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# Quantum Cosmology and Stationary States[1]

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*Abstract*

A model for quantum gravity, in which the conformal part of the metric is quantized using the path integral formalism, is presented. Einstein's equations can be suitably modified to take into account the effects of quantum conformal fluctuations. A closed Friedman model can be described in terms of well-defined stationary states. The "ground state" sets a lower bound (at Planck length) to the scale factor preventing the collapse. A possible explanation for matter creation and quantum nature of matter is suggested.

## §(1): *Introduction*

Classical gravity predicts singularities [1]. The existence of singularities is probably one of the most disturbing features of Einstein's gravity. The well-known examples for this kind of catastrophe are the initial and final singularities of the closed Friedman model and final stage of the collapsing dust ball.

Both the examples–and in fact all others–involve ultrastrong gravitational fields near the singular event. The gravitational tidal forces vary rapidly. Can one have faith in Einstein's equations when tested under such drastic conditions? The well-known "thumb rule" of quantum theory indicates the need for a quantum description when the action $J$ for the system satisfies the criterion

$$J \lesssim \hbar \tag{1}$$

Simple calculation shows the need for quantum gravity when the fields vary rapidly over distances, of the order of

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$$L_p = \left(\frac{G\hbar}{c^3}\right)^{1/2} \tag{2}$$

Clearly, quantum gravity is required for a meaningful discussion of singularities.

It is conceivable that a quantized theory avoids a singularity present in the classical picture, as happened with the hydrogen atom. Wheeler [2] has stressed the fact that quantum mechanics avoids the collapse of the electron in hydrogen atom by the introduction of the concept of stationary states.

We show here that the analogy can be made quantitative. In the formalism below, we quantize the conformal degree of freedom of geometry, and show that stationary states exist for a Friedman universe, thus preventing the collapse! Various approaches to quantum gravity has led to conflicting results regarding singularities. Our approach here leads to a consistent picture with a happy ending.

## §(2): *Outline of the Formalism*

Feynman's "sum over histories" approach is a royal road to quantization. Consider a space-time foliated by a series of spacelike hypersurfaces labeled by $t$. The probability amplitude for transition from an initial 3-geometry $\mathcal{G}_1$ to a final 3-geometry $\mathcal{G}_2$ is given by (symbolically!)

$$K[\mathcal{G}_2 t_2; \mathcal{G}_1 t_1] = \int \mathcal{D}\mathcal{G} \exp \frac{i}{\hbar} J[\mathcal{G}] \tag{3}$$

where $J$ is the *total* action (including the Hawking surface counter term to remove the second derivatives) [3]. The nightmares of path integral quantization–choice of measure and choice of degrees of freedom–are bypassed by restricting oneself to the conformal degree of freedom. The metric is written in the form

$$g_{ik} = \Omega^2(x)\, \bar{g}_{ik}(x)$$

and the conformal part $\Omega(x)$ is treated as a quantum variable. The choice is based on the following considerations (a detailed justification and motivation can be found in References 4–6): (i) the *quantum conformal fluctuations* (*QCF*) leaves the light-cone structure and causal structure of the space-time intact. (ii) The split up in equation (4) is generally covariant; $\Omega(x)$ can be taken to transform as a scalar. (iii) The measure for the path integral can be defined unambiguously (see below). (iv) Important cosmological solutions–Friedman models–are conformally flat, having just one degree of freedom.

When the attention is directed to quantizing the conformal part, the "kernel" in equation (3) can be written with the action (see Reference 5),



$$J = -\frac{1}{16\pi G}\int (-\bar{g})^{1/2}\, d^4x[6\Omega^i\Omega_i - \bar{R}\Omega^2] + J_{\text{matter}} + J_{\text{Hawking}} \tag{4}$$

($J_m$ = matter action; $J_{\text{Hawking}}$ = Hawking counter term). Since the action is quadratic in $\Omega(x)$, the measure for the path integral is well defined.

This formalism has been used in recent years to study the QCF around a classically singular background metric $\bar{g}_{ik}(x)$ [4, 7, 8]. It can be proved in general, that, QCF diverges at the classical singularity (i.e., the expectation value $\langle\Omega^2\rangle$ diverges) [6], leading to quantum transitions to nonsingular space-times. The effect of QCF can be incorporated into an "average" (or "effective") metric defined as [7]

$$g_{ik}^{\text{eff}} = \langle\Omega^2\rangle\, g_{ik} \tag{5}$$

The indication of a "stationary state" came from the study of $g_{ik}^{\text{eff}}$ for a Friedman universe. The background metric,

$$\bar{d}s^2 = \bar{g}_{ik}\, dx^i\, dx^k = dt^2 - S^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{6}$$

has a scale factor $S(t)$ which goes to zero as $t \to 0$ (singularity). The effective metric replaces it with an

$$S_{\text{eff}}^2(t) = \langle\Omega^2\rangle S^2(t) \longrightarrow (\hbar G/c^3) \quad \text{as} \quad t \longrightarrow 0 \tag{7}$$

[7, 9]. Thus the "effective metric" has a lower bound avoiding the singularity. This suggests that one should look at the "Schrödinger equation" corresponding to the path integral action, which is what we shall do in the next section.

