

# The role of general relativity in the uncertainty principle

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**Abstract.** The role played by general relativity in quantum mechanics (especially as regards the uncertainty principle) is investigated. It is confirmed that the validity of time-energy uncertainty *does* depend on gravitational time dilation. We also show that there exists an intrinsic lower bound to the accuracy with which acceleration due to gravity can be measured. The motion of equivalence principle in quantum mechanics is clarified.

## 1. Introduction

In the early years of quantum mechanics, Einstein suggested a 'thought experiment' which could possibly violate the energy-time uncertainty principle. The experiment ('weighing the photon') involves determining the weight of a box containing radiation before and after the emission of a photon. The emission is controlled by a clockwork shutter mechanism allowing precise determination of time. Since the energy can be calculated from the weight, the above arrangement allows one to violate the inequality

$$\Delta E \Delta T \geq \frac{1}{2} \hbar. \quad (1)$$

Bohr refuted the argument by pointing out the following fact. To make a precise determination of weight, the position of the pointer (corresponding to the length of stretched spring in the balance) has to be read off with a precision  $\Delta x$ . Because  $\Delta x$  is non-zero, the gravitational potential in which the clock is placed is uncertain by a factor

$$\Delta \phi = g \Delta x \quad (2)$$

where  $g$  is the acceleration due to gravity. This  $\Delta \phi$  causes an uncertainty  $\Delta T$ , as in general relativity:

$$\Delta T/T \approx g \Delta x/c^2. \quad (3)$$

In (3),  $T$  denotes the time taken to perform the measurement. Using (3), Bohr could show that (1) is satisfied (see, e.g., Pauli 1955).

The surprising feature in the above 'thought experiment' is the entry of general relativity. Does it mean that quantum mechanical principles would be invalid if general relativity is wrong? If so, it would be extremely surprising. A considerable amount of work done in the last three decades, has completely failed to bring together general relativity and quantum theory. Could it be that some fundamental link involving measurement theory is being overlooked?

Before such questions can be answered, one must make clear the ingredients that go into 'general relativity'. For example, the validity of (1) is certainly independent of Einstein's field equations for gravity. In fact, the only two ingredients that go into

Bohr's argument are (i) the special theory of relativity and (ii) equivalence of inertial and gravitational mass. The gravitational time dilation in (3) can be derived (at least to the lowest order) from (i) and (ii) above. Thus, at least we can conclude that the validity of the uncertainty principle depends on the validity of (i) and (ii) above. Even this (especially the dependence on (ii)) is surprising. There are claims in the literature that quantum phenomena do not respect the principle of equivalence, which makes the above results more puzzling. This particular experiment, however, is very much based on the equivalence of the 'inertial' and gravitational mass of photons. Clearly, the experiment has to be remodelled if the dependence on the equivalence principle is to be investigated.

The simplest remodelling of the above idea proceeds as follows. Measure the mass  $M$  of a body by measuring the gravitational acceleration produced by that object on a test particle at a distance. In other words, we shall measure the 'active' mass rather than the 'passive' mass. (The uncertainty in the mass determination, in the 'weighing the photon' experiment, arises only from the final mass measurement. In other words, we can replace an experiment to measure mass difference by an experiment to measure the mass itself.) We shall discuss such an experiment in the next section.

Questions involving general relativity and quantum mechanics are also relevant from two other points of view. The first point has to do with the clash between the non-locality of quantum mechanics and the local nature of the principle of equivalence. Consider a quantum mechanical experiment done in an elevator falling freely in a gravitational field. The size of the local inertial frame (elevator) is limited by the background curvature. On the other hand, the spreading of wavepackets with time sets a bound to the timescale for measurement processes. In other words we expect new limitations on the possible measurements of gravitational acceleration, etc, because of the existence of curvature.

