

LRS BIANCHI I COSMOLOGICAL UNIVERSE MODELS WITH VARYING COSMOLOGICAL TERM Λ

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Einstein's field equations with cosmological term Λ varying with time are considered in the context of general homogeneous, anisotropic universes in a way which conserves the energy–momentum tensor of matter content. It is shown that the field equations are solvable for any arbitrary cosmic scale functions. Some physical and geometrical features of the models are discussed.

1. Introduction

One of the most important and outstanding problems in cosmology is the cosmological constant problem. The smallness of the effective cosmological constant observed today ($\Lambda_0 \leq 10^{-56} \text{ cm}^{-2}$) is one of the most difficult problems involving cosmology and elementary particle physics theory. In order to explain the striking cancellation between the “bare” cosmological constant and the ordinary vacuum energy contributions of the quantum fields, several mechanisms have been proposed in the last few years.¹ The “cosmological constant problem” can be expressed as the discrepancy between the negligible value Λ has for the present universe (as can be seen by the successes of Newton's theory of gravitation²) and the values 10^{50} times larger expected by the Glashow–Salam–Weinberg model³ or by grand unified theory (GUT) where it should be 10^{107} times larger.⁴ The cosmological term Λ is then small at the present epoch simply because the universe is so old. The problem in this approach is then to try to determine the right dependence of Λ upon R or t .

In recent years, models with a relic cosmological constant Λ have received considerable attention among researchers for various reasons (see Refs. 5–7 and references therein). Some of the recent discussions on the cosmological constant “problem” and on cosmology with a time-varying cosmological constant by Ratra and Peebles,⁸ Dolgov^{9–11} and Sahni and Starobinsky¹² point out that in the absence

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of any interaction with matter or radiation, the cosmological constant would remain a “constant,” however, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress–energy with a time-varying Λ can be found. For these solutions, conservation of energy requires any decrease in the energy density of the vacuum component to be compensated by a corresponding increase in the energy density of matter or radiation. For earlier reviews on this topic, the reader is referred to Zeldovich,¹³ Weinberg¹ and Carroll, Press and Turner.¹⁴

Recently, motivated by dimensional grounds in keeping with quantum cosmology, Chen and Wu¹⁵ considered a Λ varying as R^{-2} , where R is the scale factor in the Robertson–Walker metric. Abdel–Rahman¹⁶ has recently considered a model with the same kind of variation. Such a dependence alleviates some problems in reconciling observational data with the inflationary universe scenario.

A number of authors have instead argued in favor of the dependence $\Lambda \sim t^{-2}$. Berman and Som¹⁷ pointed out that the relation $\Lambda \sim t^{-2}$ seems to play a major role in cosmology, and now we want to discuss some possibility for realizing this hypothesis. In fact, Berman, Som and Gomide¹⁸ found this relation in Brans–Dicke static models; Berman¹⁹ predicted it in a static universe with Endo–Fukui modified Brans–Dicke cosmology; Berman and Som,¹⁷ and Berman²⁰ investigated it again in general Brans–Dicke models which obey the perfect-gas equation of state; the same relation was observed by Bertolami,^{21,22} Beesham^{23,24} and Singh *et al.*²⁵

For simplification and description of the large scale behaviour of the actual universe, locally rotationally symmetric [henceforth referred to as LRS] Bianchi I spacetime have been widely studied. Mazumder²⁶ has obtained solutions of LRS Bianchi I spacetime filled with a perfect fluid. Hajj-Boutros and Sfeila²⁷ and Shri Ram²⁸ also obtained some solutions for the same field equations by using their solution-generating techniques. In a recent work, Vishwakarma and Abdussattar²⁹ investigated plane symmetric homogenous models with a perfect fluid and variable cosmological constant in LRS Bianchi type-I spacetime. Recently, Pradhan and Kumar³⁰ have obtained an exact solution of the same field equations. In the present work, in what follows, we will discuss LRS Bianchi I cosmological models obtained by augmenting the energy–momentum tensor of a perfect-fluid by a term that represents a variable cosmological constant varying with time, and we will later generalize the solutions of Refs. 26–28 and 30.

