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RELATIVISTIC GRAVITATIONAL MASS IN TOLMAN-VI SOLUTION

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Some known solutions for static charged fluid spheres of Tolman-VI type in general relativity are reexamined. The gravitational mass that appears in the Pant and Sah [1] and Tikekar [2] solutions is shown to be of electromagnetic origin in the sense that the gravitational mass, along with all other physical quantities, depends on the electromagnetic field alone. The energy condition, singularity and stability problems of the models are discussed thoroughly.

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1. Introduction

Studies of static solutions in connection with stellar interiors have been of continuing interest for many researchers in the framework of general relativity. These works can be divided into two categories: (1) static fluid spheres without charge and (2) charged static fluid spheres. Some remarkable works on the neutral cases of Einstein's field equations, related to the Schwarzschild solution, were done by Tolman [3], Wyman [4], Leibovitz [5], Whitman [6] and Bayin [7]. On the other hand, solutions of the coupled Einstein-Maxwell field equations have been obtained by several authors. Some of the interesting works in this direction with the Reissner-Nordstöm space-time can be found out in the papers by Weyl [8], Majumdar [9], Papapetrou [10], Bonnor [11], Kyle and Martin [12] and Cooperstock and de la Cruz [13].

The reason behind the work on the coupled charge-matter distributions is mostly to avert the singularities that occur in the Schwarzschild-like solutions. These singularities are of two types: (1) the coordinate singularity at $r = 2m$ which provides the event horizon of a black hole of mass m, and (2) the physical singularity at $r = 0$, the centre of the spherical system. Now, in connection with the singularity, it is observed that the gravitational collapse of a spherically symmetric distribution of matter to a point singularity may be avoided in the presence of charge. This happens since the repulsive Coulomb force due to the charge (in addition to the thermal pressure gradient in the fluid) opposes the gravitational attraction. Therefore, the issue of studying the Einstein-Maxwell space-time has always received considerable attention from the researchers [1, 14–16].

However, there was another issue in studying charged static fluid spheres, which is associated with *electromagnetic mass models*. According to Lorentz [17], in an extended electron "there is no other, no 'true' or 'material' mass", and this provides only an "electromagnetic masses of the electron". Wheeler [18] also believed that the electron has a 'mass without mass', while Feynman [19] termed this type of models the "electromagnetic mass models", where mass originates from the electromagnetic field alone and hence gives a phenomenological relationship between the gravitational and electromagnetic fields [13, 20–24].

Therefore, motivated by the above ideas, we have recently [25] investigated the class of astrophysical solutions previously obtained by us [26] and also by Pant and Sah [1] for a static, spherically symmetric Einstein-Maxwell space-time. While discussing the Pant and Sah solutions [1], which represent the charged analogue of the Tolman VI solution [3], we [25] have studied the case $n = 0$ only related to the electromagnetic mass models [17,19], where n is a free parameter appearing in the set of solutions. In the present investigations, however, it is shown that the electromagnetic mass models can also be constructed not only for the specific choice $n = 0$ but for the whole range of $0 \leq n \leq 2$, even

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including fractional values.

It is known that the Tikekar solutions [2] represent a generalization of the Pant and Sah solutions [1], so we have also studied them as a continuation of the verification scheme. It is possible to show, from the expression for the charge and the equation of state, that the gravitational mass depends on the electric charge alone.

In the concluding part, it is discussed that there are some other solutions available in the literature where, especially in the case (λ, n) in section X of Ivanov's paper [16], our requirement for a solution to describe an electromagnetic mass model, is also satisfied. For physical validity related to all cases studied in this article, the singularity problem, stability situation and the energy conditions are discussed in connection with de Sitter space-time.

2. The Einstein-Maxwell field equations

Consider the spherically symmetric line element

$$ds^2 = g_{ij}dx^i dx^j = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2 d\Omega^2, \quad (1)$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ in the standard coordinates $x^{(0,1,2,3)} = (t, r, \theta, \phi)$, where ν and λ are two functions of the radial coordinate r.