The second point has to do with the uncertainties in the measurement of gravitational field which arise from the uncertainties in the position and velocity of the source. A measurement of the gravitational acceleration and its time derivative can be related to the position and velocity of the source particle. It is necessary to make sure that the uncertainty principle is not violated in such experiments. Clearly, the analysis of the above situation has important implications for quantum gravity.

Most of the previous work on this subject was done in connection with neutron interference experiments (see Greenberger 1968, 1983, Greenberger and Overhauser 1979). Some other papers dealing with the uncertainty principle and gravity are those of Kuchar (1980), Unruh (1984) and Padmanabhan (1982).

## 2. Measuring the active gravitational mass

Consider a gravitating body ('star') of mass  $M$  located at the origin of the coordinate system. We set up a 'measuring system' at a distance  $R$ . It is assumed that

$$\varepsilon_1 \equiv GM/c^2 R \ll 1 \quad (4)$$

so that general relativistic effects need to be retained only to the lowest order in  $\varepsilon_1$ . The 'measuring system' essentially consists of a test particle of mass  $m$  and two clocks kept stationary at distances  $(R - \frac{1}{2}x)$  and  $(R + \frac{1}{2}x)$ . It will be assumed that

$$\varepsilon_2 \equiv x/R \ll 1 \quad (5)$$

so that only leading terms in  $\varepsilon_2$  need to be retained. Because of (5) we may use experiments taking place over a length  $x$  to measure 'local' properties at  $R$ . Let us suppose that the clock at  $(R + \frac{1}{2}x)$  reads  $t_1$  when the clock at  $(R - \frac{1}{2}x)$  reads  $t_2$ . From the formula for gravitational time dilation, it follows that

$$t_2 \approx t_1(1 + gx/c^2) + O(\varepsilon_2^2) \tag{6}$$

where

$$g = GM/R^2 \tag{7}$$

is the acceleration due to gravity. Equation (6) allows the 'book-keeping' between times read off by the two clocks so that the readings can be compared.

Here and elsewhere in this paper, we shall ignore uncertainties which can be made arbitrarily small. For example, one may ask questions like, 'how exactly can the position of the clock itself be known?', 'how correctly can the synchronisation of clocks be effected?', etc. These considerations, of course, can be a source of more uncertainty unless elaborate procedures are developed. But these uncertainties are irrelevant to our discussion because the simplest source of uncertainties (which we *do* consider in the following!) themselves give us the requisite bound.

The 'experiment' consists of dropping a mass  $m$  from  $(R + \frac{1}{2}x)$  towards the central body at  $t_1 = t_2 = 0$ . The time read by the clock at  $(R - \frac{1}{2}x)$  is read off as the test particle crosses it. Suppose this reading is  $t$ . Then, to the lowest order in  $\varepsilon_2$ , we can write

$$x = \frac{1}{2}gt^2 = \frac{1}{2}(GM/R^2)t^2. \tag{8}$$

(Since (8) already contains a factor  $GM/R^2$ , the difference between  $t_1$  and  $t_2$  does not matter in the above expression. A more tedious calculation based on radial geodesics in the Schwarzschild metric confirms our conclusions.) From (8) we obtain

$$GM = 2xR^2/t^2. \tag{9}$$

Knowing  $x$ ,  $R$  and measuring  $t$ , we determine  $M$ . We shall now show that the uncertainties  $\delta M$  and  $\delta t$  in  $M$  and  $t$  satisfy the inequality

$$\delta(Mc^2) \delta t \geq \frac{1}{2}\hbar. \tag{10}$$

We begin by noticing that the position of the test particle at  $t = 0$  cannot be infinitely accurate without producing an infinitely uncertain initial velocity. Treating the test particle as a Gaussian wavepacket, we can calculate the minimum uncertainty in  $x$  at time  $t$  to be

$$[\delta x(t)]^2 = [\delta x(0)]^2 + \frac{\hbar^2 t^2}{4m[\delta x(0)]^2}. \tag{11}$$

