

Recreational Aspects in the Study of Mathematics

(The P. L. Bhatnagar Memorial Lecture)

by :

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While recognizing the great honour you have accorded me by asking me to deliver the P. L. Bhatnagar Lecture, I must confess to a feeling of inadequacy in addressing you. Like the late Professor Bhatnagar, whom I knew in my student days you are all scholars dedicated to the task of teaching of mathematics. I am, by definition, neither a mathematician, nor a teacher ! Although I use, indeed depend on, mathematics in my work, the high priests of mathematics will not call me a mathematician. As for teaching, my experience to date has been limited to lecture courses and seminars in specialized subjects. It would therefore be presumptuous on my part to offer you suggestions on improving the teaching of mathematics.

I have therefore decided to share with you my experiences as a **student** of this great subject. In particular I wish to present before you the various ways and means that made mathematics an interesting and fascinating subject for me right from my childhood. It may well be that many of you have had experiences similar to mine.

Challenge and Incentives :

Telling a man 'do this' may not necessarily evoke a positive response. But tell him 'I bet, you cannot do it' and he immediately rises to the challenge. While I have often been bored by the string of stereotyped exercises at the end of a typical maths lesson, I have always been excited by problems that look challenging.

I therefore heartily commended the 'Pravinya' and 'Pradnya' schemes of tests for school children that the Maharashtra Ganit Adhyapak Mahamandal has been operating for the last few years. Its popularity is ample testimony to the fact that many school children respond to challenges in the same way that I did.

While I did not have such schemes in my school days I was fortunate in other ways. My home environment had provided me with such challenges. My father and maternal uncle, both mathematicians, did present me with challenges and incentives through interesting mathematical problems. Let me state at the outset that I did not succeed in solving all challenge problems. For example, the proof of the following result eluded me.

Theorem : ABC is a triangle. BY and CZ are the bisectors of angles $\angle ABC$ and $\angle ACB$ respectively, where Y lies on AC and Z on AB. Given that $BY = CZ$, prove that $AB = AC$.

The proof is not straight forward although it uses elementary geometry (no trigonometry allowed). When I was told the answer*, I proudly posed the problem to my classmates and the maths teacher (I was in class IX then). They could not do it and asked me to write the proof on the board.

In today's over-stuffed school syllabi and overcrowded school rooms I wonder how many teachers will have the time and energy for such out-of-the way problems ?

Let me narrate a story about the Cambridge mathematical tripos – the ultimate in the mathematics examinations. Karl Pearson, known to statisticians for his many researches, was a bright student in the late 19th century. In those days the tripos examination contained problems of varying difficulties, the simple ones intended to be solved by sportsmen and the difficult ones by scholars. Pearson was sitting behind a sportsman and after a few minutes he noticed that this man had ticked a particular question on the question paper.

'It must be a simple question ! Let me get it out of the way as I warm up for the difficult ones' said Pearson to himself and he started on the problem.

However, he could not do it as easily as he had thought. But he felt that he had to do it since even the duffer sportsman had done it. So he put all his time and energy into it and finally solved it. But it took him the entire three hours of the examination.

'How did you do Q. 4 so easily ?' asked the exasperated Pearson to the sportsman. For, he felt that by solving only one simple question he would undoubtedly fail in the examination.

'Q. 4 ?' exclaimed the duffer looking at the paper 'Oh, I was only doodling since I could not do any problem. So I must have ticked it unknowingly !'

As it turned out, Q. 4 was the most difficult problem in the paper and being the only one to solve it Pearson got full credit for his efforts.

Puzzles :

Mathematics is after all a game played according to certain rules which are sanctified by logic. Very often a student loaded with numbers and equations loses sight of this fundamental fact. There are several puzzles which highlight mathematical reasoning without getting bogged down in details. I have enjoyed reading books on such puzzles**.

Consider the following two problems on compound interest :

Problem A : A man invested Rs. 5750 at a compound interest of $4\frac{1}{2}\%$ per annum. How much money will he collect at the end of 3 years ?

