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Modelling the non-linear gravitational clustering in the expanding Universe

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ABSTRACT

The gravitational clustering of collisionless particles in an expanding universe is modelled using some simple physical ideas. I show that it is indeed possible to understand the non-linear clustering in terms of three well-defined regimes: (1) the linear regime; (2) the quasi-linear regime which is dominated by scale-invariant radial infall; and (3) the non-linear regime, dominated by non-radial motions and mergers. Modelling each of these regimes separately, I show how the non-linear two-point correlation function can be related to the linear correlation function in hierarchical models. This analysis leads to results which are in good agreement with numerical simulations, thereby providing an explanation for numerical results. The ideas presented here will also serve as a powerful analytical tool to investigate non-linear clustering in different models. Several implications of the results are discussed.

Key words: cosmology: theory – dark matter – large-scale structure of Universe.

1 INTRODUCTION

The driving force behind the formation of large-scale structures in the Universe is the gravitational field produced by density fluctuations. Overdense regions accrete matter at the expense of underdense regions, allowing inhomogeneities in the Universe to grow. Observations suggest that the material content of the Universe is dominated by dark matter, likely to be made of collisionless elementary particles. In that case, the gravitational force is mainly due to these particles and, to first approximation, we can ignore the complications arising from baryonic physics. The evolution of density perturbations is then governed purely by the gravitational force.

When these density perturbations are small, it is possible to study their evolution using linear theory. Once the density contrast becomes comparable to unity, however, linear perturbation theory breaks down and one must use *N*-body simulations to study the growth of perturbations. While these simulations are of some value in making concrete predictions for specific models, they do not provide clear physical insight into the process of non-linear gravitational dynamics. To obtain such an insight into this complex problem, it is necessary to model the gravitational clustering of collisionless particles using simple physical concepts. I shall develop one such model in this paper, which – in spite of its extreme simplicity – reproduces the simulation results for hierarchical models fairly accurately. Further, this model also provides insight into the clustering process and can be modified to take into account more complicated situations.

The paradigm for understanding the clustering is based on the well-known behaviour of a spherically symmetric overdense region in the universe. In the behaviour of such a region, one can identify three different regimes of interest. (1) In the early stages of the evolution, when the density contrast is small, the evolution is described by linear theory. (2) Each of the spherical shells with an initial radius x_i can be parametrized by the mass contained inside the shell, $M(x_i)$, and the energy, $E(x_i)$, for the particular shell. Each shell will expand to a maximum radius $x_{\max} \propto M/|E|$ and then turn around and collapse. Such a spherical collapse and resulting evolution allow a self-similar description (Filmore & Goldreich 1984) in which each shell acts as though it has an effective radius proportional to $x_{\max}$ (Bertschinger 1985). This will be the quasi-linear phase. (3) The spherical evolution will break down during the later stages, for several reasons. First of all, non-radial motions will arise as a result of amplification of deviations from spherical symmetry. Secondly, the existence of substructure will influence the evolution in a non-spherically symmetric way. Finally, in the real Universe, there will be merging of such clusters [each of which could have been the centre of a spherical overdense region in the beginning] which will again destroy the spherical symmetry. This will be the non-linear phase.

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2 THE MODEL

The description given above is sufficiently vague and sufficiently well known that one may suspect that it cannot lead to any insight into the problem. In particular, the real Universe is not spherical. I will show that it is, however, possible to model the above process in a manner that allows direct generalization to the real Universe.

To do this we will begin by studying the evolution of a system starting from Gaussian initial fluctuations with an initial power spectrum, $P_{\mathrm{in}}(k)$. The Fourier transform of the power spectrum defines the correlation function $\xi(a,x)$, where $a \propto t^{2/3}$ is the expansion factor in a universe with $\Omega=1$. It is more convenient for our purpose to work with the average correlation function inside a sphere of radius x, defined by

$$\bar{\xi}(a,x) \equiv \frac{3}{x^3}\int_0^x \xi(a,y)\, y^2 \,\mathrm{d}y. \tag{1}$$

In the linear regime we have $\bar{\xi}_{\mathrm{L}}(a,x) \propto a^2 \bar{\xi}_{\mathrm{in}}(a_i,x)$. In the quasi-linear and non-linear regimes, we would like to have a prescription that relates the exact $\bar{\xi}$ to the mean correlation function calculated from the linear theory. One might have naïvely imagined that $\bar{\xi}(a,x)$ should be related to $\bar{\xi}_{\mathrm{L}}(a,x)$, but one can convince oneself that the relationship is likely to be non-local, by using the following analysis.

