5}$$

So far we have made no specifications about the vector v_α - one may be tempted to identify it with the velocity vector of matter that may be considered to be present in the space time. However, as we are not going to introduce any assumption about the nature of the energy stress tensor $T^{\mu\nu}$, such identification is not always possible. However v^α does represent a possible velocity field for material particles and hence the appropriateness of the term "vorticity".

To somewhat simplify our analysis without sacrificing generality, we assume the metric to be of the form

$$\begin{pmatrix} g_{00} & g_{01} & 0 & 0 \\ g_{01} & g_{01} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix} \tag{6}$$

and g_{00}, g_{01} not dependent on x^3. Then the only non-vanishing component of ω^α is ω^3 given by

$$\omega^3 = \frac{1}{2}\frac{v_{,2}}{\sqrt{(v^2 - g_{11}/g_{00})g_{22}g_{33}}} \tag{7}$$

where we have written v for g_{01}/g_{00}. Writing $dr = \sqrt{(|g_{22}|)}dx^2$ for the proper distance for the coordinate difference dx^2 and $\omega = \sqrt{|g_{33}|}\omega^3$ for the magnitude of the vorticity, we get from (7),

$$\omega dr = \frac{1}{2}\frac{v_{,2}dx^2}{\sqrt{(v^2 - g_{11}/g_{00})}}. \tag{8}$$

In the region where there no timelike interval in the three space $x^0 = constant$, g_{11}/g_{00} is negative. Timelike intervals occur only when g_{11} becomes positive. Hence integrating over an entirely space like region,

$$\int \omega dr < \frac{1}{2}\int \frac{dv}{v} < \frac{1}{2} ln v.$$

Hence $\int \omega dr$ can at most have a logarithmic divergence. Should it have a stronger divergence the above inequality will break down showing that g_{11} has turned positive. Thus a sufficient condition for occurrence of timelike lines in the 3-space is a stronger-than-logarithmic divergence of the last integral $\int \omega dr$ or in other words as $r \to \infty, \omega$ vanishes at least faster than $1/r$. As ωdr may be looked upon as the relative velocity between particles at a distance dr apart, one might have expected that closed timelike curves bringing in acausality would occur for $\int \omega dr$ exceeding unity, the velocity of light; for according to Lorentz transformation, ordering of events become acausal for such velocities. However, a comparison with situations in the published literature shows that such is not the case.

Thus with the Gödel metric, in the form [6],

$$ds^2 = 4a^2[dt^2 - dr^2 - dy^2 + (sinh^4 r - sinh^2 r)d\phi^2 + 2\sqrt{2} sinh^2 r d\phi dt]$$

the vorticity $\omega = \frac{1}{\sqrt{2}}$ in the direction of the coordinate y. However, closed timelike lines ocuur for $r > ln(1+\sqrt{2})$ is $\sqrt{2} ln(1+\sqrt{2}) \simeq 1.24$ and thus exceeds unity.

In case of the metric given by Som and Raychaudhuri [7]

$$ds^2 = dt^2 - e^{2(A^2-a^2)r^2}dr^2 + (a^2r^4 - r^2)d\phi^2 + 2ar^2 d\phi dt$$

the vorticity has magnitude $e^{-(A^2-a^2)r^2}.a$ and the proper distance $\sqrt{g_{rr}}dr$ is $e^{(A^2-a^2)r^2}dr$. Consequently $\int_0^r \omega dr = ar$. The closed timelike curves occur for $r > 1/a$, i.e., where the integral $\int_0^r \omega dr > 1$.

An example of non-occurrence of closed time like curves even though there is vorticity is provided by the Maitra metric [8]

$$ds^2 = dt^2 - e^{2\mu}(dr^2 + dz^2) - (r^2 - m^2)d\phi^2 + 2md\phi dt$$

with

$$\mu = -\frac{1}{4}x^2[(1+x^2)^{1/2} - 1] + \frac{1}{8} - \frac{1}{4}ln\frac{1}{2}[(1+x^2)^{1/2} + 1]$$

$$m = \frac{1}{2}[(1+x^2)^{1/2} - 1 - ln\frac{1}{2}[(1+x^2)^{1/2} + 1]$$

$$m = 2r/a$$

Here $\omega \to 1/r$ as $r \to \infty$, thus there is a logarithmic divergence of the integral $\int \omega dr$ and consistent with our discussion there is no closed timelike line, only the circular ϕ lines tend to become null as $r \to \infty$.

Thus in the absence of topological pecularities as in case of strings or wormholes, closed timelike lines can occur only if vorticity is present. But the mere presence of vorticity may not bring in closed time like curves. If closed time like curves do not occur we can always have an everywhere hypersurface orthogonal timelike vector field. Thus in the metric (1) if there are no timelike line, the three spaces $x^0 = constant$ are everywhere space like and the orthogonal to these spaces have the covariant components $v_\alpha = (\frac{1}{\sqrt{900}}, 0, 0, 0)$, the factor $\frac{1}{\sqrt{900}}$ normalises the vector to be of unit magnitude.

