

A formal analysis of two dimensional gravity.

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ABSTRACT

Several investigations in the study of cosmological structure formation use numerical simulations in both two and three dimensions. In this paper we address the subtle question of ambiguities in the nature of two dimensional gravity in an expanding background. We take a detailed and formal approach by deriving the equations describing gravity in $(D + 1)$ dimensions using the action principle of Einstein. We then consider the Newtonian limit of these equations and finally obtain the necessary fluid equations required to describe structure formation. These equations are solved for the density perturbation in both the linearised form and in the spherical top hat model of nonlinear growth. We find that, when the special case of $D = 2$ is considered, no structures can grow. We therefore conclude that, within the frame work of Einstein's theory of gravity in $(2 + 1)$ dimensions, formation of structures cannot take place. Finally, we indicate the different possible ways of getting around this difficulty so that growing structures can be obtained in two dimensional cosmological gravitational simulations and discuss their implications.

Subject headings: Cosmology : theory – dark matter, large scale structure of the Universe

1. Introduction

The dominant paradigm for the generation of the observed large scale structure in the universe is based on the idea that the gravitational instability amplifies the initially small density perturbations. The equations describing the growth of density perturbations in the highly nonlinear stage are analytically intractable and hence large scale numerical simulations are resorted to for exploration of this regime.

These N-Body simulations require large amount of computing resources (CPU, memory and storage space) if one is to get the requisite amount of dynamical range, *i.e.* good resolution in force and mass, large range in values of density etc. Time and resource constraints usually limit our ability to probe structure formation issues more deeply using computers, once the required resources are at the limits of technological feasibility. The key parameter which decides the feasibility level of numerical simulations is the size of a simulation, which — in turn — is characterised by: (i) The number of particles in the simulation volume, which is generally specified as N^D , where D is the dimensionality of the simulation (usually 2 or 3) and (ii) The number of mesh points (M) along any axis which determines the minimum length scale at which the results can be treated as reliable indicators of physical phenomena. In order to create a simulation volume that is a fair sample of the universe one needs about 10^7 particles and in order to have a high enough force resolution one needs to increase the number of grid points adequately (for a review, see eg. Bagla and Padmanabhan, 1996). Let us suppose we have 160^3 particles in three dimensions and our grid is 160 units on a side. Then, for the same amount of computational resources one can simulate a two dimensional situation with 2048^2 particles on a 2048^2 grid ($160^3 \approx 2048^2$). So, if we can extract useful (*i.e.* generalisable to the three

dimensional case) physical insights from results in two dimensions, then simulations of two dimensional gravity will be helpful. This hope has led to a large number of two dimensional simulations in the field of gravitational clustering (see eg Munshi et al, 1998, Bagla et al, 1998, Sathyaprakash et al, 1995).

There are three ways in which two dimensional gravity can be operationally defined and corresponding numerical simulations undertaken: (i) Consider a system of point particles in a three dimensional (expanding) background with the force of interaction being given by Newton’s law of gravitation (*i.e* $F \propto 1/r^2$). The initial positions and velocities of the particles are such that they all lie in the same plane and all the velocities are in the plane *i.e* there are no velocity components orthogonal to the plane. This system will evolve with the particles being confined to the plane with clustering occurring in the plane. Thus, we have a two dimensional clustering scenario. (ii) Another system we can consider consists of infinite, thin ‘needles’ located parallel to each other. The mass elements in the ‘needles’ still interact through the $1/r^2$ force, but the interaction between ‘needles’ (obtained by summing over the mass elements) is given by a $1/r$ force. In this case as well, the background space expands uniformly in three dimensions. The two dimensional clustering that we study is the clustering of these ‘needles’, examined by taking a slice orthogonal to the ‘needles’. (iii) The third possibility involves writing down the Einstein’s equations in two dimensions, finding the homogeneous and isotropic cosmological solution, taking the Newtonian limit (in which the potentials due to density perturbations and background metric can be superposed), finding the corresponding perfect fluid equations and solving them. In this case, we will have a background spacetime expanding in *two* dimensions unlike the other two cases. (There is yet another, fourth possibility, which can be defined only in an *ad hoc* manner. We will present this in the end.)

The first case is not of much interest for cosmological simulations since the system is anisotropic, confined to a single plane and the clustering takes place in a specific plane *only because the initial conditions were specifically selected* to give this result. Hence we will not discuss it and it is mentioned here only for completeness. The way simulations in two dimensions are carried out usually is by simulating the second case and then defining the ‘particles’ as the intersection of the ‘needles’ with any plane orthogonal to them. In this case – as in the first case – the background spacetime expands in three dimensions (for a flat dust dominated universe the scale factor $a(t)$ goes as $t^{2/3}$). The clustering that we observe and quantify in two dimensions, is basically the clustering of these needles in three dimensions. But this is also an anisotropic situation since the background spacetime is expanding in three dimensions. As an alternative we may try to write down the equations derived from Einstein’s equations in two dimensions and examine how a system of particles interacting in a two dimensional expanding background spacetime is to be simulated.

In the rest of this paper we shall examine the third alternative. We will take a very general approach by developing the formal theory of $(D + 1)$ gravity and considering $D = 2$ as a special case. (For some previous work on 2D gravity see eg. Deser, Jackiw and t’Hooft, 1984 and references cited therein).

The basic layout of the paper is as follows: In section (2) we first define the analogue of Einstein’s gravity in $(D + 1)$ dimensions, discuss the Newtonian limit and its corresponding Poisson’s equation and then go on to analyse the Friedmann metric in $(D + 1)$ dimensions for a flat universe with dust. In section (3) we write down the D dimensional fluid equations and obtain the equation governing the density perturbations. This equation is then solved in the linear approximation and using the Spherical Top Hat model. Then in section (4) and section (5) we specialise to the cases $D = 3$ and $D = 2$ respectively. Finally, in section (6) we summarise and discuss the implications of the results obtained in the earlier sections.

2. Formal $(D + 1)$ dimensional gravity

We start our analysis of $(D + 1)$ dimensional (1 time dimension and D space dimensions) gravity from the action principle which we assume has the same form as that used in $(3 + 1)$ dimensions. Using this action we construct the corresponding $(D + 1)$ dimensional Einstein equations which will be subsequently used to study structure formation and spherical collapse. Thus, we begin with the action principle,

$$S = S_g + S_m = -\frac{c^4}{2\kappa(D)} \int d^{(D+1)}x R \sqrt{|g|} + \int d^{(D+1)}x \mathcal{L}_m \quad (1)$$

where S_g is the action for the gravitational field, S_m is the action for the matter fields, g is the determinant of the metric tensor g_{ik} , R is the Ricci scalar, $\kappa(D)$ is a suitable constant which can be, in general, a function of D (when $D = 3$, $\kappa = 8\pi G$, G being the usual gravitational constant) and \mathcal{L}_m is the lagrangian density for the matter fields. The metric signature we adopt is $(+, -, -, -, \dots, -)$. We adopt the following convention regarding indices. Latin alphabets $i, j, k \dots$ are used to represent $(D + 1)$ dimensional indices which take on the values $(0, 1, 2, \dots, D)$ while greek letters are used to denote D dimensional indices taking on the values $(1, 2, \dots, D)$. Varying the total action S with respect to g_{ik} we obtain Einstein's equations,

$$G_{ik} \equiv R_{ik} - \frac{1}{2} g_{ik} R = \frac{\kappa(D)}{c^4} T_{ik} \quad (2)$$

where T_{ik} is the energy momentum tensor of the matter fields and is defined by

$$\frac{1}{2} \sqrt{|g|} T_{ik} = \frac{\partial(\sqrt{|g|} \mathcal{L}_m)}{\partial g^{ik}} - \frac{\partial}{\partial x^l} \left(\frac{\partial(\sqrt{|g|} \mathcal{L}_m)}{\partial(\partial g^{ik}/\partial x^l)} \right) \quad (3)$$

G_{ik} is the Einstein tensor and R_{ik} is the usual Ricci tensor. Note that the $(1/2)$ that appears in Einstein's equations arises due to the square root in the term $\sqrt{|g|}$ and has nothing to do with the dimension of the spacetime.

We will use the above equations in the subsections to follow. In subsection (2.1), we will study the Newtonian limit of the metric tensor and then construct the corresponding Poisson equation that relates the Newtonian gravitational field ϕ to the matter density ρ . Then, in subsection (2.2), we analyse the Friedmann metric in $(D + 1)$ dimensions and the corresponding Newtonian limit of this metric is derived.

2.1. Poisson equation in D dimensions

In this section, we derive the Poisson equation relating the gravitational potential ϕ to the matter density ρ . We keep all factors of c since the Newtonian limit involves the limit $c \rightarrow \infty$. The analysis here follows closely the treatment in Landau and Lifshitz, 1975. Consider the metric.

$$ds^2 = \left(1 + \frac{2\phi}{c^2} \right) c^2 dt^2 - dl^2 \quad (4)$$

where ϕ is a function of space and time with dimensions of velocity square. The term dl^2 is the D dimensional spatial line element given by the formula

$$dl^2 = \sum_{\alpha=1}^D (dx^\alpha)^2 \quad (5)$$

We will now show that the metric written above is the Newtonian limit of Einstein's gravitational equations. We do this by showing that, in the Newtonian limit, the equation of motion of a particle follows Newton's force law with the force $-m\nabla\phi$... In relativistic mechanics, the motion of a particle of mass m is determined by the action function S

$$S = -mc \int ds = -mc \int c dt \sqrt{\left(1 + \frac{2\phi}{c^2} - \frac{v^2}{c^2}\right)} \quad (6)$$

where v^2 is the square of the magnitude of the particle's velocity in D dimensions. In arriving at the second equality we have used the form of the metric in equation (4). In the limit $c \rightarrow \infty$, the action S can be approximated as

$$S \approx -mc^2 \int dt \left(1 + \frac{2\phi - v^2}{2c^2}\right) = \int dt \left(-mc^2 + \frac{1}{2}mv^2 - m\phi\right) \quad (7)$$

The equation of motion for the particle can be immediately written down and we obtain

$$m \frac{d\mathbf{v}}{dt} = -m\nabla\phi \quad (8)$$

where \mathbf{v} is the velocity vector in D space dimensions. Thus, Newton's force law is recovered in the non-relativistic limit and from this we conclude that the metric given in equation (4) is the Newtonian limit of Einstein's gravitational equations with ϕ acting as the Newtonian gravitational potential.

The relation between ϕ and the mass density ρ is found by taking the $c \rightarrow \infty$ limit of Einstein's equations. This procedure, in $(3+1)$ gravity, determines the constant $\kappa(D)$ since the Poisson equation is explicitly known. In other dimensions however, a definite criterion, like Gauss's law for example, must be imposed in order to determine $\kappa(D)$. We now consider the limit $c \rightarrow \infty$ of Einstein's equations in the following manner. First, we use the line element given in equation (4) to calculate the Ricci tensor component R_{00}

$$R_{00} = \frac{1}{c^2} \frac{1}{c^2 + 2\phi} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{c^2} \partial_\mu \partial^\mu \phi \quad (9)$$

where the summation convention has been invoked in the above equation and the sum over μ is only over the *spatial* dimensions. Then, using equation (2), we obtain,

$$R = -\frac{2\kappa(D)}{c^4(D-1)} T \quad (10)$$

where we have used the fact that $g_{ik}g^{ik} = D+1$ and assumed $D \neq 1$. Thus, Einstein's equations can be written in the equivalent form,

$$R_{ik} = \frac{\kappa(D)}{c^4} \left(T_{ik} - \frac{1}{D-1} g_{ik} T \right). \quad (11)$$

where T is the trace of T_{ik} . The energy momentum tensor of point particles is $T_{ik} = \rho c^2 u_i u_k$ where ρ is the mass density and u_i is the four velocity. Since, in the non-relativistic limit, the macroscopic motion is slow, the space components of u_i can be neglected and only the time component should be retained. Therefore, $u_0 = \sqrt{g_{00}}$ and $u_\mu \approx 0$ for all μ . Consequently, only $T_{00} = g_{00}\rho c^2$ is non-zero. Substituting for T_{ik} into equation (11) and using the expression in equation (9) for R_{00} , we get,

$$\frac{1}{c^2} \partial_\mu \partial^\mu \phi = -\left(\frac{D-2}{D-1}\right) \left(1 + \frac{2\phi}{c^2}\right) \frac{\kappa(D)}{c^4} \rho c^2 \approx -\left(\frac{D-2}{D-1}\right) \frac{\kappa(D)}{c^2} \rho \quad (12)$$

That is,

$$\nabla^2 \phi = \left(\frac{D-2}{D-1}\right) \kappa(D) \rho \quad (13)$$

where ∇^2 is the usual Laplacian operator in D dimensions. This equation is the Poisson equation in D dimensions. Note that when substituting for the value of R_{00} from equation (9), we neglected the first term in comparison with the second since the former is of order c^{-4} while the latter is only of order c^{-2} .

2.2. Friedmann Universe in $(1 + D)$ dimensions

Let us next consider the maximally symmetric Robertson-Walker metric in $(D + 1)$ dimensions, specialising to flat space with $k = 0$ (we set $c = 1$ in this and in subsequent sections),

$$ds^2 = dt^2 - a^2(t)dl^2 \quad (14)$$

where $a(t)$ is the scale factor and dl^2 is the D dimensional line element given in equation (5). Calculating the components of the Einstein tensor, we obtain,

$$\begin{aligned} G_{00} &= \frac{D(D-1)\dot{a}^2}{2a^2} \\ G_{11} = G_{22} = \dots = G_{DD} &= (1-D)a\ddot{a} + \left(1 - \frac{D}{2}\right)(D-1)\dot{a}^2 \end{aligned} \quad (15)$$

where \dot{a} stands for $da(t)/dt$ and similarly \ddot{a} is the second derivative of $a(t)$ with respect to time. All the other components are zero. For consistency, the energy momentum tensor must have the form $T^i_k = \text{diag}(\rho, -p, -p, -p, \dots)$ where ρ is the matter density and p is the pressure.

Substituting in Einstein's equations, we obtain,

$$\frac{D(D-1)\dot{a}^2}{2a^2} = \kappa(D)\rho \quad (16)$$

$$\frac{\ddot{a}}{a} + \frac{D-2}{2} \frac{\dot{a}^2}{a^2} = -\frac{\kappa(D)p}{D-1} \quad (17)$$

The above two equations, together with the equation of state in the form $p = p(\rho)$ completely specify the system. Solving these three equations, we can determine $a(t)$, $\rho(t)$ and subsequently $p(t)$. Combining equations (16,17), we get the single equation,

$$\frac{\ddot{a}}{a} = -\frac{\kappa(D)}{D(D-1)} [(D-2)\rho + Dp]. \quad (18)$$

We now specialise to the case of pressureless dust with the equation of state $p = 0$. Using the principle of conservation of energy and momentum expressed by the relation

$$T^k_{i;k} = 0 \quad (19)$$

we derive the following relation,

$$\begin{aligned} T^k_{i;k} &= \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^k} \left(\sqrt{|g|} T^k_i \right) - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} T^{kl} \\ &= \frac{1}{a^D} \frac{\partial}{\partial x^k} (a^D T^k_i) - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} T^{kl} = 0 \end{aligned} \quad (20)$$

Noting that the only non-zero component of T^i_k is $T^0_0 = \rho$ we finally get

$$\rho a^D = \text{constant} = C_1 \quad (21)$$

Substituting the above relation into equation (16), and solving for $a(t)$ and subsequently for $\rho(t)$, we obtain the solutions,

$$a(t) = \left(\frac{D\kappa(D)C_1}{2(D-1)} \right)^{1/D} t^{2/D} \quad ; \quad \rho(t) = \left(\frac{2(D-1)}{D\kappa(D)} \right) t^{-2} \quad (22)$$

Let us next consider the Newtonian limit of the Friedmann metric. This limit is important because the length scales of interest in structure formation are small compared to the Hubble radius and the velocities in the system are also much smaller than c . This permits us to study the formation of large scale structures in the universe in a Newtonian framework where the effective potential due to the expanding background universe, Φ_{FRW} , and the potential due to the density perturbations, φ , can be simply superposed. In order to obtain Φ_{FRW} , we first recast the Friedmann metric in equation (14) into the more convenient form

$$ds^2 = dt^2 - a^2(t) (dX^2 + X^2 d\Omega^2) \quad (23)$$

where X is the radial distance in D dimensions and Ω is the corresponding solid angle. We then apply the transformations (see Padmanabhan, 1996, pg 80,346)

$$r = Xa(t) \quad , \quad T = t - t_0 + \frac{1}{2} a\dot{a}X^2 + \mathcal{O}(R^4) \quad (24)$$

where only terms up to quadratic in R are retained. Direct calculations, correct upto this order, transforms the Friedmann line element to the form,

$$ds^2 \approx \left(1 - \frac{\ddot{a}}{a} r^2 \right) dT^2 - dr^2 - r^2 d\Omega^2 \quad (25)$$

which upon comparison with the metric in equation (4) gives the equivalent Newtonian potential Φ_{FRW} in D dimensions as

$$\Phi_{\text{FRW}} = -\frac{1}{2} \frac{\ddot{a}}{a} r^2 \quad (26)$$

We will now use the results developed in the last two subsections to study structure formation and spherical collapse using the STH model.

3. Structure formation in D dimensions

Having determined the form of the Poisson equation in the Newtonian limit and analysed the Friedmann equations in $(D+1)$ dimensions, we proceed to derive the equation for the growth of inhomogeneities in the expanding universe. After this we consider a specific model, the STH model, to study spherical collapse of matter.

3.1. Equation for density perturbations in D dimensions

Let us assume that matter in the universe is a perfect, pressureless fluid with density ρ_m and flow velocity \mathbf{U} . We can formally write down the D dimensional fluid equations describing a perfect fluid in an external potential field Φ_{tot} in a proper coordinate system labelled by the D dimensional vector \mathbf{r} . Therefore, we have,

$$\left(\frac{\partial \rho_m}{\partial t} \right)_{\mathbf{r}} + \nabla_{\mathbf{r}} \cdot (\rho_m \mathbf{U}) = 0 \quad (27)$$

$$\left(\frac{\partial \mathbf{U}}{\partial t} \right)_{\mathbf{r}} + (\mathbf{U} \cdot \nabla_{\mathbf{r}}) \mathbf{U} = -\nabla_{\mathbf{r}} \Phi_{\text{tot}} \quad (28)$$

where equation (27) is the usual continuity equation while equation (28) is the Euler equation for the fluid. The potential in equation (28), Φ_{tot} , is the total external Newtonian potential

$$\Phi_{\text{tot}} = \Phi_{\text{FRW}} + \varphi \quad (29)$$

where Φ_{FRW} is the background potential associated with the smooth background matter density ρ_{bm} and is given in equation (26) while φ is the potential caused by density perturbations ($\rho_{\text{m}} - \rho_{\text{bm}}$). The potential φ satisfies the Poisson equation given in equation (13). Thus,

$$\nabla_{\mathbf{r}}^2 \varphi = \left(\frac{D-2}{D-1} \right) \kappa(D) (\rho_{\text{m}} - \rho_{\text{bm}}) = \left(\frac{D-2}{D-1} \right) \kappa(D) \rho_{\text{bm}} \delta \quad (30)$$

where δ is the density contrast defined by

$$\delta = \frac{\rho_{\text{m}} - \rho_{\text{bm}}}{\rho_{\text{bm}}} \quad (31)$$

We now transform to comoving coordinates defined by $\mathbf{x} = \mathbf{r}/a(t)$ and define the peculiar velocity \mathbf{v} by the relation

$$\mathbf{U} = H(t)\mathbf{r} + \mathbf{v} = \dot{a}\mathbf{x} + \mathbf{v} \quad (32)$$

where $\mathbf{v} = a\dot{\mathbf{x}}$ and $H(t) = (\dot{a}/a)$. Then, equation (27) and equation (28) become

$$\left(\frac{\partial \rho_{\text{m}}}{\partial t} \right)_{\mathbf{x}} + DH\rho_{\text{m}} + \frac{1}{a} \nabla_{\mathbf{x}} \cdot (\rho_{\text{m}} \mathbf{v}) = 0 \quad (33)$$

$$\left(\frac{\partial \mathbf{v}}{\partial t} \right)_{\mathbf{x}} + H\mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla_{\mathbf{x}}) \mathbf{v} = -\frac{1}{a} \nabla_{\mathbf{x}} \varphi \quad (34)$$

where we have used equation (26) to substitute for Φ_{FRW} . Similarly, in co-moving co-ordinates, equation (30) reduces to

$$\nabla_{\mathbf{x}}^2 \varphi = \left(\frac{D-2}{D-1} \right) \kappa(D) a^2 \rho_{\text{bm}} \delta \quad (35)$$

Using $\rho_{\text{m}} = \rho_{\text{bm}}(1 + \delta)$, transforming the time variable from t to $a(t)$ and defining a new velocity variable \mathbf{u} by

$$\mathbf{u} = \frac{d\mathbf{x}}{da} = \frac{\mathbf{v}}{a\dot{a}} \quad (36)$$

we can obtain equations for $\delta(a)$ and $\mathbf{u}(a)$. Therefore, using equation (21) and performing the transformations, equation (33) and equation (34) further reduce to,

$$\frac{\partial \delta}{\partial a} + \nabla_{\mathbf{x}} \cdot [\mathbf{u}(1 + \delta)] = 0 \quad (37)$$

$$\dot{a}^2 \frac{\partial \mathbf{u}}{\partial a} + \left(\ddot{a} + 2\frac{\dot{a}^2}{a} \right) \mathbf{u} + \dot{a}^2 (\mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} = -\frac{1}{a^2} \nabla_{\mathbf{x}} \varphi \quad (38)$$

Now, we use the Friedmann equations in equations (16,17) with ρ replaced by ρ_{bm} and with $p = 0$ to substitute for \ddot{a} in the above equation. Further, we define a new potential Ψ by the relation

$$\Psi = \left(\frac{D(D-1)}{6-D} \right) \frac{1}{\kappa(D) \rho_{\text{bm}} a^3} \varphi \quad (39)$$

so that, upon using equation (35), one obtains,

$$\nabla_{\mathbf{x}}^2 \Psi = \left(\frac{D(D-2)}{6-D} \right) \frac{\delta}{a} \quad (40)$$

where all reference to $\kappa(D)$ has disappeared. Hence the final system of equations we need to tackle are,

$$\frac{\partial \delta}{\partial a} + \nabla_{\mathbf{x}} \cdot [\mathbf{u}(1 + \delta)] = 0 \quad (41)$$

$$\frac{\partial \mathbf{u}}{\partial a} + (\mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} = -\frac{6-D}{2a} A [\nabla_{\mathbf{x}} \Psi + \mathbf{u}] \quad (42)$$

where A is given by the relation

$$A = \left(\frac{2\kappa(D)}{D(D-1)} \right) \frac{a^2}{\dot{a}^2} \rho_{\text{bm}} = \frac{\rho_{\text{bm}}(t)}{\rho_c(t)} \quad ; \quad \rho_c \equiv \left(\frac{D(D-1)}{2\kappa(D)} \right) \frac{\dot{a}^2}{a^2} \quad (43)$$

For the $k = 0$ universe, we will set $A = 1$.

To proceed further and determine the equation satisfied by δ , we decompose the term $\partial_{\alpha} u_{\beta}$ (where u_{β} is the β th covariant component of the vector \mathbf{u} and ∂_{α} is short for $\partial/\partial x^{\alpha}$) as

$$\partial_{\alpha} u_{\beta} = \sigma_{\alpha\beta} + \Omega_{\alpha\beta} + \frac{1}{D} \delta_{\alpha\beta} \theta \quad \alpha, \beta = (1, 2, \dots, D) \quad (44)$$

where $\sigma_{\alpha\beta}$ is the traceless, symmetric shear tensor, $\Omega_{\alpha\beta}$ is the antisymmetric rotation tensor, θ is the (trace) expansion and $\delta_{\alpha\beta}$ is the Kronecker delta symbol. Then, equation (41) and equation (42) are combined by taking the divergence of equation (42) and using the above decomposition of $\partial_{\alpha} u_{\beta}$ to obtain a single equation for δ . Straight forward algebra gives,

$$\frac{d^2 \delta}{da^2} + \left(\frac{6-D}{2a} \right) \frac{d\delta}{da} - \left(\frac{D(D-2)}{2a^2} \right) \delta(1+\delta) = \left(\frac{D+1}{D} \right) \frac{1}{(1+\delta)} \left(\frac{d\delta}{da} \right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2) \quad (45)$$

where $\sigma^2 = \sigma_{\alpha\beta} \sigma^{\alpha\beta}$ and $\Omega^2 = (1/2) \Omega_{\alpha\beta} \Omega^{\alpha\beta}$. This equation is the full non-linear equation for δ . Apart from the obvious nonlinear terms containing δ^2 and $(d\delta/da)^2$, the term $(1+\delta)(\sigma^2 - 2\Omega^2)$, which is the contribution from the shear and rotation, is also non-linear. The non-linear terms in δ in the above equation render the equation unsolvable in general. Ignoring these non-linear terms to a first approximation, we can get a linear equation for $\delta(a)$,

$$\frac{d^2 \delta}{da^2} + \left(\frac{6-D}{2a} \right) \frac{d\delta}{da} - \left(\frac{D(D-2)}{2a^2} \right) \delta = 0. \quad (46)$$

Assuming a power law solution for delta in the form $\delta \propto a^p$, we get,

$$p = \frac{D-4}{4} \pm \frac{1}{4} \sqrt{9D^2 - 24D + 16} \quad (47)$$

as the required values for p . Notice that δ has a growing mode as well as a decaying mode in general. The above solutions hold for all values of $D > 1$ in the linear regime.

Though the full non-linear equation is not solvable, by neglecting the contribution from the shear and rotation terms and by using a suitable ansatz for δ , the resulting non-linear equation *can* be solved. We proceed to do this within the framework of the STH model in the next section.

3.2. The Spherical Top Hat (STH) model

In the STH (spherical collapse) model, we assume spherical symmetry by neglecting the shear and rotation terms in the equation for δ . With this assumption the δ equation can be exactly solved.

Transforming equation (45) by changing the independent variable back to t , dropping the rotation and shear terms and using the Friedmann equations given in equations (16,17), we get,

$$\frac{d^2\delta}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\delta}{dt} - \left(\frac{D+1}{D}\right)\frac{1}{(1+\delta)}\left(\frac{d\delta}{dt}\right)^2 = \left(\frac{D-2}{D-1}\right)\kappa(D)\rho_{\text{bm}}\delta(1+\delta) \quad (48)$$

We now define a function $R(t)$ by the relation

$$1 + \delta = \frac{\rho}{\rho_{\text{bm}}} = \frac{M}{C_D R^D(t) \rho_{\text{bm}}} \quad (49)$$

where $C_D = 2\pi^{D/2}/(D\Gamma[D/2])$ is the volume of a unit sphere in D dimensions introduced for later convenience and M is a constant. The expression for δ above can be rewritten using the relation $\rho_{\text{bm}} a^D = \rho_0 a_0^D$ from equation (21):

$$1 + \delta = \frac{M}{C_D \rho_0 a_0^D} \left[\frac{a}{R}\right]^D = \lambda \frac{a^D}{R^D} \quad (50)$$

where ρ_0 and a_0 are the matter density and scale factor at some (arbitrarily chosen) ‘‘present’’ epoch t_0 . Substituting equation (50) in equation (48), we get an equation for the growth of $R(t)$ as,

$$\frac{d^2 R}{dt^2} = -\frac{D-2}{D(D-1)} \frac{\kappa(D)}{C_D} \frac{M}{R^{D-1}} \quad (51)$$

[As an aside we may note that if the universe contains matter or fields with equations of state other than $p = 0$, the equation for $R(t)$ becomes

$$\frac{d^2 R}{dt^2} = -\frac{D-2}{D(D-1)} \frac{\kappa(D)}{C_D} \frac{M}{R^{D-1}} - \frac{\kappa(D)}{D(D-1)} ((D-2)\rho + Dp)_{\text{rest}} R \quad (52)$$

where the term $((D-2)\rho + Dp)_{\text{rest}}$ comes from the smoothly distributed component with $p \neq 0$.]

From the form of the equation of motion of $R(t)$ we can give the following interpretation. Since the entire system considered above is spherically symmetric, we interpret R as the radius of a D dimensional spherical region containing a mass M . The equation of motion of R determines the motion of the surface of this region. In general, a spherical overdense region will be expected to initially expand because of the expansion of the background universe till the excess gravitational force due to the overdensity of enclosed matter stops the expansion and causes the region to collapse back on itself. We will discuss the cases $D = 3$ and $D = 2$ in the subsequent sections and determine the differences in the behaviour of the growth of inhomogeneities.

4. Summary of standard results in 3-dimensions

When $D = 3$, all the standard equations are recovered. First the Poisson equation satisfied by the Newtonian gravitational potential given by equation(13) reduces to the standard form,

$$\nabla^2 \phi = 4\pi G \rho \quad (53)$$

where we have defined G by relating it to $\kappa(3)$ by $\kappa(3) = 8\pi G$. Similarly, equations (41) and equation (42) reduce to (with $A = 1$),

$$\frac{\partial \delta}{\partial a} + \nabla_{\mathbf{x}} \cdot [\mathbf{u}(1 + \delta)] = 0 \quad (54)$$

$$\frac{\partial \mathbf{u}}{\partial a} + (\mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} = -\frac{3}{2a} [\nabla_{\mathbf{x}} \Psi + \mathbf{u}] \quad (55)$$

while the equation for Ψ becomes,

$$\nabla_{\mathbf{x}}^2 \Psi = \frac{\delta}{a}. \quad (56)$$

In a similar manner, the δ equation reduces to,

$$\frac{d^2 \delta}{da^2} + \frac{3}{2a} \frac{d\delta}{da} - \frac{3}{2a^2} \delta(1 + \delta) = \frac{4}{3(1 + \delta)} \left(\frac{d\delta}{da} \right)^2 + (1 + \delta)(\sigma^2 - 2\Omega^2) \quad (57)$$

and the solutions to the linear perturbation equation which is obtained by dropping the nonlinear terms and the $(\sigma^2 - 2\Omega^2)$ term, are

$$\delta \propto a^p, \quad p = 1, -\frac{3}{2} \quad (58)$$

which are well known. The STH model for $D = 3$ also reduces to the standard form

$$\frac{d^2 R}{dt^2} = -\frac{GM}{R^2} - \frac{4\pi G}{3} (\rho + 3p)_{\text{rest}} R \quad (59)$$

which is again well known (see Padmanabhan, 1996). Therefore, it is seen that the full D dimensional equations reduce to the correct equations in three dimensions. Now we will go on to discuss the important case of $D = 2$.

5. Two dimensional Gravity

If we naively consider the limit $D \rightarrow 2$ in the D dimensional equations, assuming that $\kappa(D)$ is finite in this limit, we obtain the following results. First, the Poisson equation (13) reduces to

$$\nabla^2 \phi = 0 \quad (60)$$

The above result shows that in two dimensions, the gravitational potential does *not* couple to the matter density ρ . In structure formation, this means that inhomogeneities cannot grow since the perturbed potential φ is not related to δ at all. The second interesting result is that the background Newtonian potential Φ_{FRW} vanishes. This occurs because, referring back to equation (18), $\ddot{a} = 0$ for pressureless dust and hence the background potential is zero. Further, the δ equation reduces to

$$\frac{d^2 \delta}{da^2} + \frac{2}{a} \frac{d\delta}{da} = \frac{3}{2(1 + \delta)} \left(\frac{d\delta}{da} \right)^2 + (1 + \delta)(\sigma^2 - 2\Omega^2). \quad (61)$$

Linearising the equation as before by dropping the $(\sigma^2 - 2\Omega^2)$ and $(d\delta/da)^2$ terms we obtain

$$\frac{d^2 \delta}{da^2} + \frac{2}{a} \frac{d\delta}{da} = 0. \quad (62)$$

The solutions to the linearised equation are

$$\delta \propto a^p, \quad p = 0, -1. \quad (63)$$

Thus only a constant or the decaying mode is present. This is consistent with the result that the perturbed gravitational potential does not couple to δ . If one considers the STH model, it is easy to see that the growth equation for $R(t)$ reduces to

$$\frac{d^2 R}{dt^2} = -\kappa(2)p_{\text{rest}} R \quad (64)$$

For, $p_{\text{rest}} = 0$, the solution to the above equation is just $R(t) = B_1 t + B_2$ where B_1, B_2 are constants. This is to be expected since there is no gravitational force which can lead to clustering and as a consequence the radius simply grows with time just like the background universe. Thus, if $\kappa(D)$ is *finite* in the limit $D \rightarrow 2$, it is not possible to have gravitational clustering that can grow with time.

We can, however, try some alternative approaches to examine whether it is possible to have a consistent physical picture of growing structures for two dimensional gravity.

One possibility is that instead of assuming $\kappa(D)$ to be finite, let us assume that the expression $(\kappa(D)(D-2))$ remains finite when $D \rightarrow 2$. This finite value can be fixed, for example, by invoking Gauss's theorem in D dimensions. This gives

$$\kappa(D) = \left(\frac{D-1}{D-2} \right) \frac{2\pi^{D/2}G}{\Gamma[D/2]} \quad (65)$$

Thus, $\kappa(D) \rightarrow \infty$ when $D \rightarrow 2$, but the Poisson equation acquires the form

$$\nabla^2 \phi = 2\pi G \rho \quad (66)$$

This is, of course, the same form which is obtained by applying Gauss's law in two dimensions. Hence, as in three dimensions, the gravitational potential is determined by the matter density and thus inhomogeneities can in principle grow. There are, however, difficulties with this approach. To begin with, the constant factor $c^4/(2\kappa(D))$ in the action \mathcal{S} in equation(1) vanishes for $D = 2$. But this is not too serious a problem. The gravitational part of the action certainly vanishes but because only the variations about the action are of significance this difficulty can be ignored. But a more serious problem arises when the solutions to the Friedmann equation are considered. The solutions for $a(t)$ and $\rho(t)$ in D dimensions are given in equation (22). Using equation (65), these reduce to,

$$a(t) = \left(\frac{D\pi^{D/2}GC_1}{(D-2)\Gamma[D/2]} \right)^{1/D} t^{2/D} \quad ; \quad \rho(t) = C_1 a^{-D} = \frac{(D-2)\Gamma[D/2]}{D\pi^{D/2}G} t^{-2} \quad (67)$$

When $D \rightarrow 2$ then, $a \rightarrow \infty$ and $\rho \rightarrow 0$ irrespective of the dependence on t . This implies that one cannot solve the equations describing the growth of structure in a consistent and non-singular way. Hence, we conclude that it is not possible to have a theoretical formulation of two dimensional gravity as the Newtonian limit to Einstein's equations in two dimensions.

An alternative that remains is to use the Newtonian fluid equations in D dimensions directly and rewrite them for an expanding background with an *arbitrary* scale factor $a(t)$. Note that $a(t)$ is not obtained from the Friedmann equations and is completely arbitrary. We can superpose the potentials for the background universe and the perturbations in this case as before. The further assumptions we need to make are (i) the potential of the background universe Φ_{bg} is of the form

$$\Phi_{\text{bg}} = -\frac{1}{2} \frac{\ddot{a}}{a} r^2 \quad (68)$$

and (ii) the Poisson equation is given by

$$\nabla^2 \phi = \kappa(D) \rho_{\text{bm}} \delta \quad (69)$$

where $\kappa(D) = 2\pi^{D/2}G/\Gamma[D/2]$. This form of $\kappa(D)$ is obtained from the use of Gauss's law in D dimensions. We also need to specify how the background density ρ_{bm} depends on time. In analogy with the usual Friedmann equations, we will assume $\rho_{\text{bm}} a^D = C_1$ where C_1 is a constant. This gives an equation for δ with an arbitrary scale factor $a(t)$ as

$$\frac{d^2 \delta}{da^2} + \left(\frac{\ddot{a}a + 2\dot{a}^2}{a\dot{a}^2} \right) \frac{d\delta}{da} - \kappa(D)C_1 \frac{1}{a^D \dot{a}^2} \delta(1 + \delta) = \left(\frac{D+1}{D} \right) \frac{1}{1 + \delta} \left(\frac{d\delta}{da} \right)^2 + (1 + \delta)(\sigma^2 - 2\Omega^2). \quad (70)$$

The above equation can be solved in any dimension D if the form of $a(t)$ is given. This gives us a non-singular way to analyse growth of structures in D dimensions, including the case $D = 2$. But in three dimensions we observe that the above equation does not correctly reduce to equation (57). We may obtain the correct equation in three dimensions by making an additional ansatz, namely, that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2\kappa(D)}{D} \rho_{\text{bm}}. \quad (71)$$

With this, equation (70) reduces to

$$\frac{d^2\delta}{da^2} + \left(\frac{6-D}{2a}\right) \frac{d\delta}{da} - \frac{D}{2a^2} \delta(1+\delta) = \left(\frac{D+1}{D}\right) \frac{1}{1+\delta} \left(\frac{d\delta}{da}\right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2) \quad (72)$$

Notice that the above equation differs from the earlier equation for δ in D dimensions, equation (45), in that there is a factor of $(D-2)$ missing from the coefficient of the $\delta(1+\delta)$ term. Therefore, the above equation does correctly reduce to equation (57) since $(D-2)$ equals unity when $D = 3$. When $D = 2$, equation (72) gives

$$\frac{d^2\delta}{da^2} + \frac{2}{a} \frac{d\delta}{da} - \frac{1}{a^2} \delta(1+\delta) = \frac{3}{2(1+\delta)} \left(\frac{d\delta}{da}\right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2) \quad (73)$$

On linearising this equation by dropping the $(\sigma^2 - 2\Omega^2)$ term and the nonlinear terms and solving it we get both a growing mode as well as a decaying mode for δ . The solutions are

$$\delta \propto a^q, \quad q = (-1 \pm \sqrt{5})/2 \quad (74)$$

(It is interesting to note that one of the power law exponents is the golden ratio).

The Spherical Top Hat (STH) equation in this case turns out to be

$$\ddot{R} = -\frac{GM}{R} + \frac{1}{2} \frac{R}{t^2} \quad (75)$$

where M is the constant mass inside a ‘spherical’ shell of radius R . This equation, unfortunately, has no simple analytic solution. While this procedure leads to nontrivial results, it has many *ad hoc* assumptions and cannot be obtained by taking appropriate limits of Einstein’s theory in a systematic manner. Consequently it cannot be applied to numerical investigations of 2 dimensional gravity with the confidence that the results will have some implications for the three dimensional case.

6. Conclusions

In this paper we have analysed the case of two dimensional gravitational clustering starting from a formulation of the D dimensional Einstein’s equations and taking the proper limits. The system of equations thus arrived at for a $(D+1)$ dimensional universe has been shown to reduce to the correct equations in three dimensions. But when the $D \rightarrow 2$ limit of these equations is taken, we are forced to conclude that irrespective of the value of $\kappa(D)$, a consistent two dimensional gravity theory in a cosmological context that supports growth of structures cannot be constructed.

If $\kappa(2)$ is assumed to be finite, we observe that the coefficient in Poisson’s equation goes to zero thus decoupling the potential from the density. This implies that perturbations do not grow but decay in

time due to the expansion of the background spacetime. The alternative which is obtained by using the expression for $\kappa(D)$ given by equation (65) gives rise to solutions for the scale factor which are singular and therefore unacceptable.

We have discussed all the ways in which two dimensional gravity may be simulated including an *ad hoc* procedure without a strong foundation which can give non-singular results as far as structure formation scenarios in two dimensions are concerned. The results presented in the paper leads us to conclude that the only way to do a numerical simulation of two dimensional gravity is to simulate infinite ‘needles’ in a background spacetime expanding in three dimensions and consider the ‘particles’ in the system to be intersections of the ‘needles’ with any plane orthogonal to them.

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