

## A LENS MAPPING ALGORITHM FOR WEAK LENSING

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### ABSTRACT

We develop an algorithm for the reconstruction of the two-dimensional mass distribution of a gravitational lens from the observable distortion of background galaxies. From the measured reduced shear  $\gamma_i/(1-\kappa)$  the lens mapping is obtained, from which a mass distribution is derived. This is unlike other methods where the convergence  $\kappa$  is directly obtained. We show that this method works best for sub-critical lenses, but can be applied to a critical lens away from the critical lines. For finite fields the usual mass-sheet degeneracy is shown to exist in this method as well. We show that the algorithm reproduces the mass distribution within acceptable limits when applied to simulated noisy data.

*Subject headings:* Cosmology: gravitational lensing; Cosmology: dark matter

### 1. INTRODUCTION

The mapping of the distribution of matter in extended regions around rich clusters of galaxies, from the systematic distortions of background galaxies due to gravitational lensing, has increasingly become feasible and popular over the last decade. This *shear* is measured from the quadrupole moments of the images of the background galaxies in terms of the local ‘polarization’ of an image compared to its assumed intrinsic circular form. In order to remove the effect of the source ellipticity distribution, the parameters are averaged over a large number of source galaxies. The two-dimensional mass distribution of the lens is reconstructed from the shear map away from the critical lines (‘weak reconstruction’). Following the pioneering work of Tyson et al. (1990), the idea was quantitatively developed by Kaiser and Squires (1993, also Kaiser 1995 and Squires and Kaiser 1996) and Seitz and Schneider (1995, 1996). Since then, several interesting variants on this theme have appeared in the literature, *e.g.* using maximum likelihood cluster reconstruction (Bartelmann et al. 1996, Seitz et al. 1998), or using methods based on the variational principle (Lombardi and Bertin 1999) or maximum entropy (Bridle et al. 1998).

Since the shear data contain complete information about the mass distribution in two independent fields  $\gamma_1/(1-\kappa)$  and  $\gamma_2/(1-\kappa)$ , it is possible to obtain several algorithms to estimate the mass distribution from the measured shear. The observed shear is available only on a finite grid. Due to the unknown intrinsic distribution of the ellipticities of the source galaxies, and the effect of the distortion of the PSF due to observing conditions (seeing, tracking etc.), the measured shear is noisy. It is therefore desirable to develop new algorithms in the hope that they might be able to deal with the noise better than other methods.

In this paper we develop an algorithm (LM: Lens Mapping algorithm) for the lens mass reconstruction from the measured reduced shear,  $g_i = \gamma_i/(1-\kappa)$ . The method involves two steps. First, the lens mapping is reconstructed from the reduced shear, which can be done uniquely with the assumption that the lens mapping goes to identity far away from the lens. As the second step, we show in §3 that for a sub-critical lens, the surface mass density can be reconstructed completely from the derived lens mapping. In reality, however, the measured shear is available

only in a finite region of the lens plane. In this case the lens mapping cannot be uniquely obtained and hence the LM algorithm exhibits a mass-sheet degeneracy, which is also present in other methods of mass reconstruction. In §3.1, we characterize this degeneracy in the context of the LM algorithm. The performance of this method with discretely sampled, noisy data is dealt with in §4, where we demonstrate the various features of the LM algorithm, by reconstructing the mass distribution for an analytically given shear field, and separately showing the effects of discrete sampling and of uniform noise (due to measurement error) on the input data.

### 2. WEAK LENSING AND THE MASS-SHEET DEGENERACY

For most practical applications of gravitational lensing the lens can be considered to be thin. Under the small angle approximation the lens equation is given by

$$\mathbf{y} = \mathbf{x} - \nabla\psi(\mathbf{x}), \quad (1)$$

where the source angular position is denoted as  $\mathbf{y}$ , the image angular position as  $\mathbf{x}$  and the relativistic potential  $\psi$  satisfies the equation  $\kappa(\mathbf{x}) \equiv \Sigma/\Sigma_{\text{crit}} = \frac{1}{2}\nabla^2\psi$ , where  $\Sigma_{\text{crit}} = (c^2/4\pi G)(d_s/d_l d_{ls})$ . Here  $d_s, d_l$  and  $d_{ls}$  are respectively the angular diameter distance to the source, to the lens and between the lens and the source. The coordinates for source and image are small compared to unity and we can treat their components as Cartesian. It is convenient to define two planes: that containing the source is called the source plane and that containing the cluster, the lens plane (or image plane).

