# A spherically symmetric higher dimensional perfect fluid space - time admitting a one parameter group of conformal motions.

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**Abstract.**

We have obtained non static exact solutions of Einstein's field equations for a higher dimensional perfect fluid sphere admitting a one - parameter group of conformal motions. Two special solutions have been obtained by selecting particular values of the constants occuring in general solutions. First solution is homogeneous representing stiff matter and it is a generalisation of solution 1 of Herrera and Leon [1] for 4 dimensional space - time. The other solution is inhomogeneous in density as well as pressure. Both the solutions are shearing and fulfil the dominant energy condition.

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## I. Introduction

There has been a large interest and effort to unify gravity with other basic interactions. Most recent attempts have been directed to studying theories [2-4] in which the space - time dimensions are greater than $(3+1)$, as the superstring theories [5] require the background space - time manifold to be $(9+1)$ dimensions. There is a large number of papers [6] in which 4 dimensional space - time has been studied and exact solutions of Einstein's field equations have been found out but exact solutions on higher dimensional space - time are comparably few in number [7-10].

In this paper we find out some exact solutions of spherically symmetric higher dimensional perfect fluid space - time under the assumption that the space - time admits a one - parameter group of conformal motions with the generator orthogonal to the fluid flow and lying along the radial direction.

We adopt the sign convention $(+ - - - .....-)$ and use the units in which $c = G = 1$.

The paper is organised as follows: In section II the field equations and their solutions with kinematical parameters are given. Particular solutions are derived and discussed in section III. Last section includes conclusions.

## Section II. FIELD EQUATIONS AND SOLUTIONS

The appropriate metric for a higher dimensional spherically symmetric space - time may be taken in comoving coordinates $x^i = (t, r, \theta^1, \theta^2, ......., \theta^{n+1})$ henceforth numbered as $0, 1, 2, ........, n+2$ respectively in the form [11].

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - e^{2\mu}dX^2_{n+1}, \tag{2.1}$$

where

$$\nu = \nu(t,r), \lambda = \lambda(t,r), \mu = \mu(t,r)$$

and

$$dX_{n+1}^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_2^2 + .......... + \sin^2\theta_1 \sin^2\theta_2 ........ \sin^2\theta_n d^2\theta_{n+1} \quad (2.2)$$

here $dX_{n+1}^2$ is the metric for unit sphere in the polar coordinates $\theta_1, \theta_2, ........, \theta_{n+1}$. The dimension D of the full space - time is $D = n+3, n = 1,2,3, .......$ The Einstein's field equations are given by

$$G_{ij} = 8\pi(\rho + p)U_i U_j - p g_{ij}. \quad (2.3)$$

Here we have assumed that the energy momentum tensor in higher dimension has the same form as in 4 dimension and isotropy of pressure is assumed in all directions [7]. The velocity of fluid is given by

$$U^i = \delta_0^i e^{-\nu}. \quad (2.4)$$

The Einstein's field equations are as follows:

$$G_0^0 \equiv (n+1)[e^{-2\nu}(\frac{n}{2}\dot{\mu}^2 + \dot{\lambda}\dot{\mu}) - e^{-2\lambda}(\mu'' + \frac{n+2}{2}\mu'^2 - \mu'\lambda') + \frac{n}{2}e^{-2\mu}] = 8\pi\rho, \quad (2.5)$$

$$G_1^1 \equiv (n+1)[e^{-2\nu}(\ddot{\mu} + \frac{n+2}{2}\dot{\mu}^2 - \dot{\mu}\dot{\nu}) - e^{-2\lambda}(\frac{n}{2}\mu'^2 + \nu'\mu') + \frac{n}{2}e^{-2\mu}] = -8\pi p, \quad (2.6)$$

$$G_2^2 \equiv G_3^3 \equiv \ldots\ldots\ldots\ldots \equiv G_{n+2}^{n+2} \equiv e^{-2\nu}[\ddot{\lambda} + n\ddot{\mu} + \dot{\lambda}^2 + \frac{n(n+1)}{2}\dot{\mu}^2 + n\dot{\mu}\dot{\lambda} - n\dot{\mu}\dot{\nu} - \dot{\lambda}\dot{\nu}]$$

$$-e^{-2\lambda}[\nu'' + n\mu'' + \nu'^2 + \frac{n(n+1)}{2}\mu'^2 - n\mu'\lambda' + n\mu'\nu' - \lambda'\nu'] + \frac{n(n+1)}{2}e^{-2\mu} = -8\pi p \quad (2.7)$$

and

$$G_0^1 \equiv (n+1)(\dot{\mu}' + \dot{\mu}\mu' - \dot{\mu}\nu' - \dot{\lambda}\mu')e^{-2\lambda} = 0 \quad (2.8)$$

where a dot and a dash overhead denote differentiation with respect to t and r coordinates respectively.

The conformal motion with the generator $\xi$ is defind by the equation

$$L_\xi g_{ij} \equiv \xi_{i;j} + \xi_{j;i} = \psi(t,r) g_{ij}, \quad (2.9)$$

L stands for the Lie derivative and $\psi$ is an arbitrary function of its arguments. The assumption about the generator yields

$$\xi^1 \neq 0 = \xi(say), \xi^0 = \xi^2 = \xi^3 = \ldots\ldots\ldots = \xi^{n+2} = 0. \quad (2.10)$$

Eq. (2.9), in view of (2.10) gives the following equations [1] namely,

$$\nu = \mu + f_1(t), \quad (2.11)$$

$$\lambda = \mu + f_2(t) + g_1(r), \tag{2.12}$$

$$\xi = A, \tag{2.13}$$

$$\psi = 2A\nu' \tag{2.14}$$

where A is a constant and $f_1, f_2$, and $g_1$ are arbitrary functions of their arguments. By a coordinate transformation of the form [12]

$$\bar{t} = \bar{t}(t), \bar{r} = \bar{r}(r), \tag{2.15}$$

one may choose

$$f_1(t) = g_1(r) = 0 \tag{2.16}$$

without loss of any generality. Let $f_2(t) = f(t)$ say, then we have

$$\nu = \mu = \lambda - f(t). \tag{2.17}$$

Equ. (2.8) with help of (2.9) and (2.17) reduces to

$$\dot{\lambda}' = \dot{\lambda}\lambda' \tag{2.18}$$

which on integration yields

$$e^{-\lambda} = h(r) + g(t), \tag{2.19}$$

h(r) and g(t) are arbitrary functions of their arguments.

Thus the equations (2.17) and (2.18) reveal that finding out metric functions or solutions is equivalent to finding out the functions f(t), h(r) and g(t) by the field equations (2.5) – (2.7). These field equations imply that f, h and g satisfy

$$(n+1)h'' + Bh = C, \tag{2.20}$$

$$\dot{g}\dot{f}e^{2f} = \frac{Bg + C}{n+1}, \tag{2.21}$$

$$\dot{f}^2 = De^{2nf} - \frac{B}{n+1}e^{-2f}. \tag{2.22}$$

where B, C and D are arbitrary constants.

The first integral of eq. (2.18) is

$$(n+1)h'^2 = 2Ch - Bh^2 + E, \tag{2.23}$$

E is a constant.

Defining a new function R(r,t) by

$$R(r,t) := \frac{e^{-f}}{h+g}, \tag{2.24}$$

one can express the density and pressure configurations in compact form as:

$$8\pi\rho = \frac{n(n+1)De^{2nf}}{2R^2} + G(t), \tag{2.25}$$

$$8\pi p = 8\pi\rho - \frac{nD(Bg+C)e^{(2n-1)f}}{R\dot{f}^2} - 2G(t) \tag{2.26}$$

where the function G(t) is given by

$$G(t) = (n+2)Cg + \frac{(n+2)}{2}(Bg^2 - E) + \frac{n+2}{2(n+1)}\frac{(Bg+C)^2}{\dot{f}^2 e^{2f}}. \tag{2.27}$$

