Derivation of the Maxwell equations and the relation between electric and magnetic charge

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Abstract

We give a derivation of the complete set of the Maxwell equations based entirely on the Lagrangian derivability of the Newtonian equation of motion for a test particle and the self adjoint character of the differential operator. In the process, we are led to a fundamental relation between electric and magnetic monopole charges and thereby establishing that there essentially exists only one kind of charge which is by convention called electric.

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1 Introduction

In recent times, derivation of the Maxwell equations has attracted considerable attention. It was triggered off by Dyson's elegant paper [1] which discussed Feynman's derivation of the homogeneous Maxwell equations. It used commutation relations between coordinates and velocities rather than

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two vector fields, what we began with. With this, not only the set becomes the exact Maxwell set (of course we go from the Galilean to the Lorentz invariance) but the most remarkable relation that emerges from consistency of the equations is a relation between scalar and pseudo-scalar charges. That is between electric and magnetic monopoles. This leads to a very profound statement that there is no essential difference between electric and magnetic monopoles and one is a mirror image of the other. We thus need to consider only one of them and could by convention call it electric or magnetic.

One may wonder, what we have really done? We began with two fields which were provided by the self-adjoint and the Lagrangian derivable equation of motion. We then further split them into four and then recombined and miraculously got the Maxwell equations together with the wonderful synthesis of electric and magnetic charges and the Lorentz invariance. This is what we have really done. Why should this lead to such a deep and profound synthesis, we do not fully comprehend?

For quantization of electric charge, one resorts to the Dirac magnetic monopole. Unfortunately theories containing magnetic monopole could not be derived from an action and nor could the monopole fit well in the quantum electrodynamics. Hence, even its theoretical existence still remains an open question. In our formulation of electrodynamics, there is no room for any other monopole charge and hence quantization will have to come from some quantum consideration. It turns out that if we use the fine structure constant in our relation for scalar and pseudo-scalar charges, then quantization of charge emerges with a proper identification of parameter. Our consideration has been purely classical and hence it is not expected to give quantization relation unless we borrow something from the quantum regime.

The paper is organized as follows. In the next Sec., we briefly recall the discussion of self adjointness of the second order differential operator and the inverse problem in classical mechanics, which lead to the Lorentz-like force with the homogeneous set of two equations. In Sec. III, we derive the intermediate set which is Galilean invariant followed by in Sec. IV the derivation of the entire set of Maxwell equations and the fundamental relation between electric and magnetic charges. We conclude with a discussion of general issues and the ones to be taken up in future.

which when substituted in eqns(6 - 8) leads to

$$\begin{aligned} \xi_{ij} + \xi_{ji} &= 0 & (10) \\ \frac{\partial \xi_{ij}}{\partial q^k} + \frac{\partial \xi_{jk}}{\partial q^i} + \frac{\partial \xi_{ki}}{\partial q^j} &= 0 & (11) \\ \frac{\partial \xi_{ij}}{\partial t} &= \frac{\partial \lambda_i}{\partial q^j} - \frac{\partial \lambda_j}{\partial q^i}. & (12) \end{aligned}$$

Eqns(9 - 12) are the necessary and sufficient conditions for existence of a Lagrangian for the Newtonian equation of motion. If we define

$$\lambda_i \equiv \mathcal{X}_i \qquad (13)$$

and

$$\xi_{ij} \equiv \epsilon_{ijk} \mathcal{Y}^k, \qquad (14)$$

then eqns(9, 11 & 12) can be written in the vector form as

$$\begin{aligned} \vec{F} &= \vec{\mathcal{X}} + \vec{v} \times \vec{\mathcal{Y}} & (15) \\ \vec{\nabla} \times \vec{\mathcal{X}} &= -\frac{\partial \vec{\mathcal{Y}}}{\partial t} & (16) \\ \vec{\nabla} . \vec{\mathcal{Y}} &= 0. & (17) \end{aligned}$$

It is important to note here that $\vec{\mathcal{X}}$ and $\vec{\mathcal{Y}}$ are any arbitrary fields experienced by a test particle and eqns (15 - 17) will hold for *any* Newtonian force which has self adjoint equation of motion. These are the equations which were derived by Feynman in 1948 by assuming the commutation relation between coordinates and velocities rather than coordinates and canonically conjugate momenta. These equations can also be obtained by assuming the similar Poisson bracket relations [4, 7].

The derivation of the above form of the force (15) and the two homogeneous equations (16 & 17) hold good if and only if the second order differential operator in the equation of motion is self adjoint. Thus the demand of existence of a Lagrangian for a self adjoint equation of motion determines the form of the force. For a non self adjoint equation of motion, there does not exist a Lagrangian and nor do the eqns (15 - 17) [6]. This is the case whenever dissipative forces are involved and then the differential equation is not self adjoint. It could however be made self adjoint by introducing appropriate Lagrange multipliers [8]. It is well known that dissipative systems

$$
\begin{aligned}
\vec{F} &= q_s(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}) + q_p(\vec{\mathcal{H}} - \vec{v} \times \vec{\mathcal{D}}) && (20)\\
\vec{\nabla} \times \vec{\mathcal{E}} &= -\frac{\partial \vec{\mathcal{B}}}{\partial t} && (21)\\
\vec{\nabla} . \vec{\mathcal{B}} &= 0 && (22)\\
\vec{\nabla} \times \vec{\mathcal{H}} &= \frac{\partial \vec{\mathcal{D}}}{\partial t} && (23)\\
\vec{\nabla} . \vec{\mathcal{D}} &= 0. && (24)
\end{aligned}
$$

This is the intermediate set which is Maxwellian like but not quite as it involves four independent vector fields. It can be easily checked that this set is invariant under the Galilean transformation because

$$
\begin{aligned}
\vec{\nabla}' &= \vec{\nabla} && (25)\\
\frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} + \vec{V} . \vec{\nabla} && (26)
\end{aligned}
$$

The covariance of the force law (15) determines the following laws of transformations for the vector fields involved.

$$
\begin{aligned}
\vec{\mathcal{E}}' &= \vec{\mathcal{E}} + \vec{V} \times \vec{\mathcal{B}} && (27)\\
\vec{\mathcal{B}}' &= \vec{\mathcal{B}} && (28)\\
\vec{\mathcal{H}}' &= \vec{\mathcal{H}} - \vec{V} \times \vec{\mathcal{D}} && (29)\\
\vec{\mathcal{D}}' &= \vec{\mathcal{D}}. && (30)
\end{aligned}
$$

4 The Maxwell equations and the fundamental relation

Clearly we can not proceed further from the intermediate set (21-24) because it is under determined, four differential relations for four vector fields. In fact twice as many would be required for the system to be solvable. For determining a vector field both its divergence and curl must be given. Thus we are led to assume relations between the two polar and two axial vectors and that we do through a one more intermediate step as follows:

$$g = (\mu\epsilon)^{-1/2} e\, tan\theta \tag{40}$$

where θ is an invariant angle for a given family of particles in the universe. Similar relation has been considered earlier by Schwinger[9] for removing one kind of charge. This relation converts scalar into pseudo-scalar and vice-versa. Also, the set of eqns(34 - 37) is incomplete in the sense that we need to know both the divergence and the curl of a vector field to determine it completely. This does not happens until and unless $\vec{\mathcal{E}}$ & $\vec{\mathcal{E}}^*$ and similarly $\vec{\mathcal{B}}^*$ & $\vec{\mathcal{B}}$ differ only by a numerical factor.