The coordinate differentials in the source plane and the corresponding differentials in the lens plane are related through  $dy_i = M_{ij}^{-1}dx_j$  and the inverse of the magnification matrix  $M$  is given by

$$M^{-1} = (1-\kappa) \begin{pmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{pmatrix}, \quad (2)$$

where  $\mathbf{g} \equiv g_1 + ig_2 = (\gamma_1 + i\gamma_2)/(1-\kappa)$ ,  $\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22})$  and  $\gamma_2 = \psi_{,12}$  and the subscripts denote differentiation with respect to the two components of the image coordinates. From the quadrupole moments of the surface brightness of the images of the background galaxies the reduced shear can be measured unambiguously in the regions where  $\kappa < 1$  (for observational details see Kaiser 1999 or Bartelmann & Schneider 2000).

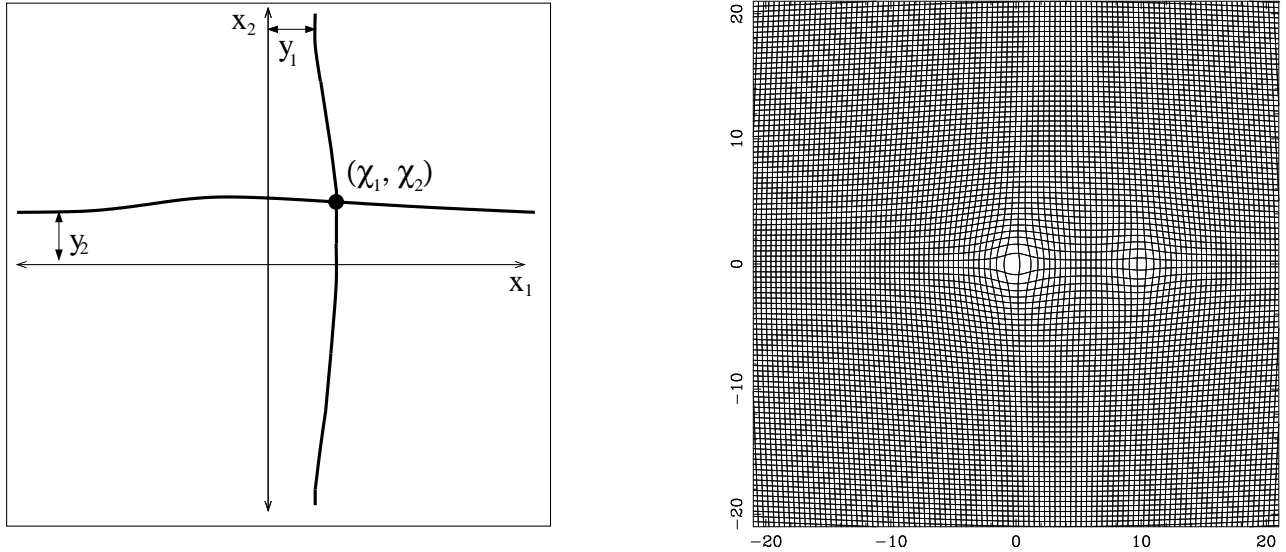


FIG. 1.— [a] The coordinate system used in this paper. We show the lens-plane images of two straight lines passing through  $(y_1, y_2)$  in the source plane; [b] An equally-spaced Cartesian grid (in the source plane) is lensed by a two-component mass model, both PICMDs with zero ellipticity and  $k_0 = 0.8$  and  $0.5$  respectively (described in §3). The axes are marked in units of the core radius  $r_c$ , assumed to be same for both the models. The separation between the centers of the two distributions is  $10r_c$ .