2. Field Equations

The usual energy–momentum tensor is modified by addition of a term

$$T_{ij}^{(\text{vac})} = -\Lambda(t)g_{ij}, \quad (1)$$

where $\Lambda(t)$ is the cosmological term and g_{ij} is the metric tensor. Thus the new energy–momentum tensor is

$$T_{ij} = (p + \rho)u_i u_j - pg_{ij} - \Lambda(t)g_{ij}, \quad (2)$$

where ρ and p are, respectively, the energy and pressure of the cosmic fluid, and u_i is the fluid four-velocity such that $u^i u_i = 1$.

We consider LRS Bianchi I spacetime

$$ds^2 = dt^2 - A^2 dx^2 - B^2 (dy^2 + dz^2), \tag{3}$$

where A and B are the functions of cosmic time t . For the energy–momentum tensor (2) and the LRS Bianchi I spacetime (3), Einstein’s field equations

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij}, \tag{4}$$

yield the following three independent equations

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} = -8\pi(p + \Lambda), \tag{5}$$

$$\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{A}}{A} = -8\pi(p + \Lambda), \tag{6}$$

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} = 8\pi(\rho - \Lambda). \tag{7}$$

An overdot indicates a derivative with respect to time t . The energy conservation equation $T^i_{j;i} = 0$ takes the form

$$\dot{\rho} + (\rho + p) \left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B} \right) = -\dot{\Lambda}. \tag{8}$$

For complete determinacy of the system, we consider a perfect-gas equation of state

$$p = \gamma\rho, \quad 0 \leq \gamma \leq 1. \tag{9}$$

It is worth noting here that our approach suffers from a lack of Lagrangian approach. There is no known way to present a consistent Lagrangian model satisfying the necessary conditions discussed in the paper.

3. Solution of the Field Equations and Discussion

From Eqs. (5) and (6), we obtain

$$\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{\ddot{A}}{A} - \frac{\dot{A}\dot{B}}{AB} = 0, \tag{10}$$

which on integration yields

$$B^2 \dot{A} - AB \dot{B} = k, \tag{11}$$

where k is an integrating constant.

Considering Eq. (11) as a linear differential equation for $A(t)$, where $B(t)$ is an arbitrary function, we obtain

$$A = k_1 B + kB \int \frac{dt}{B^3(t)}, \tag{12}$$

where k_1 is an integrating constant.

Similarly considering Eq. (11) as a linear differential equation for $B(t)$, where $A(t)$ is an arbitrary function, we obtain

$$B^2 = k_2 A^2 - 2k A^2 \int \frac{dt}{A^3(t)}, \tag{13}$$

where k_2 is a constant of integration. Therefore, for any $B(t)$ from Eq. (12), one can obtain $A(t)$. Hence, from Eqs. (5)–(9), p , ρ and Λ can be calculated, i.e. for any given function $B(t)$, the field equations are solvable. Similarly by using Eq. (13), the field Eqs. (5)–(7) can be solved for any given function $A(t)$.

Following Mazumder,²⁶ we choose

$$B = t^{\frac{1-n}{2}}, \tag{14}$$

where n is a real number satisfying $n \neq 1/3$.

Equation (12), with the help of (14) yields

$$A = k_1 t^{\frac{1}{2}(1-n)} + \frac{2k}{(3n-1)} t^n. \tag{15}$$

For this solution, the geometry of our universe is described by the line-element

$$ds^2 = -dt^2 + \left(k_1 t^{\frac{1}{2}(1-n)} + \frac{2k}{(3n-1)} t^n \right)^2 dx^2 + t^{(1-n)} (dy^2 + dz^2). \tag{16}$$

For the model (16), from Eqs. (5)–(7), we obtain

$$32\pi(p + \Lambda) = \frac{(1-n)(1+3n)}{t^2}, \tag{17}$$

$$32\pi(\rho - \Lambda) = \frac{2k(1+2n-3n^2)t^{\frac{3n-1}{2}} + 3k_1(n-1)^2(3n-1)}{t^2[2kt^{\frac{3n-1}{2}} + (3n-1)k_1]}. \tag{18}$$