The kinematical parameters of the fluid namely expansion $\theta := U^i_{;i}$, $\sigma := (\frac{1}{2}\sigma_{ij}\sigma^{ij})^{1/2}$ are found to be

$$\theta = -e^f[(n+2)\dot{g} + (n+1)(h+g)\dot{f}] \tag{2.28}$$

and

$$\sigma = \frac{1}{\sqrt{2}}(n+1)e^f(h+g)\dot{f}. \tag{2.29}$$

For relistic perfect fluids

$$\rho \geq p > 0, \tag{2.30}$$

so

$$\frac{nD(Bg+C)e^{(2n-1)f}}{R\dot{f}^2}+2G\geq 0. \tag{2.31}$$

## III. PARTICULAR SOLUTIONS

Finding out the most general solution of the space - time is to find out the first integrals of the equations (2.21) – (2.23), but we have been able to solve them explicitly only for two special cases by choosing particular values of the constants as:

**Case A:** $D\neq 0$, otherwise Equation (2.30) is not satisfied. so let us choose

$$B=C=0. \tag{3.1}$$

This choice, with the help of Equations (2.21) – (2.23), (2.27) and (2.31) yield

$$h(r)=M, g(t)=N, E=0, \tag{3.2}$$

$$e^f=(a\sin nt)^{\frac{-1}{n}}, a^2\equiv D. \tag{3.3}$$

where M and N are constants.

Pressure, density, expansion and shear have the following expressions.

$$8\pi p=8\pi\rho=\frac{n(n+1)(M+N)^2}{2a^{2/3}(\sin^2 nt)^{(1+\frac{1}{n})}}, \tag{3.4}$$

$$\theta = (n+1)(M+N)a^{-1/n}\cos nt(\sin nt)^{-(1+\frac{1}{n})}, \tag{3.5}$$

$$\sigma = \frac{-1}{\sqrt{2}}\theta. \tag{3.6}$$

For n=1, i.e. D=4, the solution reduces to the solution 1 of [1]. In this way our solution is a generalisation of [1]. It represents a homogeneous stiff matter distribution with oscillatory matter and kinematical variables.

**Case B:** In this case we choose

$$B = 0 \tag{3.7}$$

As in case A, here find the following values of the different functions as

$$e^f = (a\sin nt)^{-1/n}, \tag{3.8}$$

$$h(r) = \frac{C}{2(n+1)}(r-r_0)^2 - \frac{E}{2C}, \tag{3.9}$$

$$g(t) = \frac{Ca^{2/n}}{n(n+1)}[lnx - \frac{x^2}{2n} + \frac{1}{n}(\frac{1}{n}-1)\frac{x^4}{8} - \frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)\frac{x^6}{36} + ..............] + C_1 \tag{3.10}$$

with

$$x \equiv \cos nt; \qquad r_0, C_1 = constants. \tag{3.11}$$

Density and pressure for this case are obtained to be

$$8\pi\rho = \frac{n(n+1)\{h(r)+g(t)\}^2}{2a^{2/n}(\sin^2 nt)^{(1+\frac{1}{n})}} + G(t) \tag{3.12}$$

$$8\pi p = 8\pi\rho - \frac{nCa^2e^{(2n-1)f}}{Rf^2} - 2G(t). \tag{3.13}$$

Here

$$R = \frac{(a\sin nt)^{\frac{1}{n}}}{(h(r)+g(t))}, \tag{3.14}$$

$$G(t) = (\frac{n+2}{2})\{2Cg(t) - E + \frac{C^2a^{2/n}}{(n+1)}\sec^2 nt(\sin^2 nt)^{(1+\frac{1}{n})}\}. \tag{3.15}$$

It is obvious from these expressions that the distribution is inhomogeneous in density as well as pressure.

Other particular solutions can be obtained by other choices of the constants.

## IV. CONCLUSION

Non - static perfect fluid solutions have been obtained in higher dimensional space - time having spherical symmetry and admitting one parameter group of conformal symmetry. Two particular solutions have been explicitly found out, of which one is the generalisation of Herrera and Leon's work [1] for 4 dimension.

The other solution is inhomogeneous in matter variables. The distributions are with shear.


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