The continuity of  $\mathbf{y}(x_1, x_2)$  implies  $\partial_{jk}^2 y_i = \partial_{kj}^2 y_i$ , which, along with, eq. (2) gives

$$\mathbf{K}_{,i} \mathbf{G}_{ij}^{-1} - \mathbf{K}_{,j} \mathbf{G}_{il,j}^{-1} = \mathbf{G}_{il,j}^{-1} - \mathbf{G}_{ij,l}^{-1}, \quad (3)$$

where  $\mathbf{K} \equiv \ln(1 - \kappa)$  and  $\mathbf{G}^{-1} \equiv \mathbf{M}^{-1}/(1 - \kappa)$ . On multiplying (3) by the inverse of  $\mathbf{G}^{-1}$ , and taking the trace of the resulting equation gives us

$$\nabla_l \ln(1 - \kappa) = \sum_{i=1}^2 \sum_{j=1}^2 [\mathbf{G}_{ij} (\mathbf{G}_{il,j}^{-1} - \mathbf{G}_{ij,l}^{-1})]. \quad (4)$$

This equation was first derived by Kaiser (1995). It is clear that replacing  $1 - \kappa$  on the left hand side with  $\lambda(1 - \kappa)$ , where  $\lambda$  is a constant, does not affect the equation. Therefore any particular solution of this equation can be used to obtain a one-parameter degenerate family of functions, all of which satisfy (4). This is known as the mass-sheet degeneracy.

### 3. THE LENS MAPPING ALGORITHM

In general the lens equation (1) is a many-to-one mapping, and consequently the inverse of the lens equation  $\mathbf{x} = \mathbf{x}(\mathbf{y})$  has several branches, with no single branch completely specifying the lens mapping. However, if the lens is sub-critical, then the lens mapping is one-to-one everywhere. In this case both the mapping and its inverse are uniquely defined and completely specify the mass distribution. Here we show that in this case it is possible to obtain the lens mapping completely from the reduced shear, with the additional assumption that it goes to identity sufficiently far away from the lens.

For a non-critical lens, open curves in the source plane are mapped to open curves in the lens plane. In particular, any infinite straight line is mapped to an infinite open curve in the lens plane. In Fig. 1a, we schematically show the images of the two perpendicular straight lines passing through the source at  $(y_1, y_2)$ . The point of intersection of the two curves gives the position of the image  $(\chi_1, \chi_2)$ .

From (2) we can obtain the equation which the image of an arbitrary infinite straight line in the source plane satisfies in the lens plane. In particular, we obtain the equations which map

the coordinate grid lines of a Cartesian coordinate system in the source plane to the lens plane

$$dy_1 = 0 \mapsto \frac{dx_1}{dx_2} = \frac{g_2}{1 - g_1}; \quad dy_2 = 0 \mapsto \frac{dx_2}{dx_1} = \frac{g_2}{1 + g_1}. \quad (5)$$

These are first-order ordinary differential equations and can be uniquely integrated through any point in the lens plane. If we consider the source position as the intersection of the lines  $y_1 = \text{constant}$  and  $y_2 = \text{constant}$  in the source plane, then the image of this point will be the intersection of the images of these two curves in the lens plane. In this manner we obtain a one-to-one mapping from the lens to the source plane. Away from the lens singularities we can integrate the equations (5) to obtain the numerical solutions

$$x_1 = X(x_2; \chi); \quad x_2 = Y(x_1; \chi) \quad (6)$$

where  $X(x_2; \chi)$  is the mapped curve for  $dy_1 = 0$ , passing through the point  $\chi$  in the lens plane, and  $Y(x_1; \chi)$  is the mapped curve for  $dy_2 = 0$ , passing through the point  $\chi$  in the lens plane (see Fig. 1a). The auxiliary variable  $\chi$  in the above equations also explicitly represents the family of curves which the equations (5) generate when integrated with the initial conditions  $X(\chi_2) = \chi_1$  and  $Y(\chi_1) = \chi_2$ , respectively, for the two equations.

In Fig. 1b we show the image of a Cartesian grid that is equally spaced in the source plane, lensed by a two-component model, where both clusters are represented by Pseudo Isothermal Circular Mass Distributions (PICMD) with the same core radius  $r_c$ , separated by  $10r_c$  along the  $x_1$  axis. The central values of the dimensionless surface density for the two components are  $\kappa_0 = 0.8$  and  $\kappa_0 = 0.5$  respectively. Fig. 1b illustrates the integral curves of (5).