Recall that the conservation of pairs of particles gives an exact equation satisfied by the correlation function (Peebles 1980):

$$\frac{\partial \xi}{\partial t} + \frac{1}{ax^2}\frac{\partial}{\partial x}[x^2(1+\xi)\,v]=0, \tag{2}$$

where $v(t,x)$ denotes the mean relative velocity of pairs at separation x and epoch t. Using the mean correlation function $\bar{\xi}$ and a dimensionless pair velocity $h(a,x) \equiv -(v/\dot{a}x)$, equation (2) can be written as

$$\left(\frac{\partial}{\partial \ln a} - h\frac{\partial}{\partial \ln x}\right)(1+\bar{\xi}) = 3h(1+\bar{\xi}). \tag{3}$$

This equation can be simplified by first introducing the variables

$$A=\ln a, \qquad X=\ln x, \qquad D(X,A)=\ln(1+\bar{\xi}), \tag{4}$$

in terms of which we have (Nityananda & Padmanabhan 1994)

$$\frac{\partial D}{\partial A} - h(A,X)\frac{\partial D}{\partial X} = 3h(A,X). \tag{5}$$

Introducing further a variable $F=D+3X$, equation (5) can be written in a remarkably simple form as

$$\frac{\partial F}{\partial A} - h(A,X)\frac{\partial F}{\partial X} = 0. \tag{6}$$

The characteristic curves to this equation – on which F is a constant – are determined by $(\mathrm{d}X/\mathrm{d}A) = -h(X,A)$ which can be integrated if h is known. Note, however, that the characteristics satisfy the condition

$$F=3X+D=\ln[x^3(1+\bar{\xi})]=\text{constant} \tag{7}$$

or, equivalently,

$$x^3(1+\bar{\xi})=l^3, \tag{8}$$

where l is another length-scale. When the evolution is linear at all the relevant scales, $\bar{\xi} \ll 1$ and $l \approx x$. As clustering develops, $\bar{\xi}$ increases and x becomes considerably smaller than l. It is clear that the behaviour of clustering at some scale x is determined by the original *linear* power spectrum at the scale l through the 'flow of information' along the characteristics. This suggests that *we should actually try to express the true correlation function* $\bar{\xi}(a,x)$ *in terms of the linear correlation function* $\bar{\xi}_{\mathrm{L}}(a,l)$ *evaluated at a different point.*

Let us see how we can do this starting from the quasi-linear regime. Consider a region surrounding a density peak in the linear stage, around which we expect the clustering to take place. It is well known that the density profile around this peak can be described by

$$\rho(x) \approx \rho_{\mathrm{bg}}[1+\xi(x)]. \tag{9}$$

Hence the initial mean density contrast scales with the initial shell radius l as $\bar{\delta}_i(l) \propto \bar{\xi}_{\mathrm{L}}(l)$ in the initial epoch, when the linear theory is valid. This shell will expand to a maximum radius of $x_{\max} \propto l/\bar{\delta}_i \propto l/\bar{\xi}_{\mathrm{L}}(l)$. In scale-invariant radial collapse models, each shell may be approximated as contributing with an effective radius which is proportional to $x_{\max}$. Taking the final effective radius x as proportional to $x_{\max}$, the final mean correlation function will be

$$\bar{\xi}_{\mathrm{QL}}(x) \propto \rho \propto \frac{M}{x^3} \propto \frac{l^3}{[l^3/\bar{\xi}_{\mathrm{L}}(l)]^3} \propto \bar{\xi}_{\mathrm{L}}(l)^3. \tag{10}$$

That is, the final correlation function $\bar{\xi}_{\mathrm{QL}}$ at x is the cube of the initial correlation function at l where $l^3 \propto x^3\bar{\xi}_{\mathrm{L}}^3 \propto x^3\bar{\xi}_{\mathrm{QL}}(x)$. This is in the form demanded by equation (8) if $\bar{\xi} \gg 1$. *Note that we did not assume that the initial power spectrum is a power law to get this result.*

If the initial power spectrum is a power law, with $\bar{\xi}_{\mathrm{L}} \propto x^{-(n+3)}$, then we find that

$$\bar{\xi}_{\mathrm{QL}} \propto x^{-3(n+3)/(n+4)}. \tag{11}$$

[If the correlation function in linear theory has the power-law form $\bar{\xi}_{\mathrm{L}} \propto x^{-\alpha}$ then the process described above changes the index from α to $3\alpha/(1+\alpha)$. We shall comment more about this aspect at the end of the paper.] For the power-law case, the same result can be obtained by more explicit means. For example, in power-law models the energy of the spherical shell will scale with its radius as some power which we write as $E \propto x_i^{2-b}$. Since $M \propto x_i^3$, it follows that the maximum radius reached by the shell scales as $x_{\max} \propto (M/E) \propto x_i^{1+b}$. Taking the effective radius as $x=x_{\mathrm{eff}} \propto x_i^{1+b}$, the final density scales as

$$\rho \propto \frac{M}{x^3} \propto \frac{x_i^3}{x_i^{3(1+b)}} \propto x_i^{-3b} \propto x^{-3b/(1+b)}. \tag{12}$$

In this quasi-linear regime, $\bar{\xi}$ will scale like the density and we get $\bar{\xi}_{\mathrm{QL}} \propto x^{-3b/(1+b)}$. The index b can be related to n by assuming that the evolution starts at a moment when linear theory is valid. The gravitational potential energy (or the kinetic energy) scales as $E \propto x_i^{-(n+1)}$ in the linear theory. This may be seen as follows. The power spectrum for the velocity field, $P_v(k)$ in the linear regime, is related to that of the density by $P_v \propto P(k)/k^2 \propto k^{n-2}$. Hence the contribution to v^2 in each logarithmic scale in k-space is $k^3 P_v/2\pi^2 \propto k^{n+1} \propto x^{-(n+1)}$. Similarly, the gravitational potential energy due to *fluctuations* is

$$\phi \propto \int_0^x 4\pi y^2 \mathrm{d}y \, \frac{\xi(y)}{y} \propto x^2 \xi(x) \propto x^{-(n+1)}. \tag{13}$$

The total energy in the initial configuration therefore scales as $x_i^{-(n+1)}$, allowing us to determine $b = n+3$. This shows that the correlation function in the quasi-linear regime is the one given by equation (11).

The case with a power-law initial spectrum has no intrinsic scale, if $\Omega = 1$. It follows that the evolution has to be self-similar and $\bar{\xi}$ can only depend on $q = xa^{-2/(n+3)}$. This allows us to determine the a-dependence of $\bar{\xi}_{\mathrm{QL}}$ by substituting q for x in equation (11). We find

$$\bar{\xi}_{\mathrm{QL}}(a, x) \propto a^{6/(n+4)} x^{-3(n+3)/(n+4)}. \tag{14}$$

Direct algebra shows that

$$\bar{\xi}_{\mathrm{QL}}(a, x) \propto [\bar{\xi}_{\mathrm{L}}(a, l)]^3, \tag{15}$$

reconfirming the local dependence on a and the non-local dependence on the spatial coordinate. This result has no trace of the original assumptions (spherical evolution, scale-invariant spectrum...) left in it, and hence one strongly suspects that it will have a far more general validity.

Let us now proceed to the third and non-linear regime. If we ignore the effect of mergers, then it seems reasonable that virialized systems should maintain their densities and sizes in proper coordinates, i.e. the clustering should be 'stable'. This would require the correlation function to have the form $\bar{\xi}_{\mathrm{NL}}(a, x) = a^3 F(ax)$ (the factor a^3 arising from the decrease in background density). From our previous analysis we expect this to be a function of $\bar{\xi}_{\mathrm{L}}(a, l)$ where $l^3 \approx x^3 \bar{\xi}_{\mathrm{NL}}(a, x)$. Let us write this relation as

$$\bar{\xi}_{\mathrm{NL}}(a, x) = a^3 F(ax) = U[\bar{\xi}_{\mathrm{L}}(a, l)], \tag{16}$$

where $U[z]$ is an unknown function of its argument which needs to be determined. Since the linear correlation function evolves as a^2 we know that we can write $\bar{\xi}_{\mathrm{L}}(a, l) = a^2 Q[l^3]$ where Q is some known function of its argument. (We are using l^3 rather than l in defining this function just for future convenience of notation.) In our case $l^3 = x^3 \bar{\xi}_{\mathrm{NL}}(a, x) = (ax)^3 F(ax) = r^2 F(r)$, where we have changed variables from (a, x) to (a, r) with $r = ax$. Equation (16) now reads

$$a^3 F(r) = U[\bar{\xi}_{\mathrm{L}}(a, l)] = U\{a^2 Q[l^3]\} = U\{a^2 Q[r^3 F(r)]\}. \tag{17}$$

Consider this relation as a function of a at constant r. Clearly we need to satisfy $U[c_1 a^2] = c_2 a^3$ where c_1 and c_2 are constants. Hence we must have

$$U[z] \propto z^{3/2}. \tag{18}$$

Thus in the extreme non-linear regime we should have

$$\bar{\xi}_{\mathrm{NL}}(a, x) \propto [\bar{\xi}_{\mathrm{L}}(a, l)]^{3/2}. \tag{19}$$

[Another way of deriving this result is to note that if $\bar{\xi} = a^3 F(ax)$ then $h = 1$. Integrating equation (5) with appropriate boundary conditions leads to equation (19).] Once again we did not need to invoke the assumption that the spectrum is a power law. If it *is* a power law, then we get

$$\bar{\xi}_{\mathrm{NL}}(a, x) \propto a^{(3-\gamma)} x^{-\gamma}; \qquad \gamma = \frac{3(n+3)}{(n+5)}. \tag{20}$$

This result is based on the assumption of 'stable clustering' and was originally derived by Peebles (1965). It can be directly verified that the right-hand side of this equation can be expressed in terms of q alone, as we would have expected.

Putting all our results together, we find that the non-linear mean correlation function can be expressed in terms of the linear mean correlation function by the relation

$$\bar{\xi}(a, x) = \begin{cases} \bar{\xi}_{\mathrm{L}}(a, l) & (\text{for } \bar{\xi}_{\mathrm{L}} < 1, \bar{\xi} < 1) \\ \bar{\xi}_{\mathrm{L}}(a, l)^3 & (\text{for } 1 < \bar{\xi}_{\mathrm{L}} < 5.85, 1 < \bar{\xi} < 200) \\ 14.14 \bar{\xi}_{\mathrm{L}}(a, l)^{3/2} & (\text{for } 5.85 < \bar{\xi}_{\mathrm{L}}, 200 < \bar{\xi}). \end{cases} \tag{21}$$

The numerical coefficients have been determined by continuity arguments. We have assumed the linear result to be valid up to $\bar{\xi} = 1$ and the virialization to occur at $\bar{\xi} \approx 200$ which is a result arising from the spherical model. The exact values of the numerical coefficients can be obtained only from simulations.

The true test of such a model, of course, is N-body simulations and, remarkably enough, simulations are very well represented by relations of the above form. Fig. 1 shows the results of a cold dark matter (CDM) simulation based on the investigations carried out by Padmanabhan et al. (1995). These data can be fitted by the relations (Bagla & Padmanabhan 1993)

$$\bar{\xi}(a, x) = \begin{cases} \bar{\xi}_{\mathrm{L}}(a, l) & (\text{for } \bar{\xi}_{\mathrm{L}} < 1, \bar{\xi} < 1) \\ \bar{\xi}_{\mathrm{L}}(a, l)^3 & (\text{for } 1 < \bar{\xi}_{\mathrm{L}} < 5, 1 < \bar{\xi} < 125) \\ 11.2 \bar{\xi}_{\mathrm{L}}(a, l)^{3/2} & (\text{for } 5 < \bar{\xi}_{\mathrm{L}}, 125 < \bar{\xi}). \end{cases} \tag{22}$$

[The fact that numerical simulations show a correlation between $\bar{\xi}(a, x)$ and $\bar{\xi}_{\mathrm{L}}(a, l)$ was originally pointed out by Hamilton et al. (1991) who, however, tried to give a multi-parameter fit to the data. This fit has somewhat obscured the simple physical interpretation of the result though has the virtue of being very accurate for numerical work.]

A comparison of equations (21) and (22) shows that the physical processes which operate at different scales are well represented by our model. In other words, the processes described in the quasi-linear and non-linear regimes for an *individual* lump still model the *average* behaviour of the Universe in a statistical sense. It must be emphasized that the key point is the 'flow of information' from l to x which is an exact result. Only when the results of the specific model are recast in terms of suitably chosen variables do we get a relation that is of general validity. It would have been, for example, incorrect to use a spherical model to obtain a relation between linear and non-linear densities at the same location or to model the function h. With hindsight, it is clear why such attempts have not succeeded in the past.

In our modelling, we have characterized the quasi-linear and non-linear regimes along the following lines. In the

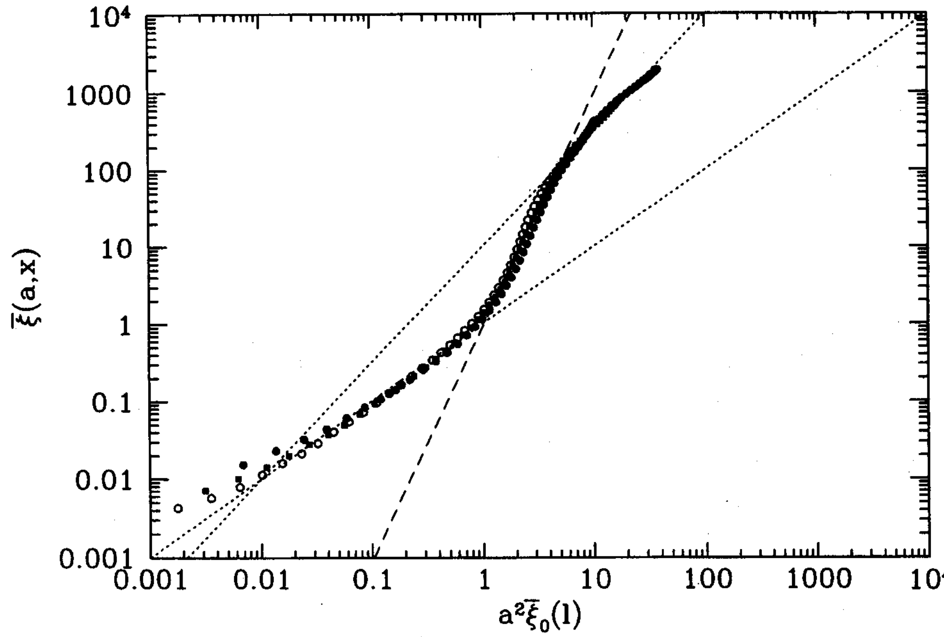

Figure 1. Plot of $\bar{\xi}(a, x)$ against $a^2\bar{\xi}_0(l)$ for a CDM model. The slopes in the three different regimes are indicated. The data points are for three different redshifts (0.1, 0.5 and 1.0) and are based on the simulations described by Padmanabhan et al. (1995).

quasi-linear regime, shells of different radii turn around and collapse, contributing with an effective radius which is proportional to the turn-around radius. Such a radial, self-similar evolution will eventually give way to virialization when non-radial motion starts to contribute significantly. Since the spherical top hat model virializes around a density contrast of 200, we had assumed in (21) that the transition from the quasi-linear to the non-linear regime takes place at this value. It is interesting to note that such a simple assumption is vindicated reasonably well by simulations.