(This equation (11), which represents the spread of a Gaussian wavepacket for a free particle, is valid even in the presence of a constant gravitational field. In general, two potentials which differ only in a term linear in the position will have the same spread. This point is discussed in the appendix.) By carefully choosing  $\delta x(0)$  one can minimise (11). It is seen that

$$[\delta x(t)]^2 \geq [\delta x(t)]_{\min}^2 = \hbar t/m. \tag{12}$$

Since the final position has an uncertainty  $\delta x(t)$ , the clock readings are off the correct value by an amount

$$\delta t \geq (t/c^2)g[\delta x]_{\min}. \tag{13}$$

From (9) we obtain

$$G \delta M \geq (2R^2/t^2)[\delta x]_{\min}. \quad (14)$$

Combining (13) and (14) and using (7) and (12) we obtain

$$G \delta M \delta t \geq \frac{2R^2}{t^2} \frac{t}{c^2} \frac{GM}{R^2} \frac{\hbar t}{m} = \frac{2GM}{c^2 m} \hbar \quad (15)$$

so that

$$\delta(Mc^2) \delta t \geq 2M\hbar/m. \quad (16)$$

Since  $M \geq m$ , (16) clearly implies (10). We shall now discuss many possible extensions of (16).

To begin with, the bound in (16) can be strengthened by taking into account the uncertainty  $\delta R(t)$  in  $R$ . Notice that the experiment demands a measurement of this distance which, in turn, requires knowledge about the position of the star. Again, it is not possible to have vanishing  $\delta R(0)$  and analysis similar to the one that led to (12) will provide the bound

$$[\delta R(t)]^2 \geq [\delta R(t)]_{\min}^2 = \hbar t/m. \quad (17)$$

When both  $\delta x$  and  $\delta R$  are considered, (13) and (14) are replaced by

$$\frac{\delta t}{t} = \frac{1}{c^2} \delta \left( \frac{GM}{R^2} x \right) = \frac{G}{c^2} \left( \frac{M}{R^2} \delta x + \frac{2Mx}{R^3} \delta R \right) \quad (18)$$

and

$$G \delta M = \frac{2R^2}{t^2} \left( \delta x + \frac{2x}{R} \delta R \right) \quad (19)$$

giving

$$\begin{aligned} \delta(Mc^2) \delta t &= \frac{2M}{t} \left( (\delta x)^2 + \frac{4x^2}{R^2} (\delta R)^2 + \frac{4x}{R} \delta x \delta R \right) \\ &\geq \frac{2M}{m} \hbar \left[ 1 + 4 \left( \frac{x}{R} \right) \left( \frac{m}{M} \right)^{1/2} + \frac{4x^2}{R^2} \left( \frac{m}{M} \right) \right]. \end{aligned} \quad (20)$$

Clearly the leading uncertainty arises from  $(\delta x)^2$ . The other two terms are suppressed by  $(x/R)$  factors. In most physical situations we need retain only the leading contribution.

We have derived (16) in a very simple-minded measurement process. Should it not be possible to improve the accuracy by more ingenious methods? We shall discuss briefly another method for determining  $M$  which, unfortunately, does no better.

This method consists of putting the test particle  $m$  into a circular orbit of radius  $r$  and measuring the time period  $T$  for one revolution. The mass  $M$  is given by

$$GM = 4\pi^2 r^3 / T^2. \quad (21)$$

The uncertainties originate in a manner similar to the above case. If the initial radial position of  $m$  is known with an accuracy  $\delta r(0)$ , then the initial radial velocity will have an uncertainty  $\sim (\hbar/2m\delta r(0))$ . This will make  $\delta r(T)^2$  become

$$[\delta r(T)]^2 = [\delta r(0)]^2 + \frac{\hbar^2 T^2}{4m^2 [\delta r(0)]^2} \geq \frac{\hbar T}{m}. \quad (22)$$