To obtain the source position corresponding to the image position  $\chi$  we use the assumption that far away from the lens the lens mapping is  $y_i \cong x_i$  (see Fig. 1a). Therefore the integrated curves should go to the original unperturbed source grid lines at large  $|\mathbf{x}|$ . This is ensured by the following asymptotic conditions

$$\lim_{x_2 \rightarrow \pm\infty} X(x_2; \chi) = y_1; \quad \lim_{x_1 \rightarrow \pm\infty} Y(x_1; \chi) = y_2. \quad (7)$$

The lens equation can now be formally written as

$$y_1 = X(\infty; \mathbf{x}); \quad y_2 = Y(\infty; \mathbf{x}), \quad (8)$$

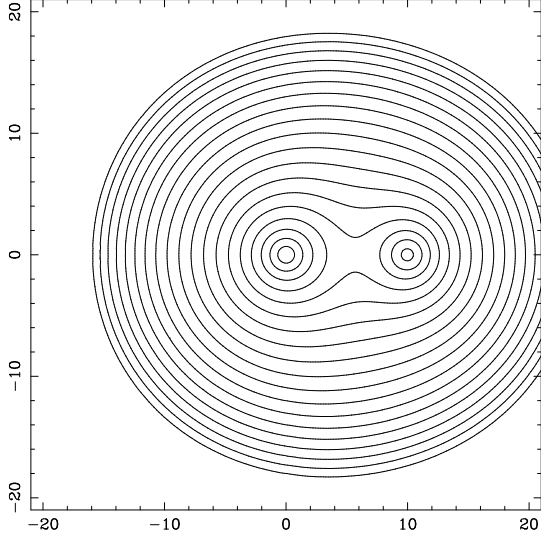


FIG. 2.— The reconstructed mass distribution, using the Lens Mapping Algorithm, for the model used in Fig. 1(b), shown as contours of equal surface density  $\kappa$ . The axes are marked in units of the core radius  $r_c$ .

where the auxiliary variables  $\chi$  are now interpreted as the image coordinates  $\mathbf{x}$ . The trace of the magnification matrix which is given in terms of  $\kappa$  as

$$\frac{\partial X(\infty; \mathbf{x})}{\partial x_1} + \frac{\partial Y(\infty; \mathbf{x})}{\partial x_2} = 2 [1 - \kappa(\mathbf{x})], \quad (9)$$

provides an estimate of  $\kappa$  at any point. Fig. 2 shows the lens mass distribution thus reconstructed from the shear represented by Fig. 1b, assuming that the shear is noise-free and is known everywhere.

It might appear surprising that we have been able to obtain the lens mapping solely from shear. To understand how, let us recall that our assumption that the lens mapping becomes identity far away from the lens removes the mass-sheet degeneracy. We also note that it is essentially equivalent to the assumption in Kaiser's method that the mass density vanishes far away from the center, though quantitatively the notion of "far" is different in the two methods. For a finite field we will see that the degeneracy reappears.

The above discussion applies to points away from the critical lines of the lens mapping. At these singularities, the eqs. (5) become non-integrable. This becomes apparent by writing the eqs. (5) in terms of  $\gamma_i$ s and  $\kappa$ ,

$$\frac{dx_1}{dx_2} = \frac{\gamma_2}{1 - \kappa - \gamma_1}; \quad \frac{dx_2}{dx_1} = \frac{\gamma_2}{1 - \kappa + \gamma_1}. \quad (10)$$

For a sufficiently smooth mass distribution, the potential and consequently  $\gamma_2$  is non-singular everywhere. Since the Jacobian of the lens mapping,  $(1 - \kappa)^2 - \gamma_2^2$ , is positive far away from the lens, we conclude that  $(1 - \kappa)^2 - \gamma_1^2 > \gamma_2^2$ . From this inequality it is clear that  $(1 - \kappa)^2 - \gamma_1^2$  can become zero only when the Jacobian  $(1 - \kappa)^2 - \gamma_2^2$  also becomes zero. Since  $(1 - \kappa)^2 - \gamma_1^2 = 0$  implies that either  $1 - \kappa - \gamma_1 = 0$  or  $1 - \kappa + \gamma_1 = 0$ , we conclude that both the equations (5) cannot be integrated simultaneously, which implies that a correspondence between the lens plane and source plane cannot be obtained. Thus the LM algorithm cannot be applied in regions very close to the critical lines of the Lens mapping.

