

# Quantum states whose particle content is invariant under Bogoliubov transformation

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**Abstract.** It is shown that, for any given Bogoliubov transformation, there exists a class of quantum states with the following property. The particle content of these states does not change under the Bogoliubov transformation. We emphasise the importance of such states in the study of quantum fields in curved spacetime.

## 1. Motivation

The quantisation of a free field in a flat spacetime is usually carried out by introducing a Fock basis via the expansion

$$\varphi(\mathbf{x}, t) = \sum_k (a_k f_k + \text{HC}) \quad f_k = \frac{1}{\sqrt{2\omega_k}} \exp[-i(\omega_k t - \mathbf{k} \cdot \mathbf{x})]. \quad (1)$$

Canonical commutation rules between  $\varphi(\mathbf{x}, t)$  and its conjugate momentum  $\pi(\mathbf{x}, t)$  lead to the standard commutation rules for  $a_k$ :

$$[a_k, a_p] = [a_k^+, a_p^+] = 0 \quad [a_k, a_p^+] = \delta_{kp}. \quad (2)$$

The basis states can now be labelled by a set of integers  $n_k$ . In particular, there exists a ground state  $|0\rangle$  (usually called 'vacuum') with all  $n_k$  set to zero. This state is annihilated by all the  $a_k$ .

The basis functions  $f_k(\mathbf{x}, t)$  in (1) are the most natural choice for geodesic observers in flat spacetime. However, non-geodesic observers may often find an entirely different mode decomposition to be more natural (see, e.g., Fulling 1973, Unruh 1976). Consider a particular non-geodesic observer using a coordinate chart  $(t', \mathbf{x}')$ ; further assume that the metric in terms of  $(t', \mathbf{x}')$  is independent of  $t'$ . (A well known example is that of a uniformly accelerated observer; in general, any observer moving along a trajectory which is an integral curve of any timelike Killing field of the flat spacetime will exhibit this feature (Letaw 1981, Padmanabhan 1982).) For such an observer, it is natural to use the decomposition,

$$\varphi(t, \mathbf{x}) = \varphi'(t', \mathbf{x}') = \sum_m (A_m F_m(t', \mathbf{x}') + \text{HC}) \quad (3)$$

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where  $F_m(t', x') \sim \exp(-i\nu_m t')g_m(x')$ . The new annihilation ( $A_m$ ) and creation ( $A_m^+$ ) operators are related to the old ones by a ‘Bogoliubov transformation’

$$A_m = \sum_k (\alpha_{mk} a_k + \beta_{mk} a_k^+) \tag{4}$$

$$A_m^+ = \sum_k (\alpha_{mk}^* a_k^+ + \beta_{mk}^* a_k). \tag{5}$$

It is easy to verify that  $A_m, A_m^+$  satisfy the same commutation rules as  $a_k, a_k^+$ , provided

$$\alpha\beta^T - \beta\alpha^T = 0 \quad \alpha\alpha^+ - \beta\beta^+ = 1. \tag{6}$$

(Here and in what follows  $\alpha = \{\alpha_{mp}\}$  and  $\beta = \{\beta_{mp}\}$  are treated as matrices.) A new ground state  $|0\rangle$  can be defined by  $A_m|0\rangle = 0$ .

It follows immediately that the ‘particle content’ of the Fock basis is not invariant under Bogoliubov transformations. For example, the old ground state  $|0\rangle$  will contain ‘particles’ in the sense that

$$\sum_m \langle 0, \text{old} | A_m^+ A_m | \text{old}, 0 \rangle = \text{Tr}(\beta\beta^+) \neq 0. \tag{7}$$

This result demonstrates the fact that the concept of a ‘particle’ based on Fock states is observer dependent.