(This formula is strictly true only for the linear motion, but it can be applied to the component of motion along a chosen axis. In our case, it is definitely admissible to use (22) for an order of magnitude estimate.) Now, as in the previous case,

$$\frac{\delta T}{T} = \frac{GM}{c^2 r^2} \delta r \tag{23}$$

and

$$G \delta M = 12 \pi^2 r^2 \delta r / T^2 \tag{24}$$

so that

$$\delta(Mc^2)\delta T = 12 \pi^2 (M/T) [\delta r(T)]^2 \geq 12 \pi^2 \frac{M \hbar T}{T m} = 12 \pi^2 \left(\frac{M}{m}\right) \hbar. \tag{25}$$

Even without considering the uncertainty in the source position, we get a bound stricter than (16). If a measurement procedure is devised bettering the bounds in (16) and (25) to the limit of  $(\hbar/2)$ , such a procedure is likely to be much more complicated than a simple free fall experiment.

The results in (16) and (25) depend on the test particle mass  $m$ . This is due to the simple fact that the wavefunction of a particle in a gravitational field depends on its mass. For example, the Gaussian wavepacket,  $\psi(x, t)$ , with

$$|\psi(x, t)|^2 = \left(\frac{1}{2\pi\sigma^2(t)}\right)^{1/2} \exp\left(-\frac{(x - \bar{x}(t))^2}{2\sigma^2(t)}\right) \tag{26}$$

will satisfy the Schrödinger equation in a gravitational field (see the appendix):

$$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + mgx\psi \tag{27}$$

provided

$$\bar{x}(t) = \bar{x}(0) - \frac{1}{2}gt^2 \tag{28}$$

and

$$\sigma^2(t) = \sigma^2(0) + \frac{\hbar^2 t^2}{4m^2 \sigma^2(0)}. \tag{29}$$

We see that, even though  $\bar{x}(t)$  is independent of  $m$ ,  $\sigma^2(t)$  does depend on  $m$  via  $(\hbar/m)$ . (This dependence has no classical analogue and vanishes for  $\hbar \rightarrow 0$ .) Suppose wavepackets are constructed for two particles of masses  $m_1$  and  $m_2$  with the same  $\sigma(0)$ . By measuring the expectation value  $\langle x^2 \rangle$  at a later time  $t$ , we can determine  $m_1$  and  $m_2$ . Classically, of course, it is impossible to deduce the mass of a particle from the trajectory in a gravitational field. No ‘deep significance’ can be attached to the above fact because this result has nothing to do with gravity. Consider a free particle in the absence of any potential. It is impossible to determine its mass from its classical trajectory. But a quantum mechanical measurement of  $\langle x^2 \rangle$  will allow the determination of the free particle mass. Since the spread in (29) is the same as the free particle spread, no special (gravity-related) significance can be attached to (29).

The form of the Schrödinger equation in a uniformly accelerated frame is the same as (27). Thus it is impossible to distinguish between gravitational field and an accelerated frame by the above procedure. The principle of equivalence, stated in this form, continues to be valid in quantum mechanics.

Because of the above features, it is doubtful whether any special significance can be attached to the  $m$  dependence of various uncertainty products either.

### 3. Measurement of gravitational acceleration

We shall now study certain intrinsic limitations that arise in the measurement of gravitational field parameters due to quantum mechanics.

Let  $S$  be a laboratory located at a distance  $R$  from the centre of a star of mass  $M$ . For example, a laboratory on the surface of the Earth can be taken for  $S$ . Let  $\bar{S}$  be a similar laboratory falling freely in the gravitational field in the same vicinity. The linear extent of the laboratories (say,  $L$ ) is assumed to satisfy the condition

$$\varepsilon \equiv \mathcal{R}L^2 \ll 1 \quad (30)$$

where  $\mathcal{R}$  is the (maximum) value of any component of curvature tensor in the location of the laboratory. It is possible to set up inertial coordinates in  $\bar{S}$  to zeroth order in  $\varepsilon$ . The line element in  $\bar{S}$  will be that of an inertial system

$$\begin{aligned} \overline{ds}^2 &= \{1 + O(\varepsilon)\} \overline{dt}^2 - \{1 + O(\varepsilon)\} (d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) \\ &\approx d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2. \end{aligned} \quad (31)$$

