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Stellar orbits in triaxial clusters around black holes in galactic nuclei

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ABSTRACT

We investigate the orbital structure of a model triaxial star cluster, centered around a supermassive black hole (BH), appropriate to galactic nuclei. Sridhar and Touma (1999) proved that the presence of the BH enforces some regularity in the dynamics within the radius of influence of the BH. We employ their averaging method to reduce the degrees of freedom from three to two. Numerical orbit integrations, together with Poincaré surfaces of section allow us to draw a global portrait of the orbital structure; in our calculations we employ a model cluster potential that is triaxial and harmonic. The averaged dynamics of the axisymmetric case is integrable, and we present a detailed comparison of orbits in oblate and prolate axisymmetric potentials. Both cases support resonant orbits with fixed values of eccentricity, inclination, and periapse, whose lines of nodes rotates steadily. These occur for all values of oblateness, but only for axis ratio greater than two, in the prolate case; we identify this phenomenon with the (in)stability of the long axis orbit. We then systematically explore significantly triaxial potentials, possessing small oblateness, or prolateness. Resonant orbits and their families are studied both numerically, and through secular perturbation theory. Chaos is highly suppressed for all the cases we studied, and we obtain effective third integrals. Some of the orbits appear to reinforce the shape of the potential; we provide phase space, as well as real space portraits of these orbits. A particularly promising resonant orbit exists in highly prolate, triaxial potentials.

Subject headings: black hole physics – celestial mechanics, stellar dynamics – galaxies: nuclei

1. Introduction

Evidence for the presence of central black holes (BH) of mass, $10^6 - 10^{9.5}\,M_\odot$, appears strong for about a dozen galaxies (c.f. Kormendy and Richstone 1995). The masses of these objects is consistent with the mass in BHs needed to produce the observed energy in light from quasars (Soltan 1982, Chokshi and Turner 1992). Recent studies suggest that the dynamical influence of the BH extends to well outside the nuclear regions of its host galaxy, if the orbits of stars carry them close to the centre (c.f. Merritt 1998 for a review). In particular, dynamical phenomena in the central regions are strongly influenced by the BH. Stars move in the combined gravitational potentials of the BH, and the surrounding mass of stars and gas (for brevity, these will be referred to as the "cluster"). The radius of influence of the BH (r_h) may be defined as radius of the sphere within which the mass of the surrounding stars and gas is less that of the BH. Within r_h, the potential of the BH dominates the dynamics, and the potential of the cluster may be considered to be a small perturbation. Sridhar and Touma (1999; hereafter referred to as ST99) studied stellar dynamics within r_h, and arrived at the following conclusions:

(i) Stellar orbits may be thought of as Keplerian ellipses that deform and precess over times that are longer than orbital times by a factor (M/M_c), where M_c is the mass of the cluster enclosed by the orbit. Well within r_h, $M/M_c \gg 1$.

(ii) The slow dynamics of precessional motions may be understood by *averaging* over the Keplerian orbital periods of stars, which are the shortest time scales in the problem.

(iii) Averaging gives rise to a secular integral of motion, $I \simeq \sqrt{GMa}$, where a is the semi–major axis of the orbit. Thus the BH enforces some degree of regularity in the structure of orbits that lie within r_h.

(iv) A stellar orbit in a time–independent cluster conserves two integrals of motion. These are the energy, which is an exact invariant, and the semi–major axis, which is secularly conserved.

(v) Thus clusters which may be modelled as razor–thin discs, or as three dimensional, axisymmetric objects, give rise to averaged dynamics that is integrable.

ST99 considered a family of model potentials for razor–thin discs, and classified orbit families. Here we take up the thread, and consider orbits in three dimensional clusters. In § 2, we provide a quick introduction to action–angle variables for the Kepler problem, and averaging over the fast orbital frequency. We introduce a family of triaxial, harmonic potentials, and average over the fast orbital motion, to obtain an expression for the Hamiltonian governing slow dynamics. Averaged dynamics for oblate and prolate axisymmetric cases is discussed in § 3. We numerically integrate the equations of motion, and take Poincare sections. In § 4, we discuss the global orbital structure for the triaxial case, when departures from spherical symmetry are small (small oblateness or prolateness, but quite large triaxiality). The results are interpreted through secular perturbation theory.

The reader who is interested in getting a quick glimpse of the real–space structure of those orbits that reinforce triaxiality, as well as oblateness/prolateness of the potential, is advised to consult the list of relevant figures and comments provided at the end of the discussion in § 5.