## §(3): *"Schrödinger Equation" and Stationary States*

To set the scene, consider a metric of the form,

$$ds^2 = \Omega^2(t)\left[dt^2 - \frac{dr^2}{1 - r^2/a^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{8}$$

in which $\Omega(t)$ is treated as a quantum variable (the restriction of isotropy and homogeneity rules out space dependence of $\Omega$). The kernel is determined through the action in equation (4), which has the form (the source is taken to be radiation; for details, see Reference 10)

$$J = -\tfrac{1}{2}M\int dt(\dot{q}^2 - w^2q^2) \tag{9}$$

where ($c \neq 1$!)

$$q = a\Omega \qquad w = \frac{c}{a} \qquad M = \frac{Vc^2}{Ga^2} \tag{10}$$

(where $V$ may be taken to be the volume of the closed background 3-space). The action in equation (9), classically, leads to the dependence,

$$\Omega(t) \sim \sin wt \tag{11}$$

which correctly represents a radiation-filled universe. Quantum theory leads to a set of stationary states, $\Psi_n(\Omega)$ parameterized by the integer $n$, for the harmonic oscillator in equation (9). The quantum geometry in the $n$th stationary state is given by a metric,

$$ds_{(n)}^2 = \langle\Omega^2\rangle_n \, \bar{d}s^2 = L_p^2(n + \tfrac{1}{2}) \, [d\eta^2 - d\chi^2 - \sin^2 \chi(d\theta^2 + \sin^2 \theta \, d\phi^2)] \tag{12}$$

The natural lower bound at the Planck length is evident.

How can one recover the classical limit [equation (11)] from such a strange "quantum metric"? This can be done exactly as in the case of the transition from the quantum harmonic oscillator to its classical counterpart. If the universe is in a state with large $n$, we know the probability density $|\psi_n(a)|^2$ will mimic the classical evolution [11, p. 73]. However, there is a more elegant way of achieving the classical limit. For the harmonic oscillator there exists a set of "coherent states" (see Reference 11, p. 74; we have adjusted the origin of $t$), with

$$|\psi(q, t)|^2 = N \exp\left[-\frac{1}{2L_p^2}(q - q_0 \sin wt)^2\right] \tag{13}$$

such that the expectation value of $\Omega$ has the classical value. If the universe is in the coherent state, the metric has the form ("almost classical")

$$ds_{q_0}^2 = (q_0^2 \sin^2 \eta + L_p^2) \, [d\eta^2 - d\chi^2 - \sin^2 \chi(d\theta^2 + \sin^2 \theta \, d\phi^2)] \tag{14}$$

The metric, of course, is nonsingular and has the correct classical limit for $\hbar \to 0$.

Are the stationary states a peculiar feature of Friedman universes? The answer seems to be "no." Detailed analysis of the homogeneous (Bianchi) cosmologies shows the existence of similar stationary states—and singularity avoidance—for most of the cosmological models [12].

## §(4): *Toward a More Complete Theory*

Classical gravity must appear as the limiting case of quantum gravity. As far as the conformal part—which is quantized—is concerned, the classical limit follows naturally. But what determines the background metric $\bar{g}_{ik}$? So far, we



have not presented a method for deciding this choice. We attempt here to incorporate the effect of QCF on the background to arrive at a self-consistent picture. Consider the classical action,

$$J = \frac{1}{16\pi}\int R(-g)^{1/2} \, d^4x + \int L_m(-g)^{1/2} \, d^4x \tag{15}$$