### 3.1. Finite Field, Critical Lens

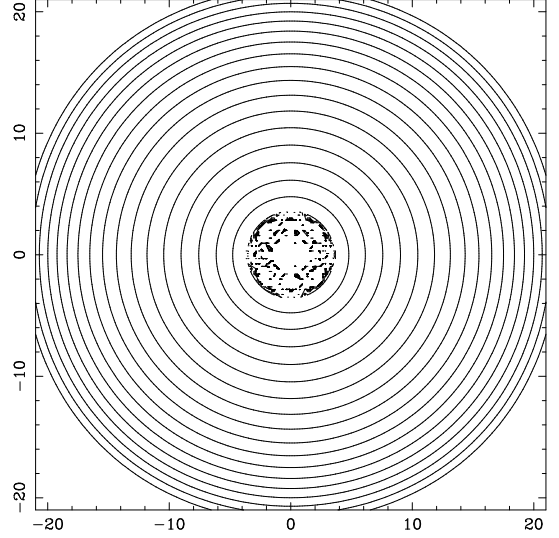


FIG. 3.— The reconstructed mass distribution for the  $\kappa_0 = 2$  case. The reconstruction works well outside the critical region. The "reconstructed" mass inside the critical lines shows up as noise, since the method fails there.

The algorithm as described in the previous section is applicable to cases where the reduced shear is known everywhere. In practice, this information is available only in a limited region of the lens plane. We show here that the LM algorithm can be generalized to this case. For a finite field, eqs. (7) can be applied only up to the edge of region in which data is available, and the correct limit cannot be evaluated. It is easy to see that due to the unknown magnification factor at the edge, these equations provide an incorrect measure of the derivatives in eq. (9).

Let us consider the differentials of  $X$  and  $Y$  (as defined in eq. 6) at the edge  $\mathbf{x}^e$ ,

$$\Delta X(x_2^e, \chi) = \frac{\Delta X}{\Delta y_1} \Delta y_1; \quad \Delta Y(x_1^e, \chi) = \frac{\Delta Y}{\Delta y_2} \Delta y_2. \quad (11)$$

Since at the edge of the image the derivatives of  $X$  and  $Y$  are  $> 1$  we obtain slightly higher values of  $\Delta y_i$ . Substituting the value of the derivatives in eq. (11) we obtain

$$\Delta X(x_2^e, \chi) = [1 - \kappa(\mathbf{x}^e)][1 - g_1(\mathbf{x}^e)] \Delta y_1, \quad (12)$$

$$\Delta Y(x_1^e, \chi) = [1 - \kappa(\mathbf{x}^e)][1 + g_1(\mathbf{x}^e)] \Delta y_2. \quad (13)$$

This gives us a way of obtaining the exact value of the derivatives of the lens mapping, and thus completely removing the mass-sheet degeneracy. However, in reality we only know the reduced shear, and consequently the correction factors at the edges, only up to a factor of  $1 - \kappa$ . For finite fields we can impose the boundary condition  $\kappa = 0$  to obtain the mass distribution. This corrects the generalized degeneracy up to a factor of  $(1 - \kappa)$ .

This algorithm can be applied to a critical lens as well. Since the algorithm depends on our ability to integrate equations (5) without hitting a singularity, we see that the method can be used for the region which lie outside the critical lines. In Fig. 3, where the model has  $\kappa_0 = 2$ , we see that our algorithm works well outside the critical region.