2. Formulation of orbit–averaged dynamics

Let us locate the BH, of mass M, at the origin (this is a valid assumption, so along as the centre of mass of the surrounding cluster does not vary with time). We introduce action–angle variables, appropriate to the Kepler problem, that allow us to conveniently carry out orbit–averaging. These are the Delaunay variables (c.f. Plummer 1960, Goldstein 1980). $I = \sqrt{GMa}$, L, and L_z are the actions, where L and L_z are the magnitude and z–component, respectively, of the angular momentum. We denote the conjugate angles by w, g and h, respectively. w is the orbital phase ("mean anomaly") which varies from 0 at pericentre to 2π after one orbit. g is the angle to the pericentre, measured from the ascending node, and h is the longitude of the ascending node (see Figure 1).[1]

Let $\Phi(x,y,z)$ be the perturbing gravitational potential of the cluster. To average Φ over the orbital phase, it is convenient to express the spatial coordinates in terms of the Delaunay variables (c.f. Goldstein 1985):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} C_gC_h - C_iS_hS_g & -S_gC_h - C_iS_hC_g & S_iS_h \\ C_gS_h + C_iC_hS_g & -S_gS_h + C_iC_hC_g & -S_iC_h \\ S_iS_g & S_iC_g & C_i \end{pmatrix} \begin{pmatrix} a(C_\eta - e) \\ a\sqrt{1-e^2}\,S_\eta \\ 0 \end{pmatrix} \qquad (1)$$

where S and C are shorthand for sine and cosine of the angles given as subscript. i is the angle of inclination determined by $\cos i = L_z/L$, and $e = \sqrt{1 - L^2/I^2}$ is the eccentricity; as before, a is the semi–major axis. η is the eccentric anomaly, related to the orbital phase through $w = \eta - e\sin\eta$. Having substituted the expressions for (x,y,z), given in equation (1), in $\Phi(x,y,z)$, averaging is conveniently carried out by integrating over η:

$$\overline{\Phi} = \frac{1}{2\pi}\oint d\eta(1 - e\cos\eta)\,\Phi\,. \qquad (2)$$

This orbit–averaged potential, $\overline{\Phi}$, plays the role of the Hamiltonian for slow, precessional dynamics (see ST99). Superimposed on the slow time variation of averaged dynamics are fast oscillations (with frequency of order $\sqrt{GM/a^3}$), whose fractional amplitudes are of order $\varepsilon \sim a\,(\Phi - \overline{\Phi})/GM$. Henceforth we assume that $\varepsilon \ll 1$, and ignore these fast oscillations.[2] Because $\overline{\Phi}$ is independent of w, the conjugate variable, I, is secularly (i.e. on the average) conserved.

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[1] For the Kepler problem, the only Delaunay variable that is time dependent is w: $w(t) = w_0 + (G^2M^2/I^3)\,t$.

[2] For fixed a, ε is larger for more eccentric orbits, because these sample a larger spatial variation of Φ.

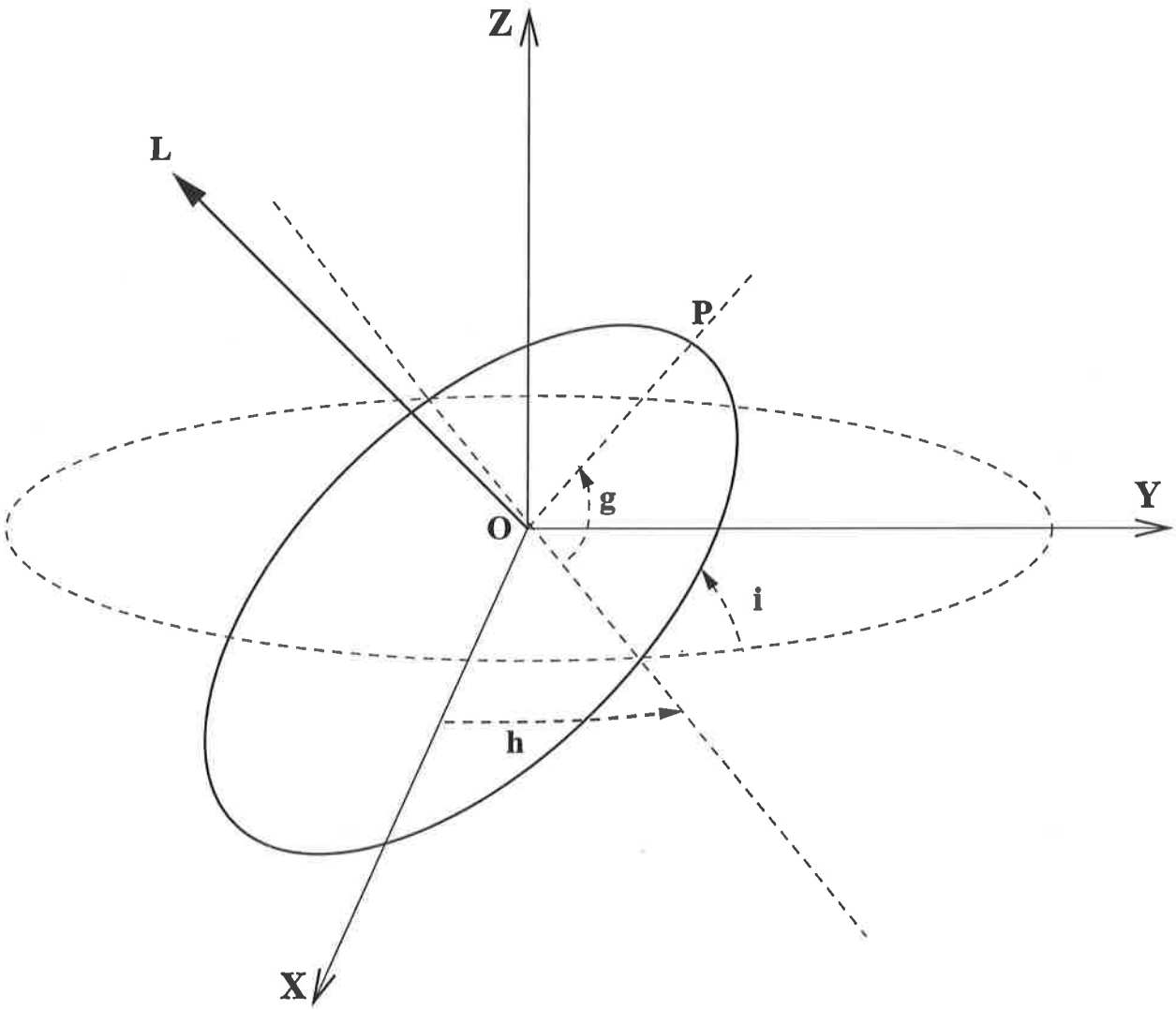

Fig. 1.— Orbital elements of the Kepler problem (Delaunay variables): the orbit is represented by the ellipse, drawn in bold line. **P** marks the pericentre. The figure shows only the four variables of precessional dynamics. The other two refer to the semi–major axis of the ellipse, and orbital phase.

For investigation of orbital structure, we consider a family of triaxial, perturbing potentials of the form

$$\Phi(x,y,z) = \frac{\Phi_0}{r_0^2}\left(x^2 + \frac{y^2}{b_\varphi^2} + \frac{z^2}{c_\varphi^2}\right), \tag{3}$$

Physically, this gravitational potential may be assumed to arise from a homogeneous, triaxial cluster whose centre coincides with the location of the BH. The shape parameters of the potential (viz. b_φ and c_φ) are related to those of the density (say, b_ϱ and c_ϱ) and can be found in the literature (c.f. Chandrasekhar 1969, Binney and Tremaine 1987). The case when the potential is axisymmetric may be studied by setting $b_\varphi = 1$; oblateness, or prolateness is achieved by choosing c_φ to be less than, or greater than unity, respectively. More realistic, inhomogeneous mass distributions could be considered, but at the expense of considerably greater algebraic complexity.

Let $\mu = (\Phi_0 a^2/I r_0^2)$ be a measure of the precession frequency for orbits with semi–major axis a. As in ST99 , it proves to be convenient to define a dimensionless time $\tau = \mu t$, Hamiltonian $H = (\overline{\Phi}/\mu I)$, and angular momenta $\ell = L/I$ and $\ell_z = L_z/I$. Averaging the potential of equation (3) over orbital phase, we obtain the dimensionless Hamiltonian

$$H(\ell,\ell_z,g,h) = \frac{1}{2}(5 - 3\ell^2) \; + \; \epsilon_c H_c(\ell,\ell_z,g) \; + \; \epsilon_b H_b(\ell,\ell_z,g,h)\,, \tag{4}$$

where

$$\begin{aligned} H_c(\ell,\ell_z,g) &= \frac{1}{4}\left(1 - \frac{\ell_z^2}{\ell^2}\right)\left(5 - 3\ell^2 - 5(1-\ell^2)\,C_{2g}\right), \\ H_b(\ell,\ell_z,g,h) &= \left(\frac{5}{2} - 2\ell^2\right)\left(C_g S_h + \frac{\ell_z}{\ell} C_h S_g\right)^2 + \frac{\ell^2}{2}\left(-S_g S_h + \frac{\ell_z}{\ell} C_h C_g\right)^2, \end{aligned} \tag{5}$$