In a genuine curved spacetime, the situation becomes more complicated (Birrell and Davies 1982). Let us suppose that, in some suitably defined asymptotic past ( $t \rightarrow -\infty$ ), a set of creation and annihilation operators  $a_k, a_k^+$  are chosen. In the Heisenberg picture, these operators will evolve in time. In the asymptotic future, they will, in general, end up as a different set  $A_m, A_m^+$  with a Bogoliubov transformation (4), connecting them to the original set. Let the quantum state of the field (which does not change in time) be the ‘in vacuum’,  $|0, \text{in}\rangle$ , which is annihilated by all the  $a_k$ . In the asymptotic future, this state  $|0, \text{in}\rangle$  will appear to contain ‘particles’ in the sense that

$$\sum_k \langle 0, \text{in} | A_k^+ A_k | \text{in}, 0 \rangle = \text{Tr}(\beta\beta^+) \neq 0. \tag{8}$$

The expectation value of  $A^+ A$  will be different from the expectation value of  $a^+ a$  in any Fock basis state. Thus the particle content of any Fock basis state will change under a Bogoliubov transformation.

We may however ask whether there is any state  $|\psi\rangle$  in the Hilbert space such that

$$A|\psi\rangle = e^{i\theta} a|\psi\rangle \equiv \varepsilon a|\psi\rangle. \tag{9}$$

(Here  $\theta$  is a real angle and  $\varepsilon \equiv e^{i\theta}$ .) In other words, we are looking for states on which the action of  $A$  and  $a$  is the same, except for the phase. As a direct consequence of (9) we obtain

$$\langle \psi | A^+ A | \psi \rangle = \langle \psi | a^+ a | \psi \rangle. \tag{10}$$

The importance of  $|\psi\rangle$  originates from (10). Suppose a quantum field is in the state  $|\psi\rangle$ ; then both the inertial observer (using  $a_k$ ) and the non-inertial observer (using  $A_k$ ) will attribute the same number of particles to the state  $|\psi\rangle$ . Similarly if a quantum field in a black hole spacetime was in the state  $|\psi\rangle$  in the asymptotic past, then the particle content of this state will appear to be the same at asymptotic future.

We shall show in this paper that such states do exist and will construct them explicitly. The result is demonstrated for a single degree of freedom in § 2 and is generalised to the infinite degrees of freedom in § 3.

## 2. Harmonic oscillator and Bogoliubov transformations

Consider a harmonic oscillator with the Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) = \frac{1}{2}\omega(2a^+ a + 1) \quad (11)$$

where

$$a = \frac{1}{\sqrt{2\omega}}(p - i\omega q) = \frac{1}{\sqrt{2\omega}}\left(-i\frac{\partial}{\partial q} - i\omega q\right) \quad (12)$$

$$a^+ = \frac{1}{\sqrt{2\omega}}(p + i\omega q) = \frac{1}{\sqrt{2\omega}}\left(-i\frac{\partial}{\partial q} + i\omega q\right). \quad (13)$$

We now make a Bogoliubov transformation to a new set of  $A$  and  $A^+$

$$A = \alpha a + \beta a^+ \quad A^+ = \alpha a^+ + \beta a. \quad (14)$$

For simplicity we have assumed  $(\alpha, \beta)$  to be real. The condition

$$[A, A^+] = 1 \quad (15)$$

implies that

$$\alpha^2 - \beta^2 = 1 \quad \alpha = (1 + \beta^2)^{1/2}. \quad (16)$$

Thus our Bogoliubov transformation is characterised by a single real parameter  $\beta$ .

We now look for states  $|\psi\rangle$  such that the following relation is satisfied:

$$A|\psi\rangle = (\alpha a + \beta a^+)|\psi\rangle = e^{i\theta} a|\psi\rangle. \quad (17)$$

In the coordinate representation (using (12) and (13)) we can reduce (17) to the differential equation

$$-i(\alpha - \varepsilon + \beta)\frac{\partial\psi}{\partial q} = i\omega q(\alpha - \varepsilon - \beta)\psi \quad (18)$$

where  $\varepsilon = e^{i\theta}$ . Integrating, we get

$$\psi(\alpha, \beta, \theta, q) = N \exp(-\frac{1}{2}z\omega q^2) \quad (19)$$

with

$$z = \left(\frac{\alpha - \beta - \varepsilon}{\alpha + \beta - \varepsilon}\right). \quad (20)$$