* It is to be found in the Mathematical Gazette XVI, p. 200, 1932:

** The latest in the list of such books include Martin Gardner's books 'Aha' and 'Gotcha' published by W. H. Freeman and Co. U. S. A.

Problem B : A man invested Rs. 100 at a fixed compound interest for a fixed period at the end of which he collected Rs. 150. If he had invested the same sum at the same interest for twice that period, how much would he have collected ?

By the time the child has gone to the end of Problem A, he would have developed a nausea against compound interest. Even assuming that he got the answer right and that it tallied with the answer given at the end (—a miracle in itself considering the printing errors in school books) he would not have grasped the essential multiplicative property of the compound interest. Problem B brings it out neatly without going into gory decimals.

Before the digital watches completely replace the conventional watches with minute and hour hands, the following problem may interest the students looking for the unusual :

Problem : I left the house between 3 p. m. and 4 p. m. and returned between 7 p. m. and 8 p. m. I found that the hour and minute hands had interchanged their positions during this interval. When did I leave the house ?

This problem can be solved algebraically by setting up two simultaneous equations. It is even more interesting to do it completely arithmetically.

The logical reasoning and elimination of possible alternatives to the correct solution are important aspects of mathematics that can be illustrated through problems of the following kind.

Problem : There are only six houses in Nursery Street, Rhymetown, three on one side and three facing them on the other. All six are inhabited by notorious characters.

The soldier has the tinker and Mr. Davies as his nextdoor neighbours. The sailor lives next door to the thief, and Mr. Eastley has one of the end houses. The beggarman lives opposite Mr. Barlow, but Mr. Foley does not live opposite the thief. The tailor and Mr. Coutts are both neighbours of Mr. Anderson.

Can you tell from these data the profession of each of the named persons?

There are no numbers in the above problem nor are numbers needed in its solution. Yet, the problem is mathematical and it uses the well known method of *reductio-ad-absurdum* so common to the proofs in other branches of mathematics.

Paradoxes :

While the above method relies on there being only two alternatives true/false to a statement paradoxes do arise when the statement is not properly phrased.

As a school boy I was not exposed to the mixed blessing of 'new maths.' Today, I believe set theory is supposed to be known to school children. Yet how many are familiar with the Russell paradox about the set of all sets? How many connect it with the paradox confronting the village barber who has to shave those and all those who do not shave themselves?

The question 'who shaves the barber?' does receive a definite answer under certain circumstances! The hint is: in the United States women barbers are becoming increasingly common in men's barber shops.

The comic opera ' Ruddigore ' by Gilbert and Sullivan contains another paradoxical situation. Baron Sir Despard Murgatroyd is forced to commit a crime everyday because there is a curse on his family : if he fails to commit a crime on a given day he would die at the end of it in a most painful manner. So, though he is basically a kind man he is notorious as a wicked baron—as were all his predecessors. However, a clever man points a way out of his difficulty. The solution is that the baron should refrain from committing a crime. For, by so doing he is ensuring his own death which is tantamount to an attempt at suicide which is itself a crime. So the curse was harmless after all !

Fallacies :

While an imprecise statement of the rules of reasoning leads to paradoxes, a mistake in reasoning can lead to manifestly false conclusions. As a student of geometry one is taught the importance of being precise in the application of basic postulates and preceding theorems to proving a given result. Equally important are examples of fallacies where false conclusions emerge from the misuse of reasoning steps.

In this connection I found the fallacy in the proof of the following theorem extremely hard to detect :

Theorem : ABC is any triangle. Show that $AB = AC$,

Proof : Bisect the angle BAC and let the bisector meet the perpendicular bisector of side BC at a point X (see Figure 1). Suppose that X lies inside the triangle ABC.

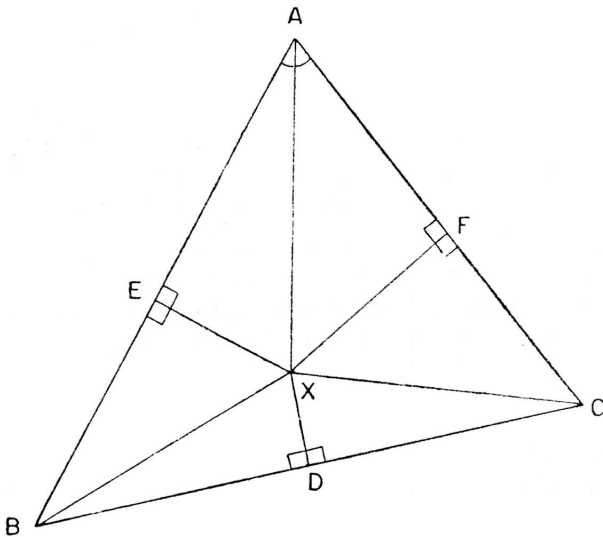


Figure 1.

$XE = XF$, $XB = XC$, $\angle XEB = \angle XFC = \text{right angle}$.

Thus these triangles are congruent and $BE = CF$.

Because we have already proved

$AE = AF$, we get $AB = AE + BE = AF + CF = AC$.

If the point X lies outside the ABC, the proof is similar except in the last step where

$AB = AE - BE = AF - CF = AC$. □

Draw perpendiculars XE and XF on AB and AC respectively. Join XB and XC. Then $\triangle XAE$ and $\triangle XAF$ are congruent because they are right angled triangles with XA common and $\angle XAE = \angle XAF$.

Therefore

$XE = XF$, $AE = AF$,

Since X lies on the perpendicular bisector of BC, we have $XB = XC$.

Therefore in $\triangle XEB$ and $\triangle XFC$ we have

There are fallacies in most branches of mathematics. In some cases they are very hard to detect. For example, many fallacious proofs of Fermat's Last Theorem and the Four Colour Problem have been given in the past. Can you find the fallacy in the following proof which appears in many text books of elementary calculus ?

Theorem : As $\theta \rightarrow 0$, $\lim \frac{\sin \theta}{\theta} = 1$.

Proof : Let the arc AB of a circle subtend an angle θ at the centre O, as shown in Figure 2. Draw perpendicular BX on OA. Let the perpendicular AY on OA meet OB (extended) in Y. Then

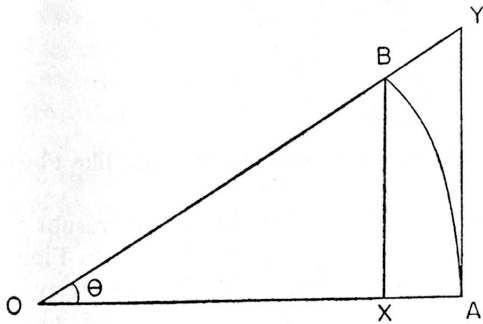


Figure 2.

Area of $\triangle BOX <$ Area of circular wedge CAB $<$ Area of $\triangle OAY$.

Suppose $OA = OB = r$. Then the above inequality becomes

$$\frac{1}{2}r^2 \sin \theta < \frac{1}{2}r^2 \theta < \frac{1}{2}r^2 \tan \theta,$$

$$\text{i.e. } 1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

As $\theta \rightarrow 0$, $1/\cos \theta \rightarrow 1$. Hence the result follows. $\square$

Intuition and Mathematical Truth :

There were occasions when I felt – as almost every student learning maths does sometime or other – that proving a theorem is an unnecessary waste of one's efforts when the result is so obvious. But human intuition can be misleading ! Just as fallacies lead to results which are false, rigorous proofs often establish truths that are unacceptable to human intuition.