The Einstein-Maxwell field equations corresponding to spherically symmetric static charged source are then given by

$$\mathrm{e}^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 = 8\pi\rho + E^2, \quad (2)$$

$$\mathrm{e}^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 = 8\pi p - E^2, \quad (3)$$

$$\mathrm{e}^{-\lambda}[\nu''/2 + \nu'^2/4 - \nu'\lambda'/4 + (\nu' - \lambda')/2r] = 8\pi p + E^2, \quad (4)$$

$$(r^2 E)' = 4\pi r^2 \sigma\, \mathrm{e}^{\lambda/2}, \quad (5)$$

where ρ, p, E and σ are the matter energy density, the fluid pressure, the electric field intensity and the electric charge density, respectively. Here, the prime denotes a derivative with respect to the the radial coordinate r.

Eq. (5) can be equivalently, in terms of the electric charge q, expressed as

$$q(r) = r^2 E(r) = \int_0^r 4\pi r^2 \sigma\, \mathrm{e}^{\lambda/2} dr. \quad (6)$$

Therefore, Eq. (2), by the use of the above equation (6), reduces to

$$\mathrm{e}^{-\lambda} = 1 - 2M(r)/r, \quad (7)$$

where $M(r)$ is the active gravitational mass, which can be expressed in terms of the effective gravitational mass, $m(r)$, in the form

$$m(r) = M(r) + \mu(r) = 4\pi \int_0^r [\rho + q^2/8\pi r^4] r^2 dr, \quad (8)$$

$\mu(r)$ being the *mass equivalent of the electric field*. It is to be mentioned here that customarily people consider $\mu(r)$ as the *mass equivalent of electromagnetic field* because of the fact that it is associated with the idea of Lorentz's conjecture of 'Electromagnetic mass' for the extended electron [17, 19]. However, the models considered here being static, the fields are not electromagnetic but only electric.

It is interesting to note that Eq. (8) provides the increase of the total gravitational mass due to inclusion of the charge [13, 20, 27], and in the absence of an electric charge it reduces to the usual active gravitational mass. In this regard, it is also to be noted that Eq. (8) is in the form of Bekenstein's [28] Eq. (31), where the part of mass related to the fluid is irreducible and charge-independent. Therefore, in the next section, one of our aims is to find out the criteria for constructing electromagnetic mass models by showing that the matter part depends on the charge, and thus the entire gravitational mass turns out to be completely electromagnetic by origin.

3. Gravitational masses of purely electromagnetic origin

3.1. The Pant-Sah models

For a static, spherically symmetric distribution of charged fluid, Pant and Sah [1] obtained a class of solutions which are as follows:

$$\mathrm{e}^{\nu} = br^{2n}, \quad (9)$$

$$\mathrm{e}^{-\lambda} = c, \quad (10)$$

$$\rho = \frac{1}{16\pi r^2}[1 - c(n-1)^2], \quad (11)$$

$$p = \frac{1}{16\pi r^2}[c(n+1)^2 - 1], \quad (12)$$

$$\sigma = \pm\frac{1}{4\pi r^2}\left[\frac{c}{2}\{1 - c(1 + 2n - n^2)\}\right]^{1/2}, \quad (13)$$

$$E^2 = \frac{1}{2r^2}[1 - c(1 + 2n - n^2)], \quad (14)$$

where

$$b = a^{-2n}\left[1 - \frac{2m}{a} + \frac{q^2}{a^2}\right], \quad (15)$$

$$c = \left[1 - \frac{2m}{a} + \frac{q^2}{a^2}\right] = \left[1 - \frac{2q^2}{a^2}\right](1 + 2n - n^2)^{-1}. \quad (16)$$

With this expression of c, the above class of solutions, for the cosmological constant $\Lambda = 0$ and also for the integration constant $B = 0$, represents a charged analogue of the Tolman VI [3] solution. It is also to be noted that the matter density, the fluid pressure, electric charge density and the radial electric field involved in Eqs. (11)–(14) follow the inverse square laws. Consequently, these physical parameters increase as the distance decreases and become infinite at the centre of the

spherical system. Therefore, due to singularities at the centre, such models are not physical as possible descriptions of stellar structure. They may be either regarded as the Reissner-Nordstöm black hole solutions, or one may think of a technique by which the singularity can be averted. We shall consider this point later.

