Draft version April 20, 2011
Preprint typeset using LaTeX style emulateapj v. 04/03/99

PSEUDO-NEWTONIAN POTENTIALS TO DESCRIBE THE TEMPORAL EFFECTS ON RELATIVISTIC ACCRETION DISKS AROUND ROTATING BLACK HOLES AND NEUTRON STARS

Banibrata Mukhopadhyay
and
Ranjeev Misra
Inter-University Center for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune-411007, India;
bm@iucaa.ernet.in; rmisra@iucaa.ernet.in
Draft version April 20, 2011

ABSTRACT

Two pseudo-Newtonian potentials, which approximate the angular and epicyclic frequencies of the relativistic accretion disk around rotating (and counter rotating) compact objects, are presented. One of them, the Logarithmically Modified Potential, is a better approximation for the frequencies while the other, the Second-order Expanded potential, also reproduces the specific energy for circular orbits in close agreement with the General Relativistic values. These potentials may be included in time dependent hydrodynamical simulations to study the temporal behavior of such accretion disks.

Subject headings: accretion, accretion disks—black hole physics—gravitation—relativity

1. INTRODUCTION

X-ray binaries are known to be powered by accretion disks around Neutron Stars and Black Holes. The rapid variability of these sources indicate that the X-ray emission arises from the inner accretion disk where the effects of strong gravity are important. High frequency ($\approx$ kHz) Quasi Periodic Oscillations (QPO) have been observed in neutron star systems (see van der Klis 2000 for a review) while slightly lower frequencies ($\approx$ 450 Hz) QPOs have been detected in black hole systems (Strohmayer 2001). For neutron star systems the kHz QPO tends to be observed in pairs. Associating these frequencies with a Keplerian frequency in the disk, leads to the conclusion that the phenomena originate at radii less than 20 gravitational radius ($r_g \equiv GM/c^2$).

A number of theoretical ideas have been proposed to explain the phenomenology of kilohertz QPO. In all these models, one of the two frequencies observed in neutron star systems, is identified as the Keplerian frequency of the innermost orbit of an accretion disk. The sonic point model (Miller et al. 1998) identifies the second frequency as the beat of the primary QPO with the spin of the neutron star, while according to the two oscillators model (Osherovich & Titarchuk 1999), the secondary frequency is due to the transformation of the primary (Keplerian) frequency in the rotating frame of the neutron star magnetosphere. On the other hand, Stella & Vietri (1999) have proposed a General Relativistic (GR) precession/apsidal motion model, wherein the primary frequency is the Keplerian frequency of a slightly eccentric orbit and the secondary is due to the relativistic apsidal motion of this orbit i.e. the secondary frequency is the Keplerian frequency minus the epicyclic one. These models in general are based on identifying the characteristic frequencies of the system with observed ones and often do not address the issue of how such oscillations occur in the accreting flow.

A complete understanding of the QPO phenomena would require a self consistent hydrodynamical simulation of the accreting flow in general relativity. While such an ambitious endeavor has been impeded for several reasons, the main difficulties can be identified to be (a) the development of a self-consistent turbulent viscosity and (b) the inclusion of GR effects. In hydrodynamical simulations, turbulent viscosity has typically been introduced in a parametric form like the α-parameterization (e.g. Taam & Lin 1984). Since the temporal behavior of accretion disks is expected to depend on the form of the viscosity law, the results of such simulations were not conclusive. A promising mechanism for driving the turbulence responsible for angular momentum and energy transport is the action of the magneto-rotational instability (MRI) that is expected to take place in such disks (Balbus & Hawley 1991). Recent 3D magneto hydrodynamical (MHD) simulations have shown that indeed the MRI can give rise to a turbulent viscosity which leads to the accretion flow in a Keplerian disk (Hawley, Balbus & Stone 2001). While presently such simulations do not include radiation (and hence do not describe optically thick accretion flow), it is expected that self-consistent simulations will be possible in the near future and the temporal behavior of accretion disks can be studied with confidence.

Despite these recent advances, it is still extremely difficult to simulate realistic accretion flows in a complete GR framework. However, relativistic effects may be approximately simulated by using modified Newtonian (or Pseudo-Newtonian) potentials in the non-relativistic radial-momentum equation. Paczyński & Wiita (1980) proposed such a pseudo-Newtonian potential which has been frequently used in simulations (e.g. Milsom & Taam 1997; Hawley & Balbus 2002). Here the Newtonian potential has been replaced by $\phi = GM/(r - 2r_g)$. The attractive feature of the potential is that it reproduces the last stable orbit exactly and the specific energies of circular orbits within 10% of the GR values (i.e. for Schwarzschild geometry). Several other pseudo-Newtonian potentials have been proposed and used in the literature (e.g. Chakrabarti & Khanna 1992). Artemova et al. (1996) have considered several such potentials and concluded that the

Paczyński-Wiita potential is better than the rest based on the above criteria for non-rotating compact objects. Recently, Mukhopadhyay (2002) has proposed a pseudo-potential which is valid for rotating compact objects. This potential reproduces the GR values of last stable orbit exactly and is a good approximation ($< 10\%$ error) for the specific energy at last stable circular orbit in case of Kerr geometry. It also reduces to the Paczyński-Wiita potential when the spin of the black hole is set to zero.

However, these potentials are not a good approximation (with error $> 50\%$) for the angular and epicyclic frequencies for radii $< 20r_g$. Thus, while they are adequate to approximate the relativistic effects for a steady state accretion disk, they can not quantitatively reproduce the temporal behavior of a disk since that is expected to depend on the disk's characteristic (i.e. the angular and epicyclic) frequencies. Nowak & Wagoner (1991) have proposed a potential for a non rotating black hole which reproduces the Keplerian frequencies (with deviations $< 15\%$) and the epicyclic frequencies (with deviations less than 45%) and hence is better than the Paczyński-Wiita potential for such applications.

In this paper, we present two pseudo-Newtonian potentials which may be used to simulate the relativistic time varying effects in accretion disks around a compact object that may be co-rotating or counter-rotating with respect to the disk with the spin parameter $a < 0.99$. For faster spin rates the predictions of these potentials deviate pronouncedly (with errors $> 200\%$) and hence are no longer a good approximation. The first has been named the *Second-order Expansion Potential (SEP)* since it contains terms up to $(r_{ms}/r)^2$, where r_{ms} is the marginally stable orbit. This potential reproduces the specific energy and the angular frequency with deviations $< 10\%$ and $< 25\%$, respectively from GR values (i.e. for Kerr geometry). The deviations in epicyclic frequency range from $25 - 170\%$ (for $a \leq 0.9$) depending on the spin rate of the compact object. When the object is not rotating, the potential reduces to the one proposed by Nowak & Wagoner (1991). The second has been named *Logarithmically Modified Potential (LMP)* since it contains a logarithmic term. This potential reproduces well the angular (with deviations $< 20\%$ for co-rotating and $< 40\%$ for counter-rotating flows) and epicyclic frequencies (with deviations $< 60\%$) but predicts specific energies which are around 30% different from the GR values.

2. PSEUDO-NEWTONIAN POTENTIALS

Since hydrodynamical code directly require the gravitational acceleration, it is practical to modify the Newtonian force instead of the potential. In terms of such a modified force per unit mass (F), the angular (Ω) and epicyclic (κ) frequencies are given by

$$\Omega^2 = \frac{F}{R} \tag{1}$$

and

$$\kappa^2 = \frac{2\Omega}{R}\frac{d}{dR}(\Omega R^2) = \frac{1}{R^3}\frac{d}{dR}(FR^3). \tag{2}$$

These Newtonian (or pseudo-Newtonian) frequencies should match with the GR ones (Ω_{GR} and κ_{GR}) as seen by an observer at infinity. In terms of the dimensionless radial coordinate ($r = R/r_g$) and spin parameter (a) these frequencies are given by (e.g. Semerák & Zácek 2000),

$$\Omega_{GR} = \frac{1}{r^{3/2} + a} \tag{3}$$

and

$$\kappa_{GR}^2 = \left(\frac{\Omega_{GR}}{r}\right)^2 [\Delta - 4(\sqrt{r} - a)^2] \tag{4}$$

where $\Delta = r^2 - 2r + a^2$. For the above equations and rest of the paper we have used dimensionless quantities by setting G, M and c to be unity.

The modified force should also reproduce the marginal stable radius (r_{ms}) given by (Bardeen 1973)

$$\begin{aligned} r_{ms} &= 3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2} \\ Z_1 &= 1 + (1 - a^2)^{1/3}[(1 + a)^{1/3} + (1 - a)^{1/3}] \\ Z_2 &= (3a^2 + Z_1^2)^{1/2}, \end{aligned} \tag{5}$$

where the '-' ('+') sign is for the co-rotating (counter-rotating) flow.

Since either one of the relativistic frequencies (Ω_{GR} and κ_{GR}) can specify the form of the required modified force, F (from Eqns. 1 or 2), it is clear that a modified Newtonian force cannot reproduce both the frequencies exactly. Hence an appropriately chosen modified force should correspond to frequencies which have minimal deviation from the GR values. Here we present two such modified (or pseudo-Newtonian) forces. Both the potentials are constructed in such a manner that the marginally stable orbit (r_{ms}) is always same as the relativistic value (i.e. in Kerr geometry).

2.1. *Second-order Expansion Potential (SEP)*

For this potential the Newtonian force per unit mass is modified to be,

$$F = \Omega^2 r = \frac{1}{r^2}\left[1 - \left(\frac{r_{ms}}{r}\right) + \left(\frac{r_{ms}}{r}\right)^2\right] \tag{6}$$

where r_{ms} is given by Eqn. (5). The corresponding epicyclic frequency is

$$\kappa = \frac{1}{r^{3/2}}\left[1 - \left(\frac{r_{ms}}{r}\right)^2\right]^{1/2}. \tag{7}$$

In Figs. 1, 2, and 3, the variation of angular and epicyclic frequencies with radii are compared with GR values for three different values of the spin parameters $a = 0, 0.5, 0.9$. Figure 4 shows the variations of these frequencies for spin parameter $a = 0.99$ where the deviations from the GR values are large and the potentials described in the work are no longer a good approximation, particularly for the epicyclic frequency. The main advantage of this potential is its relative simplicity and that the angular frequencies deviate from the GR values by less than or equals to 25%. The specific energy (i.e. the energy per unit mass for a circular orbit) is also close (error is at most $\sim 10\%$ for all values of the Kerr parameter including $a = 1$) to the relativistic values (see Fig. 5). Its disadvantage is that κ deviates from κ_{GR} by around 40% for low and by nearly 150% for high spin values ($a \approx 0.9$) of the compact object (Fig. 3). However, for higher counter-rotation of the compact object the error in κ reduces to $\sim 25\%$.

2.2. *Logarithmically Modified Potential (LMP)*

Here the Newtonian force is modified to be

$$F = \frac{1}{r^2}\left[1 + r_{ms}\left\{\frac{9}{20}\frac{(r_{ms}-1)}{r} - \frac{3}{2r}log\left(\frac{r}{(3r-r_{ms})^{2/9}}\right)\right\}\right] \quad (8)$$

and the corresponding epicyclic frequency is given by

$$\kappa^2 = \frac{3}{2r^4}\frac{(r-r_{ms})(2r-r_{ms})}{(3r-r_{ms})}. \quad (9)$$

Note that the term in the force which depends on r^{-3} does not contribute to κ (Eqn. 2). The logarithmic form of the modified force (Eqn. (8)) was obtained by integrating the epicyclic frequency expression (9) whose form was guessed to be a good approximation. The advantage of this potential is that both the angular and epicyclic frequencies are generally better comparable with the GR values than the SEP (Figs. 1, 2 and 3). Its disadvantage is that the specific energy deviates by more than 30% from the GR values which is substantially larger than the deviation for SEP (Fig. 5).

3. SUMMARY AND DISCUSSION

In this work, we have presented two pseudo-Newtonian potentials which approximate the general relativistic effects on an accretion disk around rotating compact objects. These two potentials are designed particularly to approximate the angular and epicyclic frequencies of the accretion disk as seen by an observer at infinity. Table 1 summarizes the results by comparing the maximum percentage deviations from relativistic values (in Kerr geometry) for the two potentials and comparing them with those of another standard pseudo-potential.

The SEP not only approximates the frequencies well, but also the specific energies for circular orbits turn out to be remarkably close to the relativistic values. Thus based on such criteria, this potential is better than other pseudo-Newtonian potentials given in the literature and can be used to simulate both the steady state and time varying accretion disks. The LMP while being a better approximation to the frequencies than SEP, gives rather large ($\approx 30\%$) deviation from the GR results for the specific energies. Hence its utility is perhaps limited to the time-dependent studies of accretion disks.

Which one of these two potentials should be used in a hydrodynamical simulation depends on problem being addressed. Acoustic waves (which depend on the epicyclic frequencies) would perhaps be better simulated by the LMP while the SEP may be more suited for the long term temporal behavior (which may depend also on the energy dissipation). Moreover, a temporal behavior detected in a simulation could be an artifact of the pseudo-Newtonian potential rather than true GR effects. Hence, it will be prudent to confirm the behavior using both the potentials. Since the mathematical forms of the two potentials are quite different any temporal behavior detected for both the potentials would imply that the behavior is indeed due to relativistic effects. Use of these potentials in hydrodynamical simulations of accretion disk will help in the understanding of relativistic effects and may serve as a guideline for advanced simulations in general relativity.

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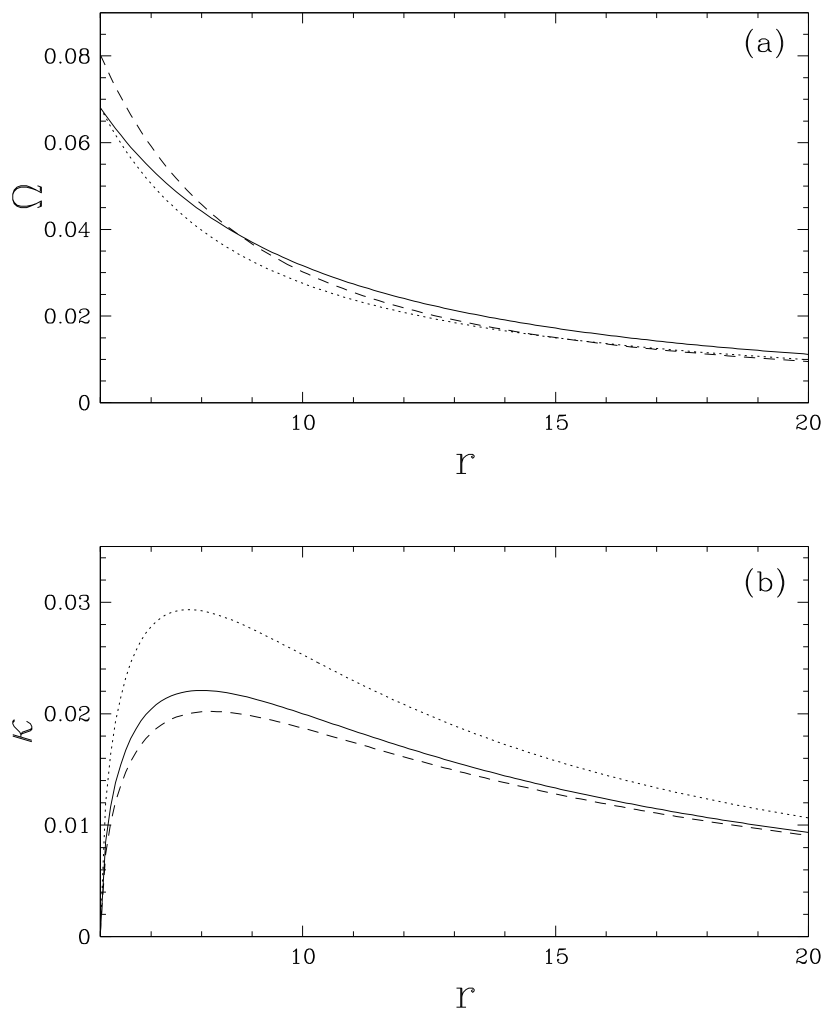

FIG. 1.— Variation of (a) angular and (b) epicyclic frequencies with radii for a non rotating compact object ($a = 0$). The solid line is for general relativity, dotted line is for the SEP (in this case same as the potential given by Nowak & Wagoner 1991) and the dashed line is for the LMP.

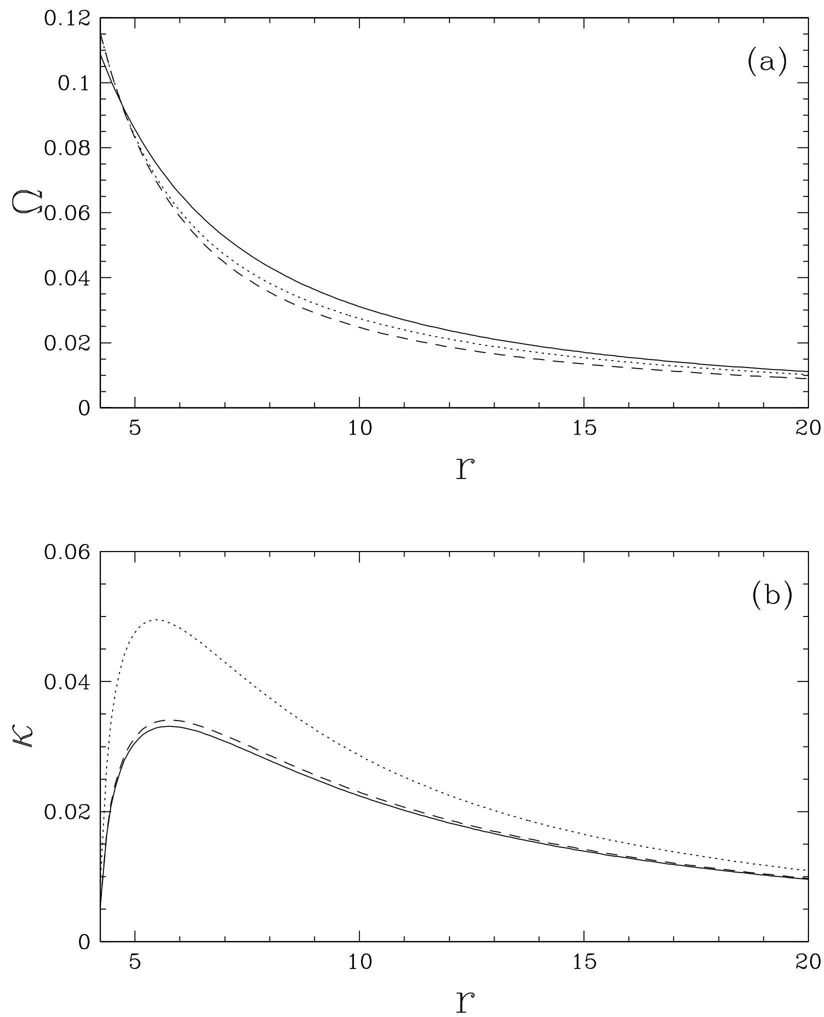

FIG. 2.— Same as in Fig. 1 except that $a = 0.5$

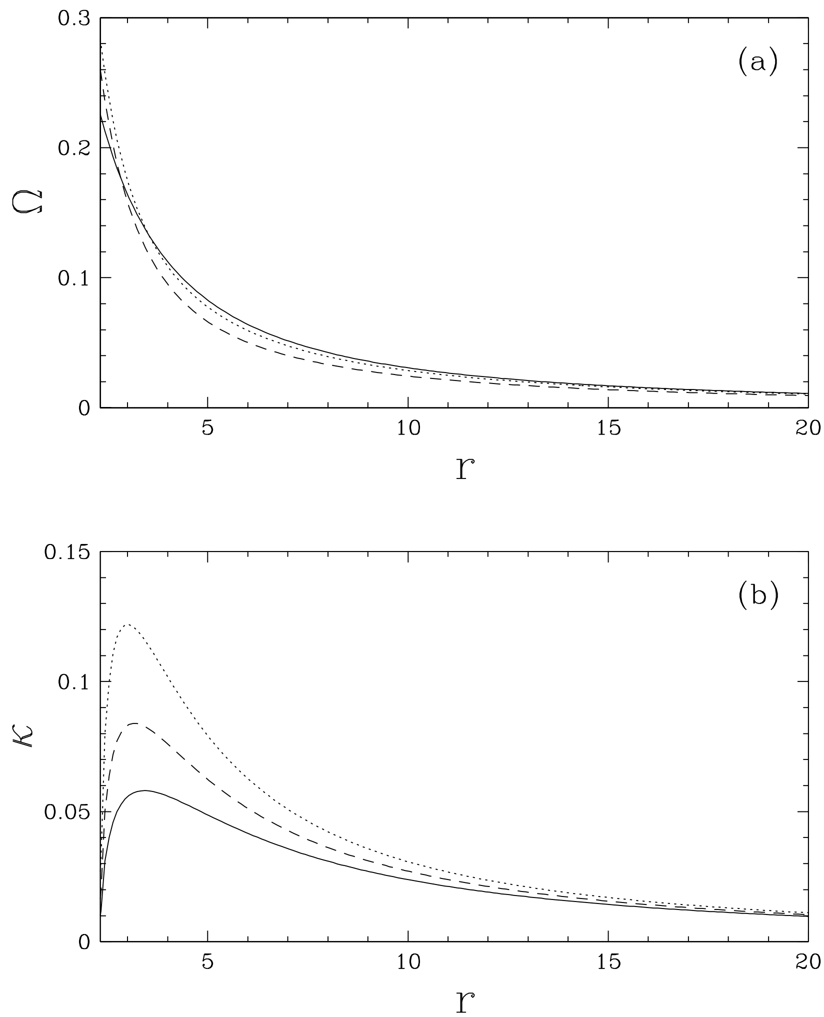

Fig. 3.— Same as in Fig. 1 except that $a = 0.9$.

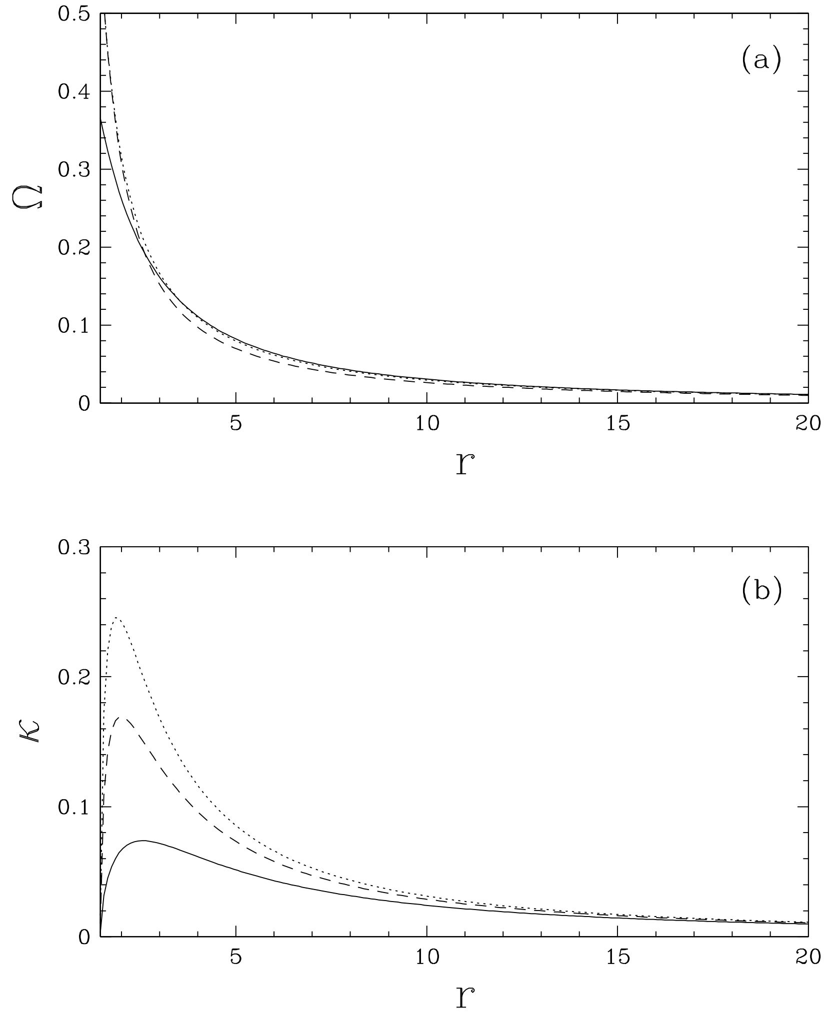

FIG. 4.— Same as in Fig. 1 except that $a = 0.99$.

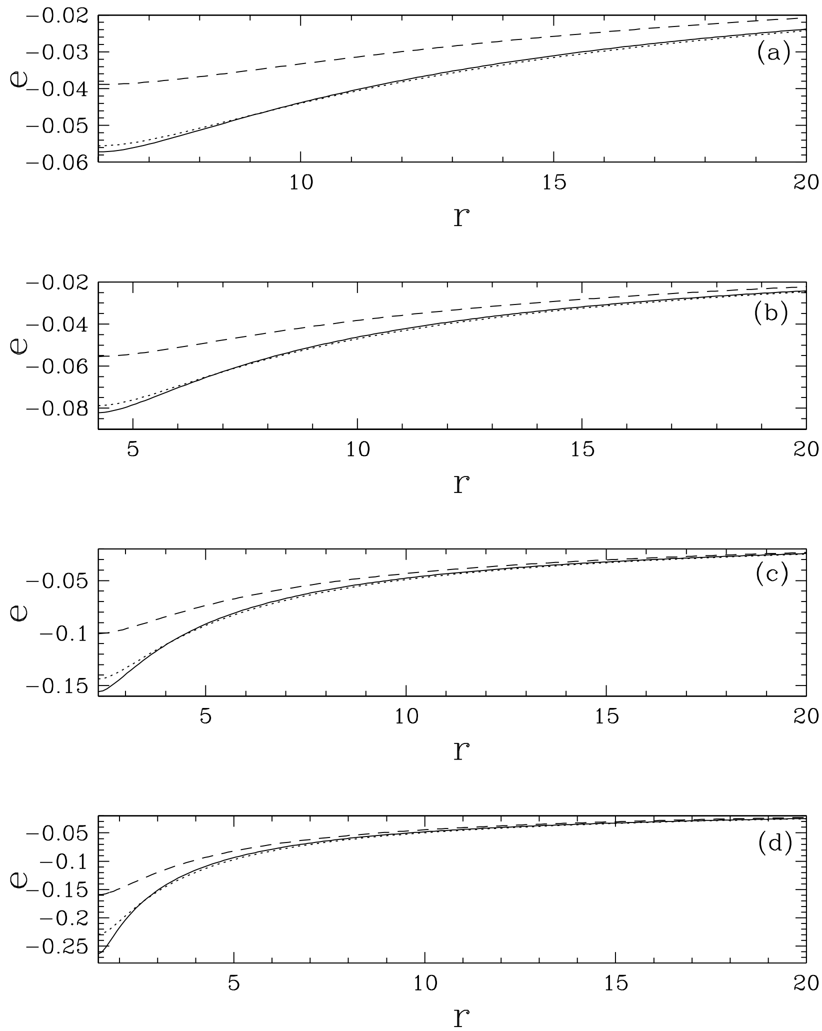

FIG. 5.— Variation of specific energy for circular orbits with radii for (a) non-rotating compact object as $a = 0$, (b) rotating compact object as $a = 0.5$, (c) rotating compact object as $a = 0.9$ and (d) rotating compact object as $a = 0.99$. The solid line is for general relativity, dotted line is for the SEP while the dashed line is for the LMP.

Table 1

Maximum percentage deviation of angular frequency ($\Delta\Omega$), epicyclic frequency ($\Delta\kappa$) and specific energy (Δe) for the Second-order Expanded Potential (SEP), Logarithmically Modified Potential (LMP) and the potential given by Mukhopadhyay (2002) (M). a is the spin parameter with positive sign indicating co-rotation and negative sign indicating counter-rotation and r_{ms} is the radius of the marginal stable orbit.

Pseudo-Newtonian Potential	a	$\Delta\Omega$ (%)	$\Delta\kappa$ (%)	Δe (%)	r_{ms}
SEP	0.99	57	445	13	1.45
LMP	0.99	57	240	40	1.45
M	0.99	180	800	14	1.45
SEP	0.9	25	168	8	2.32
LMP	0.9	22	64	35	2.32
M	0.9	100	300	12	2.32
SEP	0.5	12	65	4	4.23
LMP	0.5	22	17	32	4.23
M	0.5	63	122	10	4.23
SEP*	0.0	13	42	3	6.00
LMP	0.0	18	13	32	6.00
M**	0.0	50	84	9	6.00
SEP	-0.5	14	31	2	7.55
LMP	-0.5	31	20	31	7.55
M	-0.5	44	67	9	7.55
SEP	-0.9	14	26	2	8.72
LMP	-0.9	42	23	31	8.72
M	-0.9	42	60	9	8.72
SEP	-0.99	15	25	2	8.97
LMP	-0.99	45	23	44	8.97
M	-0.99	41	57	8.5	8.97

*SEP reduces to the one given by Nowak & Wagoner (1991) for $a = 0$.

**The potential given by Mukhopadhyay (2002) reduces to the Paczyński-Wiita potential for $a = 0$.