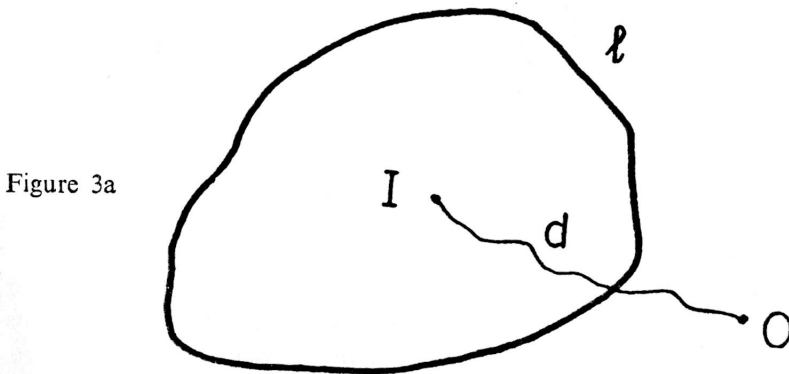
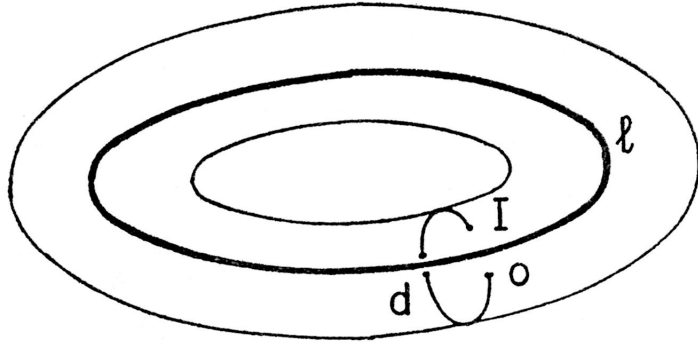


Figure 3a

Figure 3a shows a closed curve l . It has an inside and an outside. If we join a point I in the inside region to a point O in the outside region we find that the joining line d must cross l somewhere. Intuition tells us that this is obvious and needs no proof.

But not all closed curves have well defined insides and outsides. The curve 1 drawn on a torus has no inside and outside. Any two points I and O on the torus can be connected by a line d that does not intersect 1 anywhere, as shown in Figure 3b.

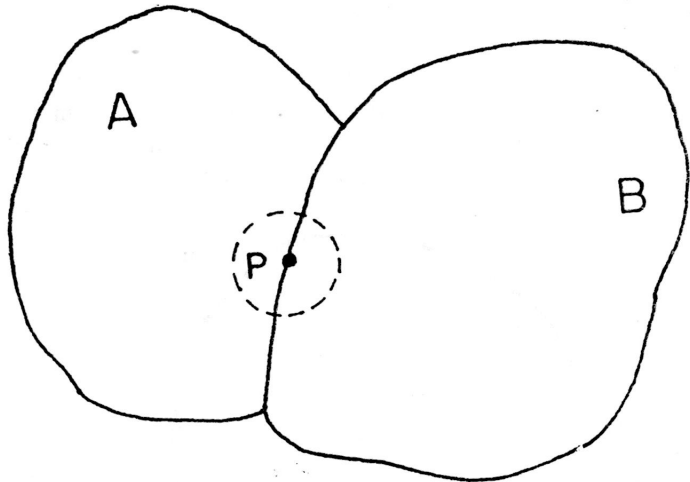
Figure 3b



A rigorous mathematical proof tells no how to separate cases like above from those shown in Figure 3a.

Take another example where intuition tells us that the stated results is impossible but which is nevertheless true and provable mathematically. In Figure 4 we see two countries A and B sharing a common border l . The property of the border is the following. Around any point P on l draw a small circle. However small the circle is, it will always contain within it points from both the countries A and B.

Figure 4



With this definition of the border is it possible for three countries A, B, C on a plane surface to share **one** common border? Intuition tells us 'no' but a rigorous mathematical construction of the border is possible*.

Can we square a circle? Can an angle be trisected with ruler and compass? Can a regular polygon of 17 sides be constructed with ruler and compass? Can every algebraic equation of degree 5 be solved with radicals?

★ See, for example the article 'The Crisis in Intuition' in 'The World of Mathematics,' by J. R. Newman, Vol. 3, p. 1956.