Now, the total gravitational mass $m(r=a)$ can be calculated from Eq. (8) using Eqs. (11) and (14):

$$m = \frac{n(2-n)a^2 + 2q^2}{2(1+2n-n^2)a}. \tag{17}$$

A close observation of Eq. (17) shows that the status of the gravitational mass depends on the parameters n, a and q. Therefore, once we fix the values of a and q for a given charged spherical system, n can be regarded as an adjustable parameter 'having real, not necessarily integral, values' [3]. However, for physical viability, the values to be assigned to this parameter are $0 \leq n \leq 2$. One can obviously assign innumerable values to n within this range. Related to the case $n = 0$, it has been already shown by us [25] that for vanishing electric charge all the physical quantities, including the gravitational mass, vanish, and the space-time becomes flat. We therefore, for our purpose, would like to study the following four subcases A–D only.

A. $n = 0.5$

This case is favourable for some historical reasons. This choice was originally made by Tolman [3] himself in his uncharged version. In this case, the gravitational mass becomes

$$m = \frac{3a^2 + 8q^2}{14a}. \tag{18}$$

The functional condition containing the energy density ρ and the pressure p, by virtue of Eqs. (11) and (12), can be written as

$$\rho + p = \frac{1}{4\pi r^2}\left[\frac{n(a^2-2q^2)}{(1+2n-n^2)a^2}\right], \tag{19}$$

which for the present case reduces to

$$\rho + p = \frac{1}{14\pi r^2}\left[1 - \frac{2q^2}{a^2}\right]. \tag{20}$$

B. $n = 1$

For this choice, the gravitational mass becomes

$$m = \frac{a^2 + 2q^2}{4a}, \tag{21}$$

whereas the functional condition containing the energy density ρ and the pressure p is

$$\rho + p = \frac{1}{8\pi r^2}\left[1 - \frac{2q^2}{a^2}\right]. \tag{22}$$

It has been pointed out by Herrera and Ponce de León [27] that this particular solution, $n = 1$, admits a one-parameter group of conformal motions and is homothetic.

C. $n = 1.5$

In this case, the gravitational mass is

$$m = \frac{3a^2 + 8q^2}{14a}, \tag{23}$$

and the related functional condition here becomes

$$\rho + p = \frac{3}{14\pi r^2}\left[1 - \frac{2q^2}{a^2}\right]. \tag{24}$$

The gravitational mass in this case is the same as for $n = 0.5$.

D. $n = 2$

The gravitational mass in this case becomes

$$m = \frac{q^2}{a}. \tag{25}$$

This expression is precisely the same as that in the $n = 0$ case of our previous work [25], it vanishes for vanishing electric charge and thus provides an 'electromagnetic mass' model [17, 19]. However, the functional condition for the present situation differs from that of the $n = 0$ case and is given by

$$\rho + p = \frac{1}{2\pi r^2}\left[1 - \frac{2q^2}{a^2}\right]. \tag{26}$$

Let us now, following Ray and Das [25, 26], and Ivanov [16], choose a relation between the electric charge and the radial coordinate of the fluid distribution as follows:

$$q(r) = Kr^s, \tag{27}$$

where K and s are two free parameters. For the specific choice $K = 1/\sqrt{2}$ and $s = 1$, the *ansatz* expressed in Eq. (27) reduces to $q(a)/a = 1/\sqrt{2}$, where a is the radius of the charged sphere. It is interesting to note that for this charge-radius ratio all the above perfect fluid functional conditions reduce to the form $\rho + p = 0$. This is known as the 'pure charge condition' [22] and also the imperfect-fluid equation of state in the literature since the matter distribution under consideration is in tension and hence the matter is named as a 'false vacuum' or 'degenerate vacuum' or 'ρ-vacuum' [29–32]. We would like to comment here that this imperfect-fluid equation of state precisely describes the cosmological constant, viz., the vacuum energy, and by the phrase 'pure charge condition' Gautreau [22] indicates the necessary condition for the 'Lorentz-type pure-charge extended electron' to be of electromagnetic origin.