As the occurrence of closed timelike curves is an intrinsic property of space time, not dependent on the choice of the coordinate system,; hypersurface orthogonal timelike vector fields can always be found if closed timelike curves do not occur and conversely if there is any everywhere hypersurface orthogonal timelike convergence, then there is no closed timelike curve. Further if there is any timelike vector whatever for which the vorticity diverges, these would indicate the presence of closed timelike fields in the space time.

3. The singularty theorem.

We prove the following theorem: The space time average of all scalars appearing in the Raychaudhuri equation vanishes if

(a) the universe is open in all directions, i.e., it has topology $R^3 \times R$.

This condition means that the ratio of the volume of any three dimensional subspace to that of the entire space time vanishes - i.e.,

$$\frac{\int\int\int \sqrt{|^3g|}dx^i dx^k dx^l}{\int\int\int \sqrt{|g|}d^4x} = 0 \qquad (9)$$

where the indices i, k, l are all different and may refer to space or time coordinate and $\sqrt{|^3g|}$ is the appropriate coefficient to give the volume for the three dimensional element.

(b) the universe is non-singular in the sense that scalars built from the Ricci tensor and its covariant derivatives remain bounded everywhere.

(c) There are no closed timelike curves.

This condition replaces the non-rotating condition used in [5] and is somewhat weaker than that.

(d) the strong energy condition is obeyed.

This means that $R_{\mu\nu}v^\mu v^\nu \geq 0$ everywhere for any unit timelike vector v^μ.

Here the kinematic scalars are built up with the timelike eigenvector of the Ricci tensor and the space time averages of any entity χ is defined as

$$\chi \equiv [\frac{\int_{-x_0}^{+x_0}\int_{-x_1}^{+x_1}\int_{-x_2}^{+x_2}\int_{-x_3}^{+x_3}\chi\sqrt{|g|}d^4x}{\int_{-x_0}^{+x_0}\int_{-x_1}^{+x_1}\int_{-x_2}^{+x_2}\int_{-x_3}^{+x_3}\sqrt{|g|}d^4x}]_{lim\ x_o,x_1,x_2,x_3\rightarrow\infty} \qquad (10)$$

Obviously, $< \chi >$ is meaningful only if the limit exists.

The Raychaudhuri equation with v^μ, the unit timelike eigenvector of the Ricci tensor, is

$$\frac{1}{3}\theta^2 + 2\sigma^2 + \kappa(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)v^\alpha v^\beta = -\dot{v}^\alpha_{;\alpha} - \dot{\theta} + 2\frac{\omega^2}{3} \qquad (11)$$

As in [5], we take the average of each term in (11) to get

$$\frac{1}{3} < \theta^2 > +2 < \sigma^2 > +\kappa < (T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)v^\alpha v^\beta >= - < \dot{v}^\alpha_{;\alpha} > - < \dot{\theta} > +2 < \frac{\omega^2}{3} > \qquad (12)$$

We repeat the argument in [5] to show that the first two terms on the right in (12) vanish. The third term is new. The absence of closed timelike curves requires that ω^2 vanishes faster than $1/r^2$ as r goes to spatial infinity. Hence in the expression for $< \omega^2 >$ the integral in the numerator will have a weaker divergence compared to the integral in the denominator and hence $< \omega^2 >= 0$. This proves our theorem.

4. A critical discussion of the conditions assumed

The openness of space time as assumed above maybe made somewhat more specific. The first term on the right of equation (12) involves an integral over a four dimensional divergence which is converted into an integral over a three dimensional hypersurface so that it reduces to the form (see ref [5])

$$\frac{\int |\dot{v}^i||d\Sigma_i|}{\int \sqrt{-g}d^4x}$$

The three space volume element $d\Sigma_i$ has a counterpart in the denominator integral. Hence the vanishing of the above requires merely the infinity of the space dimension in the direction of $\dot{v}^i$. Similarly, as is clear from [5], the vanishing of $< \dot{\theta} >$ requires only the openness in the direction of the coordinate t. As far as the vanishing of $< \omega^2 >$ is concerned, the crucial point is that the direction corresponding to the coordinate x^2 should be open. This direction is orthogonal to the vorticity vector and the spatial velocity whose curl is related to the vorticity. Thus finally the openness assumed for all directions is suffiecient but more than necessary for the validity of our theorem.

Obviously, the theorem will not hold good if the universe be spatially closed or time incomplete. A positive definite quantity can have a zero average over a finite domain only trivially.

For the Senovilla theorem, besides the space time average, the space and time averages vanish sepatately. A foliation of the space time is possible in this case as the fluid is non-rotating. In any case this result seems to be peculiar to the Senovilla solution and not a general characteristic of non-singular solutions as the following examples show.