The frame  $S$  is accelerating with respect to  $\bar{S}$  with an acceleration  $g$ . Over the size  $L$  that satisfies (30), we may assume  $g$  to be uniform. Making the usual transformation from inertial frame to a uniformly accelerated frame one may write down the line element in  $S$  to be

$$ds^2 = (1 + gx/c^2)c^2 dt^2 - (dx^2 + dy^2 + dz^2) + O(\varepsilon). \quad (32)$$

In other words, results of experiments in  $S$  can be obtained by transformation from the inertial frame  $\bar{S}$ . Since quantum mechanics in the inertial frame is known, quantum mechanics in  $S$  can be obtained.

We shall now use the experiment discussed previously to measure the acceleration due to gravity,  $g$ . If a particle falls a distance  $x$  in time  $t$  (in the frame  $S$ ) then  $g$  is given by

$$g = 2x/t^2. \quad (33)$$

Such a freely falling particle will satisfy the free-particle Schrödinger equation in  $\bar{S}$ :

$$i\hbar \frac{\partial \bar{\psi}}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} \quad (34)$$

and a Schrödinger equation with potential ( $mgx$ ) in  $S$ :

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + mgx\psi. \quad (35)$$

A wavepacket solution to (34) or (35) (connected by a coordinate transformation) is given in (27)–(29) with

$$\bar{x}(t) = \bar{x}(0) - \frac{1}{2}gt^2 \quad [\delta x(t)]^2 = [\delta x(0)]^2 + \frac{\hbar^2 t^2}{4m^2 [\delta x(0)]^2}. \quad (36)$$

We are now in a position to compute the accuracy  $\delta g$  in the measurement of  $g$  in (33). From (33) we have

$$\delta g > (2/t^2)\delta x. \quad (37)$$

We do not display the  $\delta t$  dependence of  $\delta g$  because it turns out to be unimportant. Now, from (36) we obtain

$$\delta x(t) \geq [\delta x(t)]_{\min} = (\hbar t/m)^{1/2}$$

so that

$$\delta g > (2/t^2)(\delta x)_{\min} = 2(\hbar/m)^{1/2}t^{-3/2}. \quad (38)$$

If one can increase  $t$  arbitrarily then  $\delta g$  can be measured with arbitrary accuracy. If the gravitational field is varying in time then the timescale variation of  $g(t)$  will set an upper bound on  $t$ . But even for the static case which we are considering quantum mechanical spread sets a bound. Notice that we must ‘keep the particle inside the laboratory’, requiring

$$[\delta x(t)]_{\min} = (\hbar/m)^{1/2}t^{1/2} < L. \quad (39)$$

Putting (38) and (39) together, we obtain

$$\delta g > [\delta g]_{\min} = \frac{2}{L^3} \left(\frac{\hbar}{m}\right)^2. \quad (40)$$

This result can be expressed in a more general form by using (30):

$$\delta g > (\delta g)_{\min} = 2(\hbar/m)^2 \mathcal{R}^{3/2}. \quad (41)$$

Local acceleration due to gravity cannot be measured with arbitrary accuracy in a spacetime with non-zero curvature. It should be clear that (41) arises essentially from (i) the principle of equivalence and (ii) the uncertainty principle.