Once the relationship between e and g is established, and the numerical factor taken care of in θ, one can replace $\vec{\mathcal{E}}^*$ by $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}^*$ by $\vec{\mathcal{B}}$ in eqn(33) to write force law in terms of charge 'e' as

$$\vec{F} = e(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}) + (\mu\epsilon)^{-1/2} e\, tan\theta(\vec{\mathcal{B}} - \mu\epsilon\vec{v} \times \vec{\mathcal{E}}) \tag{41}$$

or

$$\vec{F} = e(\vec{\mathcal{E}} + (\mu\epsilon)^{-1/2}\, tan\theta\, \vec{\mathcal{B}}) + e(\vec{v} \times (\vec{\mathcal{B}} - (\mu\epsilon)^{1/2}\, tan\theta\, \vec{\mathcal{E}})). \tag{42}$$

We now define two fields $\vec{E}$ & $\vec{B}$ such that

$$\vec{E} = \vec{\mathcal{E}} + (\mu\epsilon)^{-1/2}\, tan\theta\, \vec{\mathcal{B}} \tag{43}$$

$$\vec{B} = \vec{\mathcal{B}} - (\mu\epsilon)^{1/2}\, tan\theta\, \vec{\mathcal{E}} \tag{44}$$

and hence

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B}). \tag{45}$$

Inversely, one can rewrite eqns(43 & 44) as

$$\vec{\mathcal{E}} = cos^2\theta\, \vec{E} - (\mu\epsilon)^{-1/2}\, cos\theta\, sin\theta\, \vec{B} \tag{46}$$

$$\vec{\mathcal{B}} = cos^2\theta\, \vec{B} + (\mu\epsilon)^{1/2}\, cos\theta\, sin\theta\, \vec{E}. \tag{47}$$

Substituting these identifications of $\vec{\mathcal{E}}$ & $\vec{\mathcal{B}}$ in eqns(34 & 35), we obtain

$$cos^2\theta\, (\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}) = cos\theta\, sin\theta((\mu\epsilon)^{-1/2}\vec{\nabla} \times \vec{B} - (\mu\epsilon)^{1/2}\frac{\partial \vec{E}}{\partial t}) \tag{48}$$

$$cos^2\theta\, \vec{\nabla}.\vec{B} = -(\mu\epsilon)^{1/2}\, cos\theta\, sin\theta\, \vec{\nabla}.\vec{E}. \tag{49}$$

If we write $c = (\mu\epsilon)^{-1/2}$, the speed of light, eqn (40) will read as

$$g = e\, c\, tan\theta. \tag{55}$$

Finally, by taking into account the regions containing sources and currents, and using the continuity equation

$$\frac{\partial \rho_e}{\partial t} + \vec{\nabla}.\vec{J}_e = 0 \tag{56}$$

where

$$\begin{aligned} \rho_e &= \rho_g\, cot\theta & (57) \\ \vec{J}_e &= \vec{J}_g\, cot\theta. & (58) \end{aligned}$$

The Maxwell equations in the source occupied region would be given by

$$\begin{aligned} \vec{\nabla}.\vec{E} &= \frac{\rho_e}{\epsilon} & (59) \\ \vec{\nabla}.\vec{B} &= 0 & (60) \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} & (61) \\ \vec{\nabla}\times\vec{B} &= \mu\epsilon\frac{\partial \vec{E}}{\partial t} + \mu\vec{J}_e. & (62) \end{aligned}$$

This completes the derivation of the Maxwell equations and the fundamental relation connecting electric and magnetic monopole charges.

5 Discussion

We have essentially begun with the half of the Maxwell equations. We then split the two vector fields into four to write the intermediate set, which is Maxwell - like as it involves four rather than two fields. Now when we set the linear relationship between the two pairs of polar and axial fields, the intermediate set ultimately reduces to the complete set of Maxwell equations with the fundamental relation between the two kinds of monopole charges. It is therefore not very surprising that our imaginative exercise of splitting and recombining leads to the Maxwell equations, however what is unexpected is quantization but we are not fully happy with it. This is one of the questions that will engage us in future. The other important questions to be addressed would include consideration of the fundamental relation in the context of the early Universe when the electromagnetic field is unified with the other fields and generalization of the formalism to internal degrees of freedom, curved space and general relativity (GR). Most importantly, could we construct a derivation of GR on similar lines. On the face of it, it looks rather difficult and unlikely. However we do believe that some ingenious and imaginative extrapolations may lead to something worth while.

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