Human intuition has failed to be the guide in questions like these and rigorous mathematical demonstrations of 'yes' or 'no' answers have taxed the best brains in history. An appreciation of this situation makes one more receptive to the mathematical approach.

The History of Mathematics :

This point brings me to my final topic – the history of mathematics.

As a student I was fascinated by E. T. Bell's classic book 'The Men of Mathematics.' To those of you who have not seen it, I recommend this book as a masterly survey of the lives of world's greatest mathematicians from Zeno and Archimedes to Poincare and Cantor. Although mathematics is considered an abstract subject, the mathematicians are after all human. Though exceptionally gifted in their subject the world's great mathematicians had other attributes no different from the rest of us humans. Besides learning their great work it is also worth learning how they arrived at their results, what struggles they underwent not only in terms of mathematics but also in terms of their very existence and subsistence. No less interesting is to learn about the evolution of mathematical ideas from the ancient to modern times : how one discovery led to another.

To capture the spirit of excitement in these areas, it is not necessary to be an expert in mathematics. Being an expert certainly gives you a better perspective on the various issues; but even the uninitiated cannot help gaining some knowledge as he reads about the trials and tribulations of Newton, Leibnitz, Fermat, Euler, Gauss, Galois, Hilbert and Einstein. In fact, historic anecdotes raise the subject of mathematics from the status of an abstract bugbear to an enlightening human enterprise.

I cannot resist giving a few mathematical anecdotes to illustrate the point.

a) In 1696 Johann Bernoulli had posed a problem in dynamics as a challenge to mathematicians of his time. The problem was as follows. Given two points A and B, with B lying below A (but not vertically so), find the curve joining A and B such that a bead sliding along a smooth wire in the shape of that curve takes the least amount of time from A to B.

The problem remained unsolved for six months until it was brought to the notice of Newton one evening on his return home from work. Though tired, he sat down to solve it and within twenty four hours he sent the solution to the Royal Society anonymously. But looking at the solution Bernoulli clearly identified who was the author, for he remarked "I can recognize the lion from his paw."

b) Euler was, for a long time, the mathematician in the court of Russia's Empress Catherine. One day the French philosopher Diderot came to the court and produced arguments denying the existence of God. When nobody could refuse his reasoning the Empress called upon Euler. Euler wrote an equation :

$$\frac{a + b^b}{n} = x.$$

and concluded ' therefore God exists.' The philosopher, who was in awe of mathematics had no reply to this ' proof ' of God's existence !

c) When Gauss at the age of 16 found the geometrical method of constructing a regular 17-gon he was very excited. When he showed it to his teacher, that worthy did not give him any encouragement. This rankled with Gauss. Since this teacher was also fond of making poems, Gauss later remarked : " He was the greatest poet amongst mathematician and the greatest mathematician amongst poets "

d) Madame Kowalesky, the famous woman mathematician and coworker with Weierstrass had an unusual introduction to mathematics. When she was a child her family wanted to decorate their country house. They ran out of wall paper by the time they came to the children's nursery. So they tore pages from a maths book and papered the walls with them. The girls looking at the formulae every day acquired a certain familiarity with them that was helpful when they began to study mathematics at a later stage. The formulae did not appear formidable to them as they did to others.

Mathematical recreations :

To sum up then, I have emphasized the peripheral aspects of mathematics that are normally ignored in school syllabi but which I personally found of great help in my study of the subject. These are actually mathematical recreations and as such I hesitate to recommend them as parts of the text books on maths. For, we in India have a habit of approaching text books in a regimental fashion. If this material appears in a text book it will be considered important for the examination. In that case there will spring up guides and special classes on the subject. And as has happened to other text book material the students will learn how to answer questions on mathematical recreations without really understanding or enjoying them.

So my hope is that no educational authority gets carried away by my enthusiasm for the above aspects of mathematics to the extent of including them in text books. But my plea to you as teachers is to smuggle some of these aspects into your regular teaching so that at least a few bright students will share in their excitement.