## 4. NOISY AND DISCRETELY SAMPLED DATA

We now present a few examples to illustrate the application of the LM algorithm to noisy data. Since we wish to compare the original mass distribution with that reconstructed by the algorithm, we correct for the mass-sheet degeneracy in all the

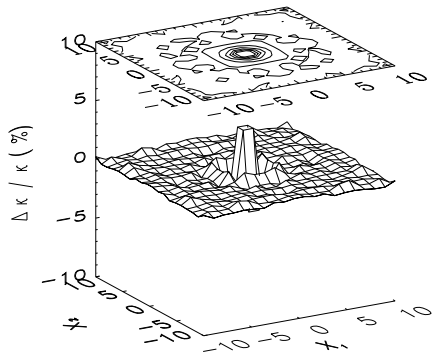


FIG. 4.— The effect of the finite sampling of shear: the percentage error in the reconstructed mass distribution (using the LM algorithm), in the case where the reduced shear is known on a grid of size  $20'' \times 20''$ , and there is no noise. The mass is reconstructed using the LM algorithm on a  $20 \times 20$  grid. The excess noise near the center of the distribution is due to the fact that the reduced shear is most rapidly varying close to the center, where the interpolated shear values are consequently noisier. Note that the rms noise outside the central region is less than 1%.

examples by the prescription given in § 3.1. We consider a single component lens modeled as PICMD. We simulate the signal by analytically calculating the reduced shear from a model. We illustrate the effect of finite sampling of data and the presence of noise with separate examples.

We first consider the case where the reduced shear is known at all the points from an analytical model. We choose a lens with the central, dimensionless surface mass density given by  $\kappa_0 = 0.7$ , and reconstruct the mass distribution by applying the LM algorithm on a  $20 \times 20$  grid. The noise in the reconstruction, which is very small everywhere ( $\lesssim 1\%$ ), can be entirely accounted for as arising from the inaccuracies in the evaluation of various integrals and derivatives required by the algorithm.

To illustrate the effect of finite sampling of shear in the image plane, we consider a model with a lens velocity dispersion of  $\sigma_v = 1100$  km/s and with a core radius  $R_c = 50$  kpc. The source galaxies are assumed to be at the redshift  $z_s = 1$  and the lens is at  $z_l = 0.2$ . We sample the reduced shear in pixels of size  $20'' \times 20''$ . Using a bicubic spline interpolation routine to evaluate the reduced shear at all the points, the mass distribution is reconstructed using the LM algorithm on a  $20 \times 20$  grid as before. The resulting percentage error in the reconstructed mass distribution (given by  $100 \times (\kappa_{\text{rec}} - \kappa_{\text{true}}) / \kappa_{\text{true}}$ ) is shown in Fig. 4. We note that the excess noise near the center of the distribution is mainly due to the fact that the reduced shear is most rapidly varying close to the center and therefore the interpolated shear values are noisier at those points. This noise will vanish if the grid on which the shear is evaluated were made finer, but then in the real world, one is limited by the number

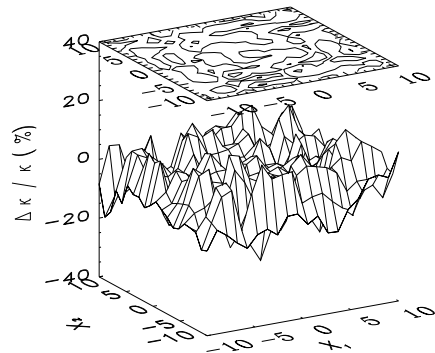


FIG. 5.— The effect of the finite sampling and noise: the percentage error in the reconstructed mass distribution in the same case as the last one, but with noise with a constant signal-to-noise ratio of 10 added to the shear field before the application of interpolation.

of galaxies over which the value of the reduced shear has to be averaged, and we cannot use a substantially finer grid without increasing shot noise.

As our last example we consider the same case as the last one, but add a uniform noise with a constant signal-to-noise ratio of 10 to the given shear field, before the application of interpolation. The noise in the reconstructed mass distribution (Fig. 5) can be seen to be around twice that of the noise in the shear field.

## 5. CONCLUSIONS

We have described a simple algorithm for mass reconstruction of a cluster lens by directly obtaining the lens mapping from the measured reduced shear and taking its derivatives to compute the surface mass density  $\kappa$ .

The method works best for sub-critical lenses, where it can give the mass distribution at all points, but it can work for critical lenses as well in limited regions of the lens plane, away from the critical lines. The algorithm is shown to have a mass-sheet degeneracy of the same type as exists in the other methods of reconstruction if the field in which the reduced shear is available is finite. We have tested the algorithm on discretely sampled noisy (simulated) data and have found that it reproduces the mass distribution within acceptable limits.

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