The above charge-radius ratio in turn makes all the total effective gravitational mass vanish for vanishing charge. Therefore, the gravitational masses here are of purely electromagnetic origin. It is to be noted here that all the above cases $n = 0.5, 1, 1.5$, including our previous case $n = 0$ [25], provide 'electromagnetic mass' models with the imperfect-fluid equation of state. The

only exception here is the last case $n = 2$ where we have obtained an electromagnetic mass model even under the perfect fluid condition. Although for the charge-radius ratio $q(a)/a = 1/\sqrt{2}$, the perfect fluid functional condition in Eq. (26) reduces to the imperfect one, but this *ansatz* is not necessary for making the gravitational mass in Eq. (25) suitably vanish as in the other cases.

Let us look at Eq. (26) differently. The components of this equation, viz., the matter energy density and the fluid pressure are given by

$$\rho = \frac{1}{8\pi r^2}\frac{q^2}{a^2}, \tag{28}$$

$$p = \frac{1}{8\pi r^2}\left[4 - \frac{9q^2}{a^2}\right]. \tag{29}$$

It can be checked that for a positive pressure the constraint on the charge-radius ratio to be imposed here is $q(a)/a \leq \pm 2/3$. The pressure vanishes for the value $\pm 2/3$, whereas it becomes positive for values less than $\pm 2/3$. Thus we have come across a case for electromagnetic mass models where the equation of state is a perfect-fluid one with a positive pressure. This case is obviously in contradiction to Ivanov's observation [16] that "...electromagnetic mass models all seem to have negative pressure". This particular aspect has been pointed out in our previous work [25] where we have mentioned that there are some examples of electromagnetic mass models where positive pressures are also available, to be shown elsewhere. However, a few more examples in this direction need a further study.

3.2. The Tikekar models

We are now interested in discussing the Tikekar [2] solutions which represent a generalization of those of Pant and Sah [1]. As a particular solution, which describes a physically plausible distribution of a charged perfect gas, Tikekar [2] has obtained the matter density and fluid pressure in the following forms:

$$\rho = \frac{b}{4\pi[(b+d+1)^2 - 4d]r^2}, \tag{30}$$

$$p = \frac{d}{4\pi[(b+d+1)^2 - 4d]r^2}, \tag{31}$$

and hence the functional condition is given by

$$\rho + p = \frac{b+d}{4\pi[(b+d+1)^2 - 4d]r^2}, \tag{32}$$

where b and d are non-negative constants. These solutions [2], as those of Pant and Sah [1], also suffer from the singularity problem since the matter density and the fluid pressure are infinite at the centre $r = 0$. This point for both solutions will be discussed later on.

The total mass and charge contained within the sphere of radius a are, respectively,

$$m = \frac{1}{2}\left[\frac{(b+d)^2 + 2(b-d)}{(b+d+1)^2 - 4d}\right]a, \tag{33}$$

$$q = \left[\frac{(b+d)^2 - 2d}{(b+d+1)^2 - 4d}\right]^{1/2} a. \tag{34}$$

Now, vanishing of the charge in Eq. (34) implies that

$$b + d = \pm\sqrt{2d}. \tag{35}$$

Again, by the use of the pure charge condition, i.e., the vacuum-fluid equation of state $\rho + p = 0$ [22] in Eq. (32), one obtains

$$b + d = 0. \tag{36}$$

Thus Eqs. (35) and (36) imply both $b = 0$ and $d = 0$. Therefore, substitutions of these values to Eqs. (30), (31) and (33) make all the physical quantities, including the gravitational mass, vanish and provide an electromagnetic mass model with the imperfect-fluid equation of state (i.e., the reduced Eq. (32) which now takes the form $\rho + p = 0$).

However, when the constants b and d are identified as

$$b = [1 - c(n-1)^2]/4c, \tag{37}$$

$$d = [c(n+1)^2 - 1]/4c, \tag{38}$$

the solutions assumes the Pant and Sah form [1] which also provides an electromagnetic mass model, as we have verified in Sec. 3.1.