Thus for a given  $(\alpha, \beta)$ , we can thus construct a series of states  $|\alpha, \beta; \theta\rangle$  parametrised by  $\theta$  such that

$$\langle\psi|A^+ A|\psi\rangle = \langle\psi|a^+ a|\psi\rangle. \quad (21)$$

What is the allowed range of  $\theta$ ? To determine this range consider the normalisability of  $\psi$ . We have

$$|\psi|^2 = N^2 \exp[-\frac{1}{2}\omega(z + z^*)q^2] \quad (22)$$

clearly

$$N = \left(\frac{\omega(z + z^*)}{2\pi}\right)^{1/4}. \quad (23)$$

The function  $|\psi|^2$  is normalisable only if

$$(z^* + z) > 0. \tag{24}$$

Using (20) in (24) we get

$$\cos \theta < \alpha^{-1}. \tag{25}$$

Thus the range of  $\theta$  is limited to  $-1 \leq \cos \theta \leq \alpha^{-1}$ , or equivalently,  $\theta_0 < \theta < (2\pi - \theta_0)$  where  $\theta_0$  is the principal root of  $\alpha \cos \theta_0 = 1$ . Note that the state with  $\theta = 0$  is not normalisable. There are no proper states in the Hilbert space on which the actions of  $a$  and  $A$  are identical.

The expectation values of various operators in the state can be easily computed. For example

$$\langle \psi | n | \psi \rangle = \langle a^+ a \rangle = \langle A^+ A \rangle = \frac{1}{2} \frac{|z-1|^2}{(z+z^*)} = \frac{1}{2} \frac{\beta^2}{(1-\alpha \cos \theta)} \tag{26}$$

and

$$\langle \psi | H | \psi \rangle = \frac{1}{2} \omega \left( \frac{1+zz^*}{z+z^*} \right) = \frac{1}{2} \omega \coth(k+k^*) \tag{27}$$

where we have defined a complex number  $k$  by  $z = \tanh k$ .

Before proceeding further, let us summarise what has been done so far. With every Bogoliubov transformation  $(\alpha, \beta) = ((1+\beta^2)^{1/2}, \beta)$  parametrised by some real  $\beta$ , we associate a one-parameter family of states  $\psi(q, \theta)$  with  $-1 < \cos \theta < \alpha^{-1}$ . (Equivalently we may parametrise these states by  $z$ , taking  $\psi = \psi(q, z)$  where  $z$  is given by (20).) The ‘particle content’ of each of these states, defined by  $\langle \theta | a^+ a | \theta \rangle$  remains invariant under Bogoliubov transformations in the sense that

$$\langle \theta | a^+ a | \theta \rangle = \langle \theta | A^+ A | \theta \rangle = \frac{1}{2} \frac{\beta^2}{(1-\alpha \cos \theta)}. \tag{28}$$

Among the various Bogoliubov transformations let us choose a particular set for which

$$\beta(T) = \frac{1}{(e^{\omega/T} - 1)^{1/2}} \quad \alpha(T) = \frac{e^{\omega/2T}}{(e^{\omega/T} - 1)^{1/2}}. \tag{29}$$

Under such a Bogoliubov transformation, the ‘old’ ground state (defined by  $a|0\rangle = 0$ ) will appear to be a Planckian state in terms of  $A$ :

$$\langle 0 | A^+ A | 0 \rangle = \beta^2 = \frac{1}{(e^{\omega/T} - 1)}. \tag{30}$$

On the other hand, the states  $|\theta\rangle$  will exhibit the invariant particle content:

$$\langle \theta | a^+ a | \theta \rangle = \langle \theta | A^+ A | \theta \rangle = \frac{1}{(e^{\omega/T} - 1)} \left( \frac{1}{2(1-\alpha \cos \theta)} \right). \tag{31}$$