We emphasise that the equality of inertial and gravitational masses has gone into the transformation connecting (35) and (34) in a crucial manner. Consider the action for a particle of mass  $m$  in the gravitational field in  $S$ :

$$A = \int dt \left( \frac{1}{2} m \dot{x}^2 - mgx \right). \quad (42)$$

Under the coordinate transformation

$$x = \bar{x} - \frac{1}{2} g \bar{t}^2 \quad t = \bar{t} \quad (43)$$

equation (42) becomes

$$\begin{aligned} A &= \int dt \left[ \frac{1}{2} m (\dot{\bar{x}} - g\bar{t})^2 - mg(\bar{x} - \frac{1}{2} g\bar{t}^2) \right] \\ &= \int dt \left[ \frac{1}{2} m \dot{\bar{x}}^2 - (d/dt)(mg\bar{x}t) + mg\bar{x} - mg\bar{x} + mg^2\bar{t}^2 \right] \\ &= \int dt \frac{1}{2} m \dot{\bar{x}}^2 + \int dt (d/dt) \left( \frac{1}{3} mg^2\bar{t}^3 - mg\bar{x}t \right). \end{aligned} \quad (44)$$

Omitting the total time derivative, this action is the same as that of a free particle in the inertial frame  $\bar{S}$ . A pure ( $m$ -independent) coordinate transformation in (43)

eliminates the gravity because of the equality of mass terms in the kinetic energy ( $\frac{1}{2}m\dot{x}^2$ ) and potential energy ( $mgx$ ). (In a constant electric field, for example, it is impossible to transform the laboratory and the particle to an inertial frame by a suitable coordinate choice.) From this point of view (41) is a special feature of gravity.

In the case of a star of mass  $M$  located at the origin, (41) can be written in the form

$$\left(\frac{\delta g}{g}\right)_{\text{at } R} > 2\left(\frac{GM}{c^2 R}\right)^{1/2} \left(\frac{\hbar}{mc}\right)^2 \frac{1}{R^2}. \quad (45)$$

As long as the Compton wavelength of the test particle is small compared to the Schwarzschild radius of  $M$  then  $(\delta g/g) \ll 1$ , even at the event horizon.

Measurement of gravitational acceleration can be used to determine the position and velocity of the source particle in an indirect manner. It is necessary to make sure that this determination does not violate the uncertainty principle either. It is easy to verify this fact as follows.

As the test particle falls towards the star, the star starts 'falling' towards the test particle with an acceleration

$$g' = g(m/M). \quad (46)$$

At the end of the experiment, the distance to the star  $l$  should be  $(R - \frac{1}{2}g't^2 - x)$  and its velocity  $u$  should be  $(m/M)v$  (where  $x, v$  are the position and velocity of the test particle). The uncertainties  $\delta l$  and  $\delta u$  satisfy

$$\delta l > \delta x \quad \delta u > (m/M) \delta v. \quad (47)$$

Thus

$$M \delta l \delta u = m \delta x \delta v \geq \frac{1}{2} \hbar. \quad (48)$$

This (expected) inequality implies that it is not possible to measure simultaneously the gravitational acceleration

$$g(t) = GM/(R - ut)^2 \quad (49)$$

and its time derivative

$$f(t) \equiv \dot{g}(t) = 2GMu/(R - ut)^3. \quad (50)$$

Using (48)-(50) evaluated at  $t=0$ , it is easy to show that

$$\delta f \delta g \geq (G^2 M / R^6) \hbar. \quad (51)$$

The inequality in (51) is usually much weaker than the ones we obtain by the specific measurement process itself. (Note that (51) arises from the quantum nature of the source while the usual uncertainties have to do with the quantum limitations of the measuring apparatus.) For this reason (51) is independent of the test particle parameter  $m$ . A more detailed discussion of these source-related uncertainties (especially their relevance to light cone fluctuations) is presented elsewhere (Padmanabhan *et al* 1985).

#### 4. Conclusions

It is clear from the above discussions that the merging of gravity and quantum mechanics provides one with unexpected dividends. The analysis has to be improved taking into account the geometric features of gravity. We hope to present a more detailed analysis in the future.