4. Conclusions

In the present article, we have shown that some of the known solutions, related to static charged fluid spheres of Tolman-VI type, are of purely electromagnetic origin. In this regard, it is also to be mentioned that Ivanov's solutions [16] with a (λ, n) classification scheme are of this category. He has obtained a set of solutions by a Bessel function technique, which recover the Pant and Sah solution [1]. However, in this set of solutions, the density and charge function are singular at the centre, as stated earlier. It is also mentioned that the Tikekar solutions [2] suffer from the same singularity problem. One of the most undesirable features of general relativity is the occurrence of spacetime singularities where the laws of physics break down, and this is inevitable according to the Hawking-Penrose singularity theorems [33]. However, following Herrera and Ponce de León [27], one can consider a sphere as being composed of (1) a central core of radius r_0 inside which all physical quantities are finite, and (2) above the core, a self-similar fluid described by any of the solutions of Sections 3.1 and 3.2. Then these solutions can be matched with any suitable interior Schwarzschild solutions across the surface $r = r_0$ where $0 < r_0 < a$. It is interesting to note that, in the context of cosmology, Trautman [34] suggested that in the Einstein-Cartan theory, where spin and torsion are taken into account inherently, a singularity is averted due to the action of torsion in a universe filled with spinning dust.

Regarding the stability of charged fluid spheres, Bonnor [11] and also independently De and Raychaudhuri [35] showed that a dust cloud of arbitrarily large mass and small radius can remain in equilibrium if it satisfies the relation between the electric charge and matter energy densities by $\sigma = \pm\rho$. However, Glazer [36], considering radial pulsations, showed that the Bonnor [11] model is electrically unstable. He [37] also explicitly established the effects of electric charge upon dynamical stability. According to Stettner [38], a fluid sphere of uniform density with a net surface charge is more stable than a neutral one. Whitman and Burch [39], for an arbitrary charge and mass distribution, showed that a charged analogue gives more stability. However, application of the pulsation equations to the charged Pant and Sah solution [1] gave an unsatisfactory result in connection with the boundary condition which is incompatible with the densities and pressures [39] (a general technique for studying the stability is provided in the Appendix.

We would like to point out that the solutions expressed by Eqs. (20), (22) and (24) (except Eq. (26) for $n = 2$ of Pant and Sah [1]), do not obey the energy condition [33]. The reason behind this is related to the equation of state which, in the present case, takes the form $\rho + p = 0$ due to application of the *ansatz* $q(a)/a = 1/\sqrt{2}$. This equation provides two situations: either $\rho > 0$ and hence p is negative or $p > 0$ which means that ρ is negative. Thus, the strong energy conditions $RijK^iK_j \geq 0$, where K^i is a timelike vector, are violated in both cases. The first possibility mentioned above requires that the system should be under tension and hence gravitationally repulsive in nature [21]. This case can be expressed in the form $g_{00}g_{11} = -1$ and hence $\lambda = -\nu$, which is also equivalent to the charged de Sitter solution. The second option, viz., $\rho < 0$ indicates a negative mass for the inside of the fluid sphere. This is possible in the case of Lorentz's extended electron of the size $\sim 10^{-16}$ cm where the spherically symmetric charged distribution of matter must contain some negative mass density [40–43].

Appendix: Stability analysis

To study the stability, one needs to match the interior solution to the exterior Reissner-Nordström black hole solution

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{q}{r^2}\right)^{-1} dr^2 - r^2 d\Omega_2^2 \qquad \text{(A.1)}$$

at the junction interface S situated outside the event horizon, $a > r_h = m \pm \sqrt{m^2 - q^2}$. To analyze the stability, it is required to use the extrinsic curvature, or second fundamental forms, associated with the two sides of the shell S as $K^{\pm}_{ij} = -u^{\pm}_{\mu;\nu} e^{\mu}_{(i)} e^{\nu}_{(j)}$, where $u^{\pm}$ are the unit normals to S and $e^{\mu}_{(i)}$ are the components of the holonomic basis vectors tangent to S. According to the Darmois-Israel formalism [44, 45], one can write the Lanczos equations for the surface stress-energy tensors S^i_j at the junction interface S as [46–50]

$$S^i_j = -\frac{1}{8\pi}([K^i_j] - \delta^i_j K), \qquad \text{(A.2)}$$

where $S^i_j = \mathrm{diag}(-\varrho, p_\theta, p_\phi)$ is the surface energy tensor with ϱ the surface density and p_θ and p_ϕ the surface pressures; $[K_{ij}] = K^+_{ij} - K^-_{ij}$ and $K = [K^i_i] = \mathrm{trace}\,[K_{ij}]$.