Using Eq. (9) in Eqs. (17) and (18), we obtain

$$\rho = \frac{(1-n)}{8\pi(1+\gamma)t^2} \left[\frac{k(3n+1)t^{\frac{3n-1}{2}} + (3n-1)k_1}{2kt^{\frac{3n-1}{2}} + (3n-1)k_1} \right], \tag{19}$$

$$\Lambda = \frac{(1-n)}{32\pi(1+\gamma)t^2} [2k(3n+1)(1-\gamma)t^{\frac{3n-1}{2}} + (3n-1)k_1\{(1-3\gamma) + 3n(1+\gamma)\}]. \tag{20}$$

The energy conditions given by Ellis³¹

- (i) $(\rho + p) > 0$,
- (ii) $(\rho + 3p) > 0$,
- (iii) $\rho > 0$,

are satisfied provided $k_1 > 0$, and $1/3 < n < 1$. The dominant energy conditions given by Hawking and Ellis³²

- (i) $(\rho - p) \geq 0$,
- (ii) $(\rho + p) \geq 0$,

are satisfied provided $k > 0$, $k_1 > 0$ and $1/3 < n < 1$.

We also observe from Eq. (20) that as t increases, Λ decreases. It is found that when $k = 0$ the cosmological parameter Λ varies as the inverse square of time, which coincides its natural units. Berman³³ suggested that the relation that is really important is $\Lambda \propto t^{-2}$. We also observe that $\Lambda > 0$ providing $k > 0$, $k_1 > 0$ and $1/3 < n < 1$. Therefore we obtained a positive cosmological Λ -term under the same conditions which also satisfy the dominant energy conditions. Recent years have witnessed a resurgence of interest in the possibility that a positive Λ -term (a cosmological constant) may dominate the total energy density in the universe.¹² Recent supernova measurements³⁴ also suggest a positive cosmological constant, allowing violation of the age constraint, and hence easing the situation.

For the velocity field u_i , the kinematical parameters are found to have the following expressions

$$V = \frac{t[k_1(3n - 1) + 2kt^{\frac{1}{2}(3n-1)}]}{(3n - 1)t^{\frac{1}{2}(3n-1)}}, \tag{21}$$

$$\theta = \frac{4kt^{\frac{1}{2}(3n-1)} + 3k_1(1 - n)(3n - 1)}{2t[2kt^{\frac{1}{2}(3n-1)} + k_1(3n - 1)]}, \tag{22}$$

$$\sigma = \left(\frac{3}{2}\right)^{\frac{1}{2}} \left\{ \frac{2k \left(n - \frac{1}{3}\right) t^{\frac{1}{2}(3n-1)}}{t[2kt^{\frac{1}{2}(3n-1)} + k_1(3n - 1)]} \right\}, \tag{23}$$

$$\omega = 0, \tag{24}$$

$$f_i = [0, 0, 0, 0], \tag{25}$$

where V , θ , σ , ω and f_i are respectively spatial volume, expansion scalar, shear tensor, rotation and acceleration vector. Hence the model (16) is expanding and shearing but non-rotating fluid which is also geodetic.

For $n = 1$, the metric (16) represents a dust universe. For $n \neq 1, -1/3, 1/3$ from Eqs. (19), (21) and (22), it is observed that at the singularity state $t = 0$, $V \rightarrow 0$ and p, ρ, θ and σ are infinitely large. As $t \rightarrow \infty$, $V \rightarrow \infty$ and p, ρ, θ and σ vanish. These results are consistent with Hubble law related with expansion of universe. Therefore, for $n \neq 1, -1/3, 1/3$, the solution (16) represents an anisotropic universe exploding from $t = 0$, which expands for $0 < t < \infty$ and after an infinitely large time t , would give essentially an isotropic empty universe.

Choosing $A = \epsilon_1 t^{\frac{1}{2}(1-n)} + \epsilon_2 t^n$ and $k = 0$ in Eq. (13), we get

$$B^2 = l_1 t^{(1-n)} + l_2 t^{\frac{1}{2}(1+n)} + l_3 t^{2n}, \tag{26}$$

where $l_1 = k_2 \epsilon_1^2$, $l_2 = 2k_2 \epsilon_1 \epsilon_2$, $l_3 = k_2 \epsilon_2^{2n}$.

Hence, in this case, the geometry of our universe is described by the line-element

$$ds^2 = -dt^2 + (\epsilon_1 t^{\frac{1}{2}(1-n)} + \epsilon_2 t^n)^2 dx^2 + (l_1 t^{(1-n)} + l_2 t^{\frac{1}{2}(1+n)} + l_3 t^{2n})(dy^2 + dz^2). \tag{27}$$

Thus, Eqs. (5), (7) and (9) give us the values of the physical parameters p and ρ and cosmological term Λ . The expressions for these parameters are quite lengthy and complicated. Therefore we shall not report them here but it is seen that the properties of the model (27) are the same as that of solution (16).

Choosing $B = t^{\frac{1}{3}}$ in Eq. (12), we obtain the geometry of the universe as

$$ds^2 = -dt^2 + t^{\frac{2}{3}}(k_1 + k \ln(t))^2 dx^2 + t^{\frac{2}{3}}(dy^2 + dz^2). \tag{28}$$

3.1. Exact solution

Let us assume

$$A = B^2. \tag{29}$$

Using Eq. (29) in (8) and then by integrating, we have

$$B^2 = 2(m_1 t + m_2)^{\frac{1}{2}}, \tag{30}$$

where m_1 and m_2 are constants of integration.

Using Eqs. (9), (29) and (30) in Eqs. (4)–(6) yield

$$\rho = \frac{5m_1^2}{64\pi(1 + \gamma)(m_1 t + m_2)^2}, \tag{31}$$

$$\Lambda = \frac{5(1 - \gamma)m_1^2}{128\pi(1 + \gamma)(m_1 t + m_2)^2}. \tag{32}$$

Therefore, the geometry of our universe is described by the line-element

$$ds^2 = dt^2 - 4(m_1 t + m_2)dx^2 - 2(m_1 t + m_2)^2(dy^2 + dz^2). \tag{33}$$

From Eq. (31), it is observed that $\rho > 0$ for all times and consequently the energy conditions³¹ are always satisfied. If we set $m_2 = 0$, the cosmological constant Λ varies as inverse square of time which matches its natural units. From Eq. (32) it is observed that the cosmological constant Λ is always positive and consequently Λ -term may dominate the total energy density in the universe.¹²

If we choose $\gamma = 1$, then from Eq. (32), $\lambda = 0$. Therefore in this case the solution (33) represents a cosmological model filled with a stiff fluid whose pressure and density are given by

$$p = \rho = \frac{5m_1^2}{128\pi(m_1 t + m_2)^2}. \tag{34}$$

Such models are important in relativistic cosmology for the description of early stages of the universe. For $m_1 = 0$, we obtain $p = 0$ and hence Eq. (33) represents a dust universe. Such type of solutions are obtained by Pradhan and Kumar.³⁰

4. Conclusions

We have investigated a new class of LRS Bianchi I cosmological models in which the cosmological constant varies with time. The nature of the cosmological constant Λ and the energy density ρ have been examined. The cosmological constant Λ gradually reduces as the universe expands. In Sec. 3, it is also observed that when $k = 0$, Λ varies as the inverse square of time, which matches its natural units. In Subsec. 3.1, it is found that when $m_2 = 0$, $\Lambda \sim t^{-2}$. This supports the views in favor of the dependence $\Lambda \sim t^{-2}$ first expressed by Bertolami^{21,22} and later on observed by several authors.^{15–20,23–25}

In this paper we have generalized the solutions of Refs. 26–28 and 30. For $\Lambda = 0$ and $n = -1/3$, from Eq. (16) one can obtain the solutions of Hajj-Boutros and Sfeila.²⁷ For $\Lambda = 0$ and $n = -1/3$ from Eq. (27) we obtain the solutions of Shri Ram.²⁸ For $\Lambda = 0$, $k = 0$ and $k_1 = 1$, the solution (28) represents the Einstein–de Sitter universe. For $\Lambda = 0$, from Eq. (16), we recover the models of Mazumder.²⁶ For $\gamma = 1$ the solution (33) represents a perfect fluid model with stiff matter investigated by Pradhan and Kumar,³⁰ and thus our solutions represent a generalization of Ref. 30.

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