Interestingly enough, there exists one particular state  $|\theta_1\rangle$  such that  $\cos \theta_1 = (2\alpha)^{-1}$ , satisfying

$$\langle \theta_1 | a^+ a | \theta_1 \rangle = \langle \theta_1 | A^+ A | \theta_1 \rangle = \frac{1}{(e^{\omega/T} - 1)}. \tag{32}$$

This state will exhibit a Planckian expectation value in terms of the 'old' as well as 'new' number operators!

We emphasise that we are here dealing with pure states and not with density matrices. The  $|\theta_1\rangle$  is just one among an infinite number of states for which the expectation value of  $(a^+a)$  is Planckian. It would, of course, be wrong to call  $|\theta_1\rangle$  a thermal state.

In general, these states do not possess any simple distinguishing feature. They are neither orthonormal nor complete over the set of square integrable functions. We can, of course, expand these states in terms of standard Fock basis:

$$|z\rangle = \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=0}^{\infty} c_n \frac{(a^+)^n}{\sqrt{n!}} |0\rangle. \tag{33}$$

Since

$$c_n = \langle n | z \rangle = \int_{-\infty}^{+\infty} dq \psi(z, q) \varphi_n^*(q) \tag{34}$$

we can write

$$\sum_{n=0}^{\infty} c_n \frac{(a^+)^n}{(n!)^{1/2}} = \int_{-\infty}^{+\infty} dq \psi(z, q) \sum_{n=0}^{\infty} \frac{\varphi_n^*(q)(a^+)^n}{(n!)^{1/2}}. \tag{35}$$

Using the relations

$$\varphi_n(q) = \left(\frac{1}{2^n n!}\right)^{1/2} \left(\frac{\omega}{\pi}\right)^{1/4} e^{-\frac{1}{2}\omega q^2} H_n((\omega q^2)^{1/2}) \tag{36}$$

and

$$\sum_{n=0}^{\infty} \frac{H_n((\omega q^2)^{1/2})(Q/\sqrt{2})^n}{(n!)^{1/2}} = \exp(-\frac{1}{2}Q^2 + \sqrt{2\omega} Qq) \tag{37}$$

equation (35) can be simplified further. After straightforward algebra we get the result:

$$|z\rangle = \left(\frac{2(z+z^*)}{(z+1)^2}\right)^{1/4} \exp\left[-\frac{1}{2}\left(\frac{z-1}{z+1}\right)(a^+)^2\right] |0\rangle. \tag{38}$$

As is to be expected  $|z\rangle$  is a superposition of  $2n$ -particle states with  $n=0, 1, 2, \dots$

All the above discussion was confined to a single instant of time. For these states to be useful it is necessary for time evolution to preserve the essential features. This is indeed true with a small modification: the value of  $z$  at time  $t$  is given by

$$z = \frac{\alpha - \varepsilon - \beta e^{-2i\omega t}}{\alpha - \varepsilon + \beta e^{-2i\omega t}} \tag{39}$$

which reduces to (20) at  $t=0$ . To derive this result consider the evolution of a Gaussian state

$$\psi(q, t) = N(t) \exp(-B(t)q^2) \tag{40}$$

in a harmonic oscillator potential. Substituting (40) into the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial q^2} + \frac{1}{2} \omega^2 q^2 \psi \tag{41}$$

and equating coefficients of  $q^2, q^1, q^0$ , we get

$$i \frac{\dot{N}}{N} = B \quad i \dot{B} = 2B^2 - \frac{1}{2}\omega^2. \tag{42}$$

Writing  $B = -\frac{1}{2}i(\dot{\sigma}/\sigma)$ , and manipulating the equations it is easy to show that

$$N = (2\pi|\sigma^2|)^{-1/4} \tag{43}$$

and

$$\sigma(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}. \tag{44}$$

Thus

$$B(t) = \frac{\omega}{2} z(t) = \frac{\omega}{2} \left( \frac{c_1 e^{i\omega t} - c_2 e^{-i\omega t}}{c_1 e^{i\omega t} + c_2 e^{-i\omega t}} \right). \tag{45}$$

Taking  $z(0)$  to be given by (20) we can set

$$c_1 = \alpha - \varepsilon \quad c_2 = \beta. \tag{46}$$

Substituting (46) into (45) and cancelling out  $e^{i\omega t}$  leads to equation (39).

All the essential properties of the state are preserved under time evolution. In particular, the number operator (which commutes with the Hamiltonian) has an expectation value which is unchanged in time.

### 3. Extension to many degrees of freedom

The basic idea of the previous section carries through for the case of many degrees of freedom as well, except for the problem of matrix manipulation. Consider a set of oscillators described by the Hamiltonian

$$H = \sum_k \frac{1}{2}(p_k^2 + \omega_k^2 q_k^2) = \sum_k \frac{1}{2}\omega_k(2a_k^+ a_k + 1). \tag{47}$$

We now make the Bogoliubov transformations

$$A_m = \sum_k (\alpha_{mk} a_k + \beta_{mk} a_k^\dagger) \tag{48}$$

$$A_m^+ = \sum_k (\alpha_{mk}^* a_k^\dagger + \beta_{mk}^* a_k). \tag{49}$$

The requirement that  $A$  satisfies the same commutation rules as  $a$  implies

$$\alpha\alpha^+ - \beta\beta^+ = 1 \quad \alpha\beta^\dagger = \beta\alpha^\dagger. \tag{50}$$

Here we have introduced the matrices  $\alpha = \{\alpha_{mk}\}$  and  $\beta = \{\beta_{mk}\}$ . Following the discussion of the previous section, we look for states  $|\psi\rangle$  which satisfy the criterion (9) for all  $k$ , with  $|\varepsilon_k| = 1$ . Let us try the ansatz

$$\langle \{q_k\} | \psi \rangle = \psi(z, q) = N \exp\left(-\frac{1}{2} \sum_{k,p} \sqrt{\omega_k} \sqrt{\omega_p} z_{kp} q_k q_p\right) \tag{51}$$

where  $z = \{z_{kp}\}$  is a symmetric matrix. Using

$$a_k = \frac{1}{\sqrt{2\omega_k}}(p_k - i\omega_k q_k) \quad a_k^+ = \frac{1}{\sqrt{2\omega_k}}(p_k + i\omega_k q_k) \tag{52}$$

and equations (48) and (51) in (9) we can obtain an equation for  $z_{km}$

$$(\alpha - \varepsilon)(z - 1) + \beta(z + 1) = 0. \quad (53)$$

(In this the matrix  $\varepsilon = \text{diag}(\varepsilon_k)$  and '1' stands for the unit matrix.) Solving (53), we get

$$z = (\alpha - \varepsilon + \beta)^{-1}(\alpha - \varepsilon - \beta). \quad (54)$$

Thus we have demonstrated the existence of states with 'invariant particle content'. For an arbitrary  $(\alpha, \beta)$  it is not possible to express  $z$  in any simpler form. Expectation values of physically relevant quantities can, of course, be expressed in terms of  $z$ . For example, the total number operator has the expectation value

$$\sum_k \langle \psi | A_k^+ A_k | z \rangle = \sum_k \langle \psi | a_k^+ a_k | \psi \rangle \quad (55)$$

$$= \frac{1}{2} \text{Tr} \{ z(z + z^*)^{-1} z^* + (z + z^*)^{-1} - 1 \}. \quad (56)$$

#### 4. Conclusions

Usually, quantum field theory in curved spacetime is discussed in the vacuum state of the quantum field. The results derived here suggest that it may be of interest to look at other quantum states of the field. It is possible, in principle, to generalise our results to curved spacetime. These results will be published elsewhere.

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