To analyze the dynamics of the junction shell, we permit the junction radius to become a function of proper time, $a \to a(\tau)$. Now, taking into account Eq. (A.2), one can find

$$\varrho = -\frac{1}{4\pi a}\left[\sqrt{1 - \frac{2m}{a} + \frac{q^2}{a^2} + \dot{a}^2} - \sqrt{c + \dot{a}^2}\right], \qquad \text{(A.3)}$$

$$p_\theta = p_\phi = p = \frac{1}{8\pi a}\frac{1 - m/a + a\ddot{a} + \dot{a}^2}{\sqrt{1 - 2m/a + q^2/a^2 + \dot{a}^2}} - \frac{(1+n)(c + \dot{a}^2) + a\ddot{a}}{\sqrt{c + \dot{a}^2}} \qquad \text{(A.4)}$$

In what follows, the overdot means a derivatives with respect to τ.

The conservation identity $S^i_{j|i} = -[\dot{\varrho} + 2\frac{\dot{a}}{a}(p + \varrho)]$ yields the following relation:

$$\varrho' = -\frac{2}{a}(p + \varrho) + Y, \qquad \text{(A.5)}$$

where

$$Y = -\frac{n}{4\pi a^2}\sqrt{c + \dot{a}^2}. \qquad \text{(A.6)}$$

After a little bit of algebra, Eq. (A.3) gives the thin shell equation of motion

$$\dot{a}^2 + V(a) = 0, \qquad \text{(A.7)}$$

where $V(a)$ is the potential and defined as

$$V(a) = \frac{1}{2}(f_1 + c) - 4\pi^2 a^2 \varrho^2 - \frac{(f_1 - c)^2}{64\pi^2 a^2 \varrho^2}, \qquad \text{(A.8)}$$

where $f_1 = 1 - 2m/a + q^2/a^2$.

Linearizing around a static solution at $a = a_0$, one can expand $V(a)$ around a_0 to obtain

$$V = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2}V''(a_0)(a - a_0)^2 + O[(a - a_0)^3], \qquad \text{(A.9)}$$

where the prime denotes a derivative with respect to a.

Since we are considering linearization around a static solution at $a = a_0$, we have $V(a_0) = 0$ and $V'(a_0) = 0$. Stable equilibrium configurations correspond to the condition $V''(a_0) > 0$, i.e., $V(a)$ has a local minimum

at a_0. Now, we define a parameter β, to be interpreted as the speed of sound [51], by the relation

$$\beta^2(\varrho) = \frac{\partial p}{\partial \varrho}\Big|_{\varrho}. \tag{A.10}$$

Following Eq. (A.5), we get

$$\beta^2(\varrho) = -1 + \frac{a}{2\varrho'}\left[\frac{2}{a^2}(p+\varrho) + Y' - \varrho''\right]. \tag{A.11}$$

The solution is stable if

$$\beta_0^2 < \frac{A - B + C - S - T + G - H}{N - L} - 1 \tag{A.12}$$

where $\beta_0 = \beta(a = a_0)$ and A, B, C, S, T, G, H, N, L are given at $a = a_0$ as follows:

$$\begin{aligned}
A &= \frac{1}{2}f_1'' = \frac{1}{2}\left[\frac{6q^2}{a^4} - \frac{4m}{a^3}\right],\\
B &= 8\pi^2(\varrho^2 - 4a\varrho\varrho' - 2a^2\varrho'^2)\\
&= \frac{1}{2a^2}(\sqrt{f_1} - \sqrt{c})^2\\
&\quad + \frac{2}{a^2}(\sqrt{f_1} - \sqrt{c})\left(\frac{f_2}{\sqrt{f_1}} - \sqrt{c}\right)\\
&\quad - \frac{1}{a^2}\left(\frac{f_2}{\sqrt{f_1}} - \sqrt{c}\right)^2,\\
C &= \frac{2}{a^2}[(p+\varrho) + Y']\left[\frac{(f_1 - c)^2}{32\pi^2\varrho^3a^2} - 8\pi^2\varrho a^2\right]\\
&= \frac{1}{8\pi a}\left(\sqrt{c}(1-n) - \frac{f_2}{\sqrt{f_1}}\right)\\
&\quad + \frac{n\sqrt{c}}{4\pi a^3}\left[\frac{4\pi}{a}\left\{(\sqrt{f_1} - \sqrt{c}) - \frac{(f_1 - c)^2}{(\sqrt{f_1} - \sqrt{c})^3}\right\}\right],\\
S &= \frac{(f_1')^2}{32\pi^2\varrho^2a^2} = \frac{1}{2}\frac{(2m/a^2 - 2q^2/a^3)^2}{(\sqrt{f_1} - \sqrt{c})^2},\\
T &= \frac{(f_1 - c)f_1''}{32\pi^2\varrho^2a^2} = \frac{1}{2}\frac{(f_1 - c)(-4m/a^3 + 6q^2/a^4)}{(\sqrt{f_1} - \sqrt{c})^2},\\
G &= \frac{(f_1 - c)f_1'}{16\pi^2\varrho^2a^2}\left[\frac{\varrho'}{\varrho} + \frac{\varrho'}{a} + \frac{2}{a}\right]\\
&= \frac{(f_1 - c)(2m/a^2 - 2q^2/a^3)}{(\sqrt{f_1} - \sqrt{c})^2}\\
&\quad \times \frac{1}{a}\frac{1}{(\sqrt{f_1} - \sqrt{c})}\left[\left(\frac{f_2}{\sqrt{f_1}} - \sqrt{c}\right)\right.\\
&\quad \left. + \frac{1}{4\pi a^3}\left(\frac{f_2}{\sqrt{f_1}} - \sqrt{c} + \frac{2}{a}\right)\right],\\
H &= \frac{(f_1 - c)^2}{16\pi^2\varrho^2a^2}\left[\frac{2\varrho'}{a\varrho} + \frac{3\varrho'^2}{2\varrho^2} + \frac{3}{2a^2}\right]\\
&= \frac{(f_1 - c)^2}{(\sqrt{f_1} - \sqrt{c})^2}\\
&\quad \times \left[\frac{3}{2a^2} - \frac{2}{a^2}\frac{1}{(\sqrt{f_1} - \sqrt{c})}\left(\frac{f_2}{\sqrt{f_1}} - \sqrt{c}\right)\right.\\
&\quad \left. + \frac{3}{2a^2}\frac{1}{(\sqrt{f_1} - \sqrt{c})}\left(\frac{f_2}{\sqrt{f_1}} - \sqrt{c}\right)^2\right],\\
N &= \frac{4\varrho'}{a^3}\frac{(f_1 - c)^2}{32\pi^2\varrho^3a^2}\\
&= \frac{1}{2\pi a^5}\left[\frac{f_2}{\sqrt{f_1}} - \sqrt{c}\right]\left[\frac{(f_1 - c)^2}{(\sqrt{f_1} - \sqrt{c})^3}\right],\\
L &= \frac{32\pi^2\varrho\varrho'}{a}\\
&= -\frac{2}{a^4}(\sqrt{f_1} - \sqrt{c})\left[\frac{f_2}{\sqrt{f_1}} - \sqrt{c}\right],
\end{aligned} \tag{A.13}$$

where (see also (A.8))

$$\begin{aligned}
f_1 &= 1 - 2m/a + q^2/a^2,\\
f_2 &= 1 - 3m/a + 2q^2/a^2.
\end{aligned}$$

Thus if a_0, u, c, m and q are specified quantities, then the stability of the configuration requires the above restriction on the parameter β_0. This means that there exists some part of the parameter space where the junction location is stable.

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