Consider a star occupying a bounded region of space and in equilibrium. The space and space-time averages vanish but the time average at points in the region occupied by the star do not vanish.

As a second example, one may take the case of Maitra rotating universe [8]. Although the matter is rotating, there is a hyper-surface orthogonal timelike killing vector and thus the space time allows a folliation into space sections. It turns out that the space and spacelike averages of the Raychaudhuri scalars vanish but not the time average.

Our proof is based on consideration of integrals over space time and specially their behaviour at infinity. It thus overlooks any local singularity that may be present without affecting the values of the infinite integrals. Consequently the converse of the theorem stated as "If the space time averages of the scalars appearing in Raychaudhuri equation vanish, the spacetime is non-singular." is not true. A simple example is to show this is the Schwarzschild metric with a $\delta-$ function in the distribution. Apparently such localised singularities are taken care of by the condition of non-existence of a closed trapped surface in the Hawking Penrose formulation. (Recall the conjecture about the absence of naked singularities.)

One may look upon the big bang as a singularity over all space but localized in time. The scalars blow up, nevertheless their space time averages vanish in case of spatially open universes. However, the universe terminates at the singularity unlike the case of singularities localized in space. As is well known, if acceleration and vorticity are absent then an expanding spacetime (open or closed) has a collapse singularity (where the expansion blows up) in the finite past. Thus non-singular space times can occur only if acceleration and/or vorticity is present. The present theorem is specially relevant in such cases.

We have identified singularities of space-time with curvature singularities and in the next section we shall consider the case of geodetic incompleteness.

5. Geodetic completeness for open models

Chinea et al [3] have shown that the Senovilla universe is non-space like geodetically complete by examining separately all possible geodesics. Here we propse a procedure by which one can conclude the geodetic completeness for open models in general.

Assume for the moment that there is a congruence of time incomplete geodesics. Taking the time coordinate along these geodesics, we have as usual the metric,

$$ds^2 = g_{00}dt^2 + 2g_{0i}dtdx^i + g_{ik}dx^i dx^k$$

The incompleteness requires that $\int \sqrt{g_{00}}dt$ should converge for either $t \to -\infty$ or $+\infty$, so that at the particular terminal point $g_{00} \to 0$ faster than $1/t^2$. First suppose that the convergence is normal so that $g_{0i} = 0$, and then Raychaudhuri equation will read:

$$\frac{1}{3}\bar{\theta}^2 + 2\bar{\sigma}^2 + \kappa(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)V^\alpha V^\beta = -\dot{\bar{\theta}}$$

where the overhead bars signify that the kinematic scalars are calculated with respect to the vector $V^\alpha = \frac{1}{\sqrt{g_{00}}}\delta^\alpha_0$. We now have, with strong energy condition,

$$-\dot{\bar{\theta}} \geq \frac{1}{3}\bar{\theta}^2$$

Consequently we obtain by the well known procedure, (if $\bar{\theta} \neq 0, T_{\mu\nu} \neq 0$) $\bar{\theta} \to \infty$ at $t \to -\infty$ and so $\sqrt{|^3g|}$ vanishes at $t \to -\infty$. This simultaneous vanishing of g_{00} and ^3g means that g itself vanishes, violating the signature requirement and indicating the occurrence of a curvature singularity.

The above argument does not hold good if the geodesic congruence be not normal, as the Raychaudhuri equation will then have an additional term on the right due to vorticity. However, when $g_{00} \to 0$, v in equation (7) blows up. A divergent $\bar{\omega}$ then indicates the presence of closed timelike curves. The divergence of $\bar{\omega}$ is also apparent from the Raychaudhuri equation - as $g_{00} \to 0 \quad V^\alpha \to \infty$ and consequently, if at that time $(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})$ do not vanish, the infinity on the left hand side has to be balanced by an infinite $\bar{\omega}$. Hence we have the theorem: In an open singularity free universe, a timelike geodesic congruence can be incomplete only if the spacetime be completely empty or there are closed timelike curves.

It should be noted that this leaves aside the case of an isolated incomplete timelike geodesic. However such a thing can occur only if it is associated with a symmetry (eg, if there be a timelike incomplete geodesic along a rotational symmetry axis) or with a singular behaviour.

6. Concluding remarks

It is interesting to compare our approach with that of Hawking and Penrose. Both assume the strong energy condition and absence of closed timelike curves. However closed trapped surfaces do not find any place in our discussion and for that we miss any localised singularity that may be present. The so-called generality condition of Hawking and Penrose is simply absent in our case. Our theorem is restricted to open universes while the Hawking-Penrose theorem is more general but the case of closed universes admit many comparatively simple singularity theorems.

A remarkable result has been our conclusion that for open universes, the non-space like geodetic incompleteness is more or less identified with curvature singularity.

On the whole our theorem seems to have two merits in its favour - it is remarkably easy to prove compared to the case of Hawking and Penrose.

Further the result seems to be more physically transparent.

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