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**Appendix**

In the text, we considered the spread of the wavepacket under the influence of a constant gravitational field. Here we summarise a general result regarding quadratic potentials which is of interest to the discussion in the text. Consider the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x, t)\psi \tag{A1}$$

in a quadratic potential of the form

$$V(x, t) = \alpha(t)x^2 + \beta(t)x + \gamma(t). \tag{A2}$$

We solve (A1) by the ansatz

$$\psi(x, t) = A(t) \exp[-B(t)(x - f(t))^2]. \tag{A3}$$

Substituting (A3) and (A2) into (A1) and equating coefficients of  $(x^2, x^1, x^0)$  we obtain the following equations:

$$i\hbar \frac{dB}{dt} = \frac{2\hbar^2}{m} B - \alpha(t) \tag{A4}$$

$$i\hbar \frac{df}{dt} = \frac{\alpha(t)}{B} f + \frac{\beta(t)}{2B} \tag{A5}$$

$$i\hbar \frac{1}{A} \frac{dA}{dt} = \frac{\hbar^2 B}{m} + \alpha f^2 + \beta f + \gamma. \tag{A6}$$

In principle these equations can be reduced to quadratic for general  $(\alpha, \beta, \gamma)$ . However, the expressions are very unwieldy. The physics can be extracted more easily by substituting

$$B = -\frac{i}{2} \frac{m \dot{\sigma}(t)}{\hbar \sigma(t)} \tag{A7}$$

in (A4), leading to

$$m\ddot{\sigma} + 2\alpha(t)\sigma = 0. \tag{A8}$$

It is also easy to verify that

$$(d/dt)[|\sigma|^2(B + B^*)] = 0 \tag{A9}$$

so that

$$2 \operatorname{Re} B(t) = \frac{|\sigma(0)|^2 2 \operatorname{Re} B(0)}{|\sigma(t)|^2} = \frac{1}{|\sigma(t)|^2}. \tag{A10}$$

The last equality follows from redefining  $\sigma(t)$  by absorbing a constant, purely for convenience. Now, taking the modulus of (A3) we obtain

$$|\psi(x, t)|^2 = N(t) \exp\left(-\frac{1}{2|\sigma(t)|^2} (x - \bar{x}(t))^2\right) \quad (\text{A11})$$

where

$$\bar{x}(t) = (Bf + B^*f^*) / (B + B^*) \quad (\text{A12})$$

and

$$N(t) = |A(t)|^2 \exp\left(- (Bf^2 + B^*f^{*2}) + \frac{Bf + B^*f^*}{B + B^*}\right). \quad (\text{A13})$$

Straightforward but tedious calculation using (A4)–(A6), (A12) and (A13) will now show that

$$N(t) = \left(\frac{1}{2\pi\sigma^2(t)}\right)^{1/2} \quad (\text{A14})$$

as it should, and that  $\bar{x}(t)$  satisfies the classical equation

$$m \, d^2\bar{x}/dt^2 = -2\alpha(t)\bar{x} - \beta(t). \quad (\text{A15})$$

Equations (A8), (A14) and (A15) completely describe the Gaussian wavepacket solution for any quadratic potential.

In the text we were concerned with the spread  $\sigma^2$  for a constant gravitational field, which corresponds to the choice ( $\alpha = 0$ ,  $\beta = mg$ ,  $\gamma = 0$ ). We see from (A8) that

$$\ddot{\sigma} = 0$$

which is the same situation as the free particle ( $\alpha = \beta = \gamma = 0$ ). In general, the linear term  $\beta(t)x$  does not affect the spread  $|\sigma|^2$  because (A8) is independent of  $\beta(t)$ . Thus the spread of the wavepacket in a constant gravitational field is the same as that for the free particle, justifying (11), (22) and (29) of the text.

## References

- Greenberger D 1968 *Ann. Phys.*, NY **47** 116  
 ——— 1983 *Rev. Mod. Phys.* **55** 875  
 Greenberger D and Overhauser A W 1979 *Rev. Mod. Phys.* **51** 43  
 Kuchar K 1980