and $\epsilon_b = (b_\varphi^{-2} - 1)$, $\epsilon_c = (c_\varphi^{-2} - 1)$. This is the model two degree–of–freedom Hamiltonian that will be studied in the rest of this paper. The equations of motion are,

$$\dot{\ell} = -\frac{\partial H}{\partial g}, \quad \dot{g} = \frac{\partial H}{\partial \ell}; \qquad \dot{\ell}_z = -\frac{\partial H}{\partial h}, \quad \dot{h} = \frac{\partial H}{\partial \ell_z}, \tag{6}$$

where the over–dots refer to $(d/d\tau)$. We integrate these equations numerically using a fourth–order, adaptive step–size, Runge–Kutta scheme, and take surfaces of section at constant values of either of the angles g, or h. Thus we follow the the deformations of Keplerian ellipses of fixed semi–major axes, over time scales much longer than the orbital periods. This technique will, in principle, reveal all of the phase space structure of the orbits. We will also supplement the numerical calcualtions with analytical estimates, wherever necessary. Note that such an assault on orbits in triaxial potentials would have been impossible, without the presence of the extra integral of motion that averaging promotes.

The Hamiltonian, given in equations (4) and (5), is a constant of motion. The orbital structure shows strong dependence on the value of this quantity, and henceforth we will refer to

it as the energy (E).[3] The simplest case is that of spherical symmetry, when $\epsilon_b = \epsilon_c = 0$. Then $H = (5 - 3\ell^2)/2$, so that $\dot{\ell} = \dot{\ell}_z = \dot{h} = 0$, and $\dot{g} = -3\ell$: the Keplerian ellipses precess rigidly, at a constant rate in a retrograde sense, maintaining an invariant orbital plane.

3. Orbits in the axisymmetric case

When $b_\varphi = 1$, the potential of equation (3) is axisymmetric about the z–axis. Hence L_z is conserved even for the unaveraged dynamics. For slow dynamics, this implies that ℓ_z is an integral of motion, and we are left with a one degree–of–freedom system, whose dynamics is obviously integrable. Setting $\epsilon_b = 0$, from equations (4) and (5), we obtain the following expression for the axisymmetric Hamiltonian,

$$H(\ell, g; \ell_z) = \frac{1}{2}(5 - 3\ell^2) \; + \; \frac{\epsilon_c}{4}\left(1 - \frac{\ell_z^2}{\ell^2}\right)\left(5 - 3\ell^2 - 5(1-\ell^2)\, C_{2g}\right) . \tag{7}$$

where ℓ_z is now regarded as a constant parameter. Note that ϵ_c is positive for oblate configurations, and negative for the prolate ones. The nodal precession rate given by

$$\dot{h} = \frac{\partial H}{\partial \ell_z} = -\frac{1}{2}\epsilon_c \frac{\ell_z}{\ell^2}\left[10\, S_g^2\,(1-\ell^2) + 2\ell^2\right] , \tag{8}$$

shows that the sign of $\dot{h}$ is the same as that of the product, $\epsilon_c \ell_z$; hence nodal precession is prograde (retrograde) for prolate (oblate) potentials.

To obtain a global picture of the dynamics, it is instructive to plot an $\ell - g$ (Poincaré) surface of section taken at some fixed value of h; Figures 2a and 2b show two such sections for oblate ($c_\varphi < 1$) and prolate ($c_\varphi > 1$) potentials, respectively. From these figures, it may be seen that the dynamics is organised around the elliptic fixed points, located at $g = \pi/2$ and $3\pi/2$. These are special orbits for which ℓ and g are constant in time, whereas h precesses steadily. Using the Hamiltonian of equation (7) in equations (6), it is straightforward to verify that these "fixed point" orbits are described by,

$$\ell^4 = \frac{5\epsilon_c \ell_z^2}{3 + 4\epsilon_c}, \qquad g = \pi/2 \;\text{ or }\; 3\pi/2\,, \qquad \dot{h} = 4\epsilon_c \ell_z - 5\epsilon_c \sqrt{\frac{3+4\epsilon_c}{5\epsilon_c}} . \tag{9}$$

Before we proceed to examine the dynamics in more detail, let us pause to regard the Poincaré section from a different perspective. For the completely integrable dynamics of the axisymmetric problem, one may invert equation (7), to express ℓ_z as a function of ℓ, g and H,

$$\ell_z = \pm \ell \left(1 - \frac{1}{\epsilon_c}\, \frac{6\ell^2 + 4H - 10}{\ell^2(5C_{2g} - 3) + 5(1 - C_{2g})}\right)^{1/2} . \tag{10}$$

[3]The Kepler energy is constant. The energy E refers to the average value of Φ over the orbit in question.