It may be noted that, to obtain the result in the non-linear regime, we needed to invoke the assumption of stable clustering which has not been deduced from any fundamental considerations. In the case that mergers of structures were important, one would consider this assumption to be suspect (Padmanabhan et al. 1995). We can, however, generalize the above argument in the following manner. If the virialized systems have reached stationarity in the statistical sense, the function h – which is the ratio between two velocities – should reach some constant value. In that case, one can integrate equation (6) and obtain the result $\bar{\xi}_{\rm NL} = a^{3h}F(a^h x)$. A similar argument will now show that

$$\bar{\xi}_{\rm NL}(a,x) \propto [\bar{\xi}_{\rm L}(a,l)]^{3h/2} \qquad (23)$$

in the general case. For the power-law spectra, one would obtain

$$\bar{\xi}(a,x) \propto a^{(3-\gamma)h}x^{-\gamma}; \qquad \gamma = \frac{3h(n+3)}{2+h(n+3)}. \qquad (24)$$

Simulations are not accurate enough to fix the value of h; in particular, the asymptotic value of h could depend on n within the accuracy of the simulations. It may be possible to determine this dependence by modelling mergers in some simplified form.

We conclude with two interesting speculations regarding the non-linear stage. If $h = 1$ asymptotically, the correlation function at the extreme non-linear end depends on the linear index n. One may feel that physics at the highly non-linear end should be independent of the linear spectral index n. This will be the case if the asymptotic value of h satisfies the scaling

$$h = \frac{3c}{n+3} \qquad (25)$$

at the non-linear end with some constant c. Only high-resolution numerical simulations can test this conjecture that $h(n+3) = \text{constant}$.

Also note that the radial, scale-invariant infall described in the quasi-linear regime has the effect of changing the linear correlation function $\bar{\xi}_{\rm L} = x^{-(n+3)} = x^{-b}$ to the quasi-linear correlation function $\bar{\xi}_{\rm QL} = x^{-3b/(1+b)}$. It is amusing to ask what will be the effect of iterating this process N times. It is easy to see that the index after N iterations can be expressed in the form

$$\gamma_N = \frac{A_N b}{1 + B_N b}; \qquad A_N = 3^N; \qquad B_N = \frac{3^N - 1}{2}. \qquad (26)$$

The fixed point, of course, is $\gamma_\infty = 2$ which is the only non-trivial fixed point for such an evolution (with the other, trivial, fixed point being zero). If one could model the evolution as a repeated application of this process, one would expect a continuum of scaling relations with the evolution being driven to a singular isothermal sphere. The quasi-linear evolution does not change the x^{-2} profile, a result which was noted earlier by Bagla & Padmanabhan (1995). It is not clear whether the clustering can indeed be modelled using equation (26).

The relations obtained in this paper will, of course, have certain limitations on their validity. To begin with, we do expect a weak n-dependence in these relations as a result of averaging over peaks of different heights. This has been discussed using a simple analytic model, as well as numerically, by Padmanabhan et al. (1995) (see Mo, Jain & White 1995 for a similar discussion). Secondly, the asymptotic behaviour will be sensitive to the value of Ω. When $\Omega < 1$, structures 'freeze out' during the late stages of evolution and 'stable clustering' is likely to be a reasonable assumption. The behaviour of scaling relations in the low-density models was previously considered by Peacock & Dodds (1994). It should be possible to obtain these results by studying the density profiles of haloes which collapse in the low-density universes. Finally, in models like hot dark matter (HDM), small-scale power is generated by the breaking of long-wavelength modes, and the evolution is best modelled by instability of shell-like regions in the universe. In general, the paradigm presented here suggests modelling the non-linear universe as a superposition of suitable 'units' with evolving density profiles. These 'units' are expected to evolve independently in the non-linear phase just as plane waves evolve independently in the linear regime (Padmanabhan 1995). These issues are under investigation.

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