Put $g_{ik} = \Omega^2 \bar{g}_{ik}$ and vary both $\Omega$ and $\bar{g}_{ik}$ independently, leading to the equations

$$\Omega^2\left(\bar{R}_{ik} - \frac{1}{2}\bar{g}_{ik}\bar{R}\right) + 6t_{ik} + \frac{16\pi G}{\Omega^2(-\bar{g})^{1/2}} \frac{\delta[L_m(-g)^{1/2}]}{\delta g^{ik}} = \alpha_{ik} \tag{16}$$

$$\Box\Omega + \frac{1}{6}\bar{R}\Omega = \frac{8\pi G}{3\Omega^3} \frac{1}{(-\bar{g})^{1/2}} \bar{g}^{ik} \frac{\delta[L_m(-g)^{1/2}]}{\delta g^{ik}} \tag{17}$$

where $\alpha_{ik}$ involves derivatives of $\Omega^2$ (which vanish in the stationary states we consider) and

$$t_{ik} = -\partial_i\Omega\partial_k\Omega + \tfrac{1}{2}\bar{g}_{ik}(\partial^a\Omega\partial_a\Omega) \tag{18}$$

Classically, it is trivial to see that equations (16)–(18) leads to Einstein's equations. Quantum theory is arrived at by the following postulates: (i) Treat $\Omega$ as a quantum variable in equation (15). In other words replace equation (17) by a suitable quantization procedure for $\Omega$. (ii) In equation (16) replace the functions of $\Omega$ by their expectation values. (iii) Choose the state for the quantum variable $\Omega$, and the classical background metric self-consistently to satisfy equation (16).

The theory in the present form is much more restrictive than Einstein's theory. The nontrivial features of the theory arise through the expectation value $\langle t_{ik}\rangle$ which has the form of a *negative* energy scalar field. It is of crucial importance to verify that the background metric assumed in the previous section [equation (8)] *is*, indeed, a self-consistent solution when the source Lagrangian represents isotropic radiation. Direct calculation with equation (16) proves this to be the case provided the energy density of radiation satisfies the quantum condition,

$$\epsilon = \frac{9}{16\pi}\frac{\hbar c}{a^4}\left(n + \frac{1}{2}\right) \tag{19}$$

Thus quantum gravity leads naturally to a quantum condition on source. We know that classical gravity determines the classical dynamics of the source. Can quantum gravity, naturally, lead to the quantum dynamics of the source? This attractive possibility remains to be explored. The result in our toy model is easily generalizable to various other sources—like dust, for example—but a model-independent proof is hard to come by.

This is not the only surprise equations (16)–(18) have in store. It is easy to see from equation (16) that [in the absence of matter ($L_m = 0$)] the flat space vacuum ($\bar{g}_{ik} = \eta_{ik}$) is unstable to conformal fluctuations! In this sense, these equations *demand* the existence of matter. Contrast it with Einstein's equations to which the matter source must be fed from outside; a flat vacuum is completely consistent with Einstein's equations. Classical gravity describes matter without demanding or explaining it; quantum gravity seems to go further.

The same equations also indicate how this creation of matter could have come about. Notice that equation (16) adds a negative energy term $\langle t_{ik}\rangle$, to the conventional energy momentum tensor (which arise from the $\delta[L_m(-g)^{1/2}]/\delta g^{ik}$). It is only this combination that is conserved; not the individual $T_{ik}$ for matter. This negative energy term allows for the creation of matter whenever quantum fluctuations are important—especially near the initial singularity. (A somewhat similar idea arises in Reference 14, though the formalisms are entirely different.) Preliminary analysis confirms these expectations [13].

## §(5): *Conclusion*

What we have presented here is far from a complete theory for quantum gravity. However, the extreme simplicity and intuitive appeal of the model allow it to be used as a preliminary probe for quantum gravity. Even at this stage, we have a glimpse of the remarkably rich structure for quantized gravity.

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# Exact Solutions for the Early Friedmann–Robertson–Walker Universe in General Scalar–Tensor Theories

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## *Abstract*

Homogeneous and isotropic cosmologies with trace-free energy-momentum tensors are studied in general scalar–tensor theories. A method is presented which allows one to construct exact solutions for theories with arbitrary coupling function $\omega(\phi)$. Particular attention is paid to Schwinger's theory.

## §(1): *Introduction*

Recent increasing interest in general scalar–tensor theories seems to stem mainly from attempts to extend the principle of (local) conformal invariance to include also gravitational phenomena, gravity becoming itself the manifestation of a broken symmetry [1, 2]. It also has been shown that the use of scalar fields enables one to construct nonsingular field-theoretical models for elementary particles [3].

In this paper we will concentrate on those scalar–tensor theories which are singled out by the requirement that test particles should move on geodesics *when* observed in particle units. Hence theories of the kind studied by Dirac, Canuto, and others, will not be taken into account. This leaves one still with numerous models in which the gravitational constant $G$, or the elementary particle masses $m$ are assumed to be varying "constants" of nature.

All of these theories, however, can be put into a standard, conformally invariant form by the introduction of two coscalars $\lambda$ and $\beta$, both of power $-1$, describing the behavior of $m$ and $G$, respectively: