

TIME-KEEPING IN SPECIAL RELATIVITY

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Abstract

An elementary discussion on how time measurements differ for different inertial observers according to Einstein's special theory of relativity. A comparison is made with the Newtonian (pre-relativity) method of time measurement.

1. Introduction

In 1905 Einstein published three papers, each of which turned out to be a trend-setter. One was his discussion of the phenomenon of Brownian motion, another was on the photo-electric effect and the third introduced the special theory of relativity. This simple discussion presents a highlight of the special relativity, one which baffled and confused most people, including physicists, but which is factually borne out by experiments.

The principle of relativity of motion implies that two inertial observers 'see' the same basic physical laws. Here an inertial observer is one on whom no force acts. By Newton's first law of motion such an observer has uniform velocity. Thus two inertial observers will in general have a uniform *relative velocity*. And two such observers should find the same mathematical structure for all basic physical laws. In other words, the relative velocity plays no role if! the fundamental formulation of physical laws.

So consider two inertial observers A and A' , with A' having a uniform velocity v relative to A . Both conduct electromagnetic experiments in their labs, and discover (or, rather, rediscover!) Maxwell's equations. Suppose A writes the Maxwell equations in vacuum as

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (1)$$

Here $\mathbf{E}$ is the electric field vector and $\mathbf{B}$ the magnetic field vector while c is a fundamental constant. (Note that in vacuum $\mathbf{B} \equiv \mathbf{H}$, $\mathbf{E} \equiv \mathbf{D}$ in the usual notation). Given this and the principle of relativity, we can assert that A' will find *exactly the same set of equations* connecting the measured quantities $\mathbf{E}'$ and $\mathbf{B}'$

So far this seems reasonable. One could argue that there exists a linear transformation between $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{E}', \mathbf{B}')$ that preserves the form (1). However, let us use (1) for a little manipulation. Taking curl of the third equation of (1) and using the first and fourth ones, we get

$$\begin{aligned} 0 \equiv \nabla \times \left[\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right] &\equiv \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} - \frac{1}{c} \nabla \times \frac{\partial \mathbf{B}}{\partial t} \\ &\equiv -\nabla^2 \mathbf{B} - \frac{1}{c} \frac{\partial}{\partial t} \nabla \times \mathbf{E} \\ &\equiv -\nabla^2 \mathbf{B} + \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} \end{aligned} \quad (2)$$

Likewise we also get

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \quad (3)$$

Both the equations (2) and (3) are wave equations and they tell us that $\mathbf{E}$ as well as $\mathbf{B}$ propagate with a fundamental speed c . What is this constant c ? Experiments identify it with the speed of light in vacuum. Thus, working in their respective labs both A' and A will conclude that any light wave they see must travel with speed c .

This conclusion drawn from a reasonable looking basic principle appears counter-intuitive to anyone accustomed to thinking in the Newtonian way. For, if A sees an object C coming with velocity u , then A' should find its velocity as $u - v$. And this result applied to a light wave would tell us that A and A' would find different values for their measurements of the speed of light.

So, we have two possibilities to us. Either we carry on with Maxwell's equations *as they are* and *revise* our Newton-based system of spacetime measurements or we *revise* Maxwell's equations so that the above discrepancy is removed. Einstein opted for the first recourse and was led to the special theory of relativity.

2. Transformations between inertial frames

Briefly, we may denote the two possibilities above by the *Galilean* and *Lorentz transformations* between the spacetime coordinates of an event measured by A and A' . Let us consider the Galilean transformation first. It is given by

$$x' = x - vt, \quad t' = t \quad (4)$$

Here we have no *change of time coordinates*, while the space coordinate suffers a translational change. This transformation gives expression to the Newtonian concepts of absolute space and absolute time. Note that,

although the spatial coordinates changes, the spatial separations between two events occurring at the same time do not. Also, accelerations do not change.

$$\frac{d^2 x'}{dt'^2} = \frac{d^2 x}{dt^2}; \quad (5)$$

so that Newton's second law of motion remains unchanged. But, as we saw earlier, Maxwell's equations cannot retain the same form under the Galilean transformation.

The second alternative of Lorentz transformation looks, at first sight, very messy. Corresponding to (4) we have

$$\begin{aligned} (x' \cdot v) &= \gamma(x' - vt) \cdot v, \\ x' \times v &= x \times v, \\ t' &= \gamma \left(t - \frac{v \cdot x}{c^2} \right), \end{aligned} \quad (6)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (7)$$

This transformation leaves the components of space coordinates normal to v unchanged, but alters the longitudinal spatial component (parallel to v) and also the time coordinate. Because of the factor γ , the displacement in space (or time) of two events at the same time (or space) is not the same, but is multiplied by the factor γ . Note, however, that a little algebra applied to the equations (6) will lead to the result

$$c^2 t'^2 - x'^2 = c^2 t^2 - x^2 \quad (8)$$

Thus if a light wave propagated along $|x| = ct$ as seen by A , it will be seen by A' as propagating along $|x'| = ct'$, i.e., with the same speed c .

3. A practical example

In case above discussion appears too pedantic here is a simple example, which I first came across in a public lecture given by Hermann Bondi.

We consider two observers A and A' separating out with a uniform speed v . We count times t, t' as measured by A and A' from the instant they were together. Thus we have $t = t' = 0$ at that instant. The tracks of A and A' in space-time diagram are shown in the Figure 1.

To have a practical measurement of each other's time coordinates, A and A' send out light signals towards each other. Suppose A sends a signal at time $t_1 > 0$ by his watch. When A' receives it, at his measured time t'_0 , say, he sends back a reply light signal immediately, reaching A at his recorded time t_2 . However, A estimates from his timings that A would have received his signal at

$$t_0 = \frac{1}{2}(t_1 + t_2) \quad (9)$$

FOUR GIANT STEPS TO KNOW THE WORLD BETTER

1. We turn back to ancient Greek times, where the motion of Euclidean geometry was introduced. That was the beginning of an appreciation of the crucial role of mathematical reasoning in our insights into Nature.
2. We now leap to the seventeenth century, when Galileo and Newton told us how the motion of ponderable bodies could be understood in terms of force between their constituent particles and the accelerations that these forces engender.
3. The nineteenth century gave us the third revolution, when Faraday and Maxwell showed us that particles were not enough, and we must consider, also, that there are continuous fields pervading space, with a reality as great as that of the particles themselves. These fields were combined into a single all pervasive entity, referred to as the electromagnetic field, and the behavior of light could be beautifully explained in terms of its self-propagating oscillations.
4. In the 20th century, it is particularly remarkable that a single physicist – Albert Einstein – had such extraordinary deep perceptions of the workings of Nature that he laid the foundation of Relativity and

as measured by his (A's watch. Let us assume that

$$t'_0 = \lambda t_0. \quad (10)$$

In the Newtonian (Galilean) system we would expect $\lambda = 1$.

However, in the special relativistic system, the speed of light is c for both the signal sent out by A and A' . So we have the distance of A and A' at the time t_0 (or t'_0 as measured by A and A') as d , where

$$2d = c(t_2 - t_1). \quad (11)$$

Now A estimates that travelling velocity v away from him for a time t_0 , A' would have moved to a distance

$$d = vt_0 = \frac{(t_1 + t_2)}{2} \times v. \quad (12)$$

From (11) and (12) therefore we get

$$\frac{v}{c} = \frac{t_2 - t_1}{t_1 + t_2}. \quad (13)$$

So far we have not used the principle of relativity fully. We do so now. Consider the light ray sent by A and t_1 which reached A' at t'_0 . Suppose we write,

$$t'_0 = \alpha t_1. \quad (14)$$

By similarity of geometrical shapes and the reciprocity between the observers, a typical signal sent out by A at any instant t_1 would reach A' at time αt_1 as measured by the watch of A' , with the same α , and vice versa. That is, the signal sent by A' at t'_0 would reach A at time $\alpha t'_0$ as measured by A . We thus have

$$t_2 = \alpha t'_0 = \alpha^2 t_1. \quad (15)$$

From (13) and (15) we therefore get

$$\alpha = \left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \right)^{1/2} \quad (16)$$

How do we compare the time measurements by A and A' ? Suppose A has stationed another colleague B at rest relative to him but located at the point where A' will be when the signal from A reaches him, that is, at time t_0 by the watches A and B (which are supposed to be synchronized). So as A' passes him, B will record that the watch carried by A' showed the time t'_0 . How does t'_0 compare with his own time t_0 ? Using (9) and (15) we get

$$t_0 = \frac{1}{2}(\alpha^2 + 1)t_1, \quad t'_0 = \alpha t_1 \Rightarrow \frac{t'_0}{t_0} = \frac{2\alpha}{\alpha^2 + 1}. \quad (17)$$

Using (16) for α we get for λ as defined in (10),

$$\lambda \equiv \frac{t'_0}{t_0} = \sqrt{1 - \frac{v^2}{c^2}} = \gamma^{-1}. \quad (18)$$

FOUR GIANT STEPS

Quantum Theories which revolutionized modern Science. This, he has done just in a single year 1905.

5. In this same year Einstein also provided fundamental new insights into two other areas, with his doctoral dissertation on the determination of molecular dimensions and with his analysis of the nature of Brownian motion. This latter analysis alone would have earned Einstein a place in history. Indeed, his work on Brownian motion (together with the independent and parallel work of Smolunchowski) laid the foundation of an important piece of statistical understanding, which has enormous implications in numerous other fields.

Thus $\lambda \neq 1$ as expected by the Galilean transformation. Rather we have $\lambda < 1$. To B it would appear that the watch carried by A' as he moves past is running slow.

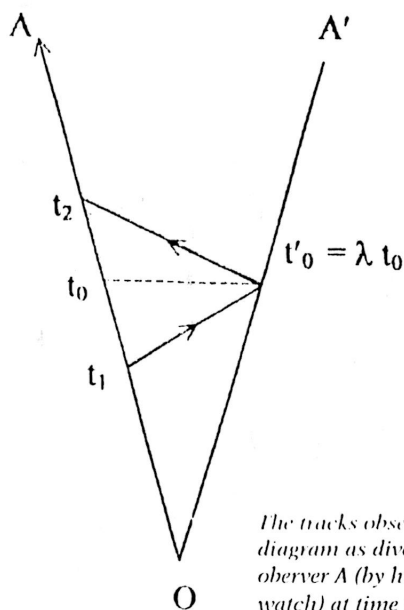
Let us recall the reciprocity of the two observers. In fact if we interchanged the roles of A and A' and let A' bounce a light signal off A , his colleague B' would come to the conclusion that the watch carried by A is going slow!

4. Conclusion

There was considerable discussion in the early days of relativity, whether time dilatation or slowing down of time implied a logical contradiction. 'If the watch A' goes slower than the watch of A by factor λ and vice versa, surely the two conclusions cannot both be right. So special relativity be!' This was the argument made by the opponents of special relativity.

This conclusion is, however, logically erroneous. For, in the first experiment the slowness of the watch of A' was detected by two observers A and B at rest relative to each other and located at different points of the trajectory followed by A' . In the second experiment A' and B' together measured and compared the times kept by their watches with the watch of A . So the two experiments were not the same and there was no contradiction between their differing conclusions. In fact the result underscores the symmetry or reciprocity of any two inertial observers.

The decay of the μ -meson (now commonly known as muon), a particle present in cosmic rays, established that the result of time dilatation is factually correct. Typically, a muon in its own frame of reference decays with a half life of 1.5×10^{-6} second. Cosmic rays take longer than this to travel to the Earth. In fact if we compare the number of muons that actually reach the sea level, with the number of particles expected to survive, we find that there are far too many muons reaching the sea level. This phenomenon is explained by the relation (18). In the rest frame of the muon, the time runs slower compared to the Earth-bound observer's frame, with $\lambda \approx 1/9$. This makes it possible for the muons to have survived.



The tracks observers A and A' are shown in the spacetime diagram as diverging from point O . A light signal sent by the observer A (by his watch) at time t_1 is received by A' (by his watch) at time t'_0 refer to the text details.

ALBERT EINSTEIN'S THOUGHTS

I want to know how God created this world. I am not interested in this or that phenomenon, in the spectrum of this or that element; I want to know his thoughts; the rest are details.

...

I am truly a 'lone traveler' and have never belonged to my country, my home, my friends, or even my immediate family, with my whole heart; in the face of all these ties, I have never lost a sense of distance and a need for solitude- feelings which increase with the years.

...

We are faced with the dismaying fact that the politicians, the practical men of affairs, have become the exponents of international ideas.

...

Where the world ceases to be the scene of our personal hopes and wishes, where we face it as free beings admiring, asking, and observing, there we enter the realm of Art and Science.

...

The man who regards his own life and that of his fellow creatures as meaningless is not merely unhappy but hardly fit for life.

...

The most beautiful experience we can have is the mysterious. It is the fundamental emotion which stands at the cradle of true art and true science--- I am satisfied with the mystery of the eternity of life.

...

A hundred times every day I remind myself that my inner and outer lives are based on the labors of other men, living and dead, and that I must exert myself in order to give in the same measure as I have received and am still receiving.

...

The true value of a human being is determined primarily by the measure and the sense in which he has attained liberation from the self.

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