

PHASE VOLUME OCCUPIED BY A TEST PARTICLE AROUND AN INCIPIENT BLACK HOLE

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The volume of phase space $g(E)$ available for a system with a definite energy E plays an important role in statistical mechanics. We compute $g(E)$ for a test particle in Schwarzschild geometry and show that it diverges as the source evolves to form a black hole. The implications of the result are discussed.

1. Introduction

In classical statistical mechanics we make extensive use of the quantity called "density of states" $g(E)$. This is proportional to the number of microstates available for the system consistent with the external constraints which are imposed on it (like constant energy, volume etc.). The entropy is given by $\log g(E)$ and all other thermodynamic parameters can be derived from this expression. In an attempt to understand certain peculiar features of spacetimes endowed with horizons (and, possibly, entropy), it may be useful to study the density of states in such spacetimes. I discuss in this paper a peculiar behaviour of this phase volume in Schwarzschild geometry. It is possible that this result will throw more light on the origin of entropy in spacetimes with horizons.

2. Phase volume in non-relativistic potentials

Consider a system of N non-interacting particles described by a Hamiltonian $H(p_i, q_i)$. If the q_i and p_i are specified at some instant and the equations of motion are solved, we can describe the evolution of this system as a curve in a phase space of dimension $6N$. Usually – especially for $N \gg 1$ – we resort to a statistical treatment of such a system wherein we are not given the values of all the q_i and p_i at an instant.

Instead, we merely know that the system is confined to a volume V and has an energy E . If this is the case, then the phase point could be (a priori) located anywhere on the constant energy surface given by

$$H(p_i, q_i) = E. \quad (1)$$

The bigger the volume of this surface, the larger are the number of "microscopic" configurations accessible to the system and the larger is our uncertainty about the system. The volume of this surface

$$g_N(E) = \int dp_i dq_i \delta(E - H(p_i, q_i)) \quad (2)$$

quantifies our ignorance. The entropy of the system is given by

$$S(E) = \log g(E). \quad (3)$$

All of classical statistical mechanics can be derived from this. *As long as we consider only non-interacting particles, we can define $g_i(E)$ for even a single particle.* This is defined exactly in analogy with (2) where E denotes the single particle energy and the integration is six-dimensional. The density of states for the full system can be easily computed as

$$g_N(E) = \int \prod_i^N g_i(E_i) dE_i \delta\left(E - \sum E_i\right). \quad (4)$$

Hereafter we will concentrate on $g_1(E)$ and – for simplicity – will denote it as $g(E)$. It can be easily verified that for an ideal gas of particles $g(E) \sim$

$VE^{1/2}$ and $g_N(E) \sim E^{3N/2-1}$. Consider now the phase volume in the presence of an externally imposed potential. For example, consider a spherical box of large radius R with an infinitely heavy particle located at the centre of the box producing a $-r^{-1}$ potential throughout the region. (To be precise, we can enclose the central mass inside a small box of radius R_1). There is an ideal gas of N particles roaming about in the region $R_1 < r < R$. We are interested in the phase volume of this gas. Because of (4) above we only need to compute $g_1(E)$. It is somewhat easier to compute first the quantity

$$\Gamma(E) = \int dp dq \Theta(E - H(p, q)) \quad (5)$$

and obtain $g(E)$ as $d\Gamma/dE$. Denoting the potential by $U(r)$ we have to calculate

$$\Gamma(E) = \int dp_r dp_\theta dp_\phi dr d\theta d\phi \Theta \left(E - U(r) - \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) \right), \quad (6)$$

where $U(r) = -U_0(L/r)$, say (note that integrating variables are just $dp_i dq^i$ without any " \sqrt{g} " factors. They will automatically arise out of the integration). We can easily show that

$$\Gamma(E) = \frac{16\pi^2}{3} \int r^2 dr \{2m[E - U(r)]\}^{3/2}, \quad (7)$$

from which $g(E)$ can be computed. We get, after detailed algebra:

$$\begin{aligned} g(E) &= c|E|^{-5/2} f(1, -1), \quad -\infty < E \leq -U_0(L/R), \\ &= cf(R/L, E/U_0), \quad -U_0(L/R) \leq E < \infty, \end{aligned} \quad (8)$$

where

$$f(x, y) = \int_0^x t^{3/2} (1 + yt)^{1/2} dt \quad (9)$$

and $c = 8\pi^2 (2mU_0)^{3/2} L^3 U_0^{-1}$. It can be verified that $g(E)$ is continuous and goes as $|E|^{-5/2}$ for large negative E and as $E^{1/2}$ for large positive E . We can also compute $g_N(E)$; it is also finite and continuous. Note that we owe the finiteness of $g(E)$ to the fact

that the potential diverges only as r^{-1} near the origin. From (7) it is clear that $\Gamma(E)$ will have the integrand $r^2 [-U(r)]^{3/2}$ near $r=0$. The integral will diverge for all $n \geq 2$. In other words a potential can open up an infinite amount of phase volume for particles interacting with it. We stress that the potential here is externally imposed and not due to self-interaction between the particles.

3. Phase volume in general relativity

Consider now the relativistic version of the above problem: a star of mass m and radius R_1 is located at the origin. We surround it by a concentric sphere of large radius R and fill the region between R_1 and R by N non-interacting particles. In this region the spacetime is described by the Schwarzschild metric:

$$ds^2 = (1 - 2M/r) dt^2 - \frac{dr^2}{1 - 2M/r} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (10)$$

$$= g_{00} dt^2 - r_{\alpha\beta} dx^\alpha dx^\beta. \quad (11)$$

(Though we are interested in Schwarzschild metric we will work with the form in (12) with $\dot{g}_{00} = \dot{\gamma}_{\alpha\beta} = 0$. This allows us to understand the physics better.) We want to compute the volume of phase space below an energy E for a single particle (i.e. $\Gamma(E)$). To do so in a covariant manner we have to use a covariant definition of energy. Since $\xi^t = (1, \mathbf{0})$ is a timelike Killing vector for (12) we can define the conserved, covariant, energy to be $E = \xi^t p_t$. The phase volume element $dx^\alpha dp_\alpha$ can also be easily shown to be invariant. Therefore, we need to evaluate

$$\Gamma(E) = \int dx^\alpha dp_\alpha \Theta(E - \xi^t p_t). \quad (12)$$

Now, using $p^t p_t = m^2$ we can write

$$\xi^t p_t = g_{00} p^0 = g_{00}^{1/2} (m^2 + \gamma^{\alpha\beta} p_\alpha p_\beta)^{1/2}. \quad (13)$$

The p_α integrations give the volume of the surface

$$\frac{\gamma^{\alpha\beta} p_\alpha p_\beta}{E^2/g_{00} - m^2} = 1, \quad (14)$$

which is

$$\frac{4}{3} \pi (\det \gamma^{-1})^{1/2} (E^2/g_{00} - m^2)^{3/2}.$$

So we get

$$\Gamma(E) = \frac{4}{3}\pi \int \sqrt{\gamma} d^3x^\alpha (E^2/g_{00} - m^2)^{3/2}. \quad (15)$$

For the Schwarzschild geometry this expression becomes

$$\Gamma(E) = \frac{16\pi^2}{3} \int_{r_1}^{R_{\max}} \frac{r^2 dr}{(1-2M/r)^{1/2}} \times \left(\frac{E^2}{1-2M/r} - m^2 \right)^{3/2}, \quad (16)$$

where R_{\max} is the maximum radius allowed for the energy E . The peculiar feature is that this expression diverges as $R_1 \rightarrow 2M$, the Schwarzschild radius. The $g(E)$ which can be obtained by differentiating $\Gamma(E)$ also diverges as $R_1 \rightarrow 2M$. The leading behaviour of $\Gamma(E)$ and $g(E)$ for R_1 near $2M$ is

$$\Gamma(E) = \frac{16\pi^2}{3} E^3 \frac{(2M)^4}{R_1 - 2M}, \quad (17)$$

$$g(E) = 16\pi^2 E^2 \frac{(2M)^4}{R_1 - 2M}. \quad (18)$$

Suppose our star – which was surrounded by the gas of particles – starts to collapse, decreasing R_1 . The result above shows that *this collapse increases the available number of microstates (and the entropy) of the outside gas*; as R_1 approaches $2M$, this entropy diverges. We stress that our result has been purely classical and that we have not introduced any (dubious) physics pertaining to what happens at $r < 2M$. (Somewhat similar features came up in a previous semiclassical analysis of the number of modes accessible for a scalar field outside a black hole [1,2]. The analysis here seems to be more straightforward.)

4. Discussion

In spite of the peculiarity of the result, it possesses a simple mathematical origin. Consider an observer at some event $P(t, x^\alpha)$ with an infinitesimal box of size dx^α . He can use a local inertial frame at P and ask: “What is the allowed momentum space volume for a particle at x^α if its energy at infinity is E ?” The

locally measured energy E_{loc} is $u^\mu p_\mu = E(g_{00})^{-1/2}$; therefore, in the locally inertial frame

$$V_{p\text{-space}}(x^\alpha) = \frac{4}{3}\pi p_{\max}^3 = \frac{4}{3}\pi [E_{\text{loc}}^2(x^\alpha) - m^2]^{3/2} = \frac{4}{3}\pi (E_\infty^2/g_{00} - m^2)^{3/2}, \quad (19)$$

giving the total phase volume as

$$\Gamma(E) = \int \sqrt{\gamma} d^3x^\alpha V_{p\text{-space}}(x^\alpha) = \frac{4}{3}\pi \int \sqrt{\gamma} d^3x^\alpha [E^2/g_{00} - m^2]^{3/2}. \quad (20)$$

This is precisely the result (15). The divergence is due to the divergence of locally measured momentum space volume near $R_1 \approx 2M$ (and not due to any coordinate space divergence). It is usual to dismiss the divergence of g^{00} at $r=2M$, as a “mere coordinate effect”; our analysis persuades one to rethink about such “mere” coordinate effects. (Incidentally, the same divergence occurs [2] in de Sitter space-time and in Rindler coordinates if we define E using corresponding Killing vectors.) In an attempt to make $g(E)$ finite we may impose the following ad hoc restriction: E_{loc} has to be always less than the Planck energy E_P . This would prevent a particle of energy E from reaching radius less than $2M[1 - (E/E_P)^2]^{-1}$. Under this condition the maximum value for $g(E)$ will be

$$g(E) \simeq \frac{16\pi^2}{E_P} \left(\frac{2M}{L_P} \right)^3 \quad (21)$$

for $E \ll E_P$. It is interesting to see the “lattice” factor $2M/L_P$ per each space dimension. Even though we are computing $g(E)$ for a gas outside the star (with $R_1 \approx 2M$) the $g(E)$ seems to depend on the number of Planck volumes inside the region $r < 2M$. The entropy of the gas corresponding to this $g(E)$ has a term $3 \log M$ giving rise to a “temperature” $3M$. This temperature goes to zero as M goes to zero unlike the conventional black hole temperature $(8\pi M)^{-1}$. Further investigations are necessary to see whether these results are mere curiosities or whether they signify anything important.

References

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- [2] T. Padmanabhan, Phys. Lett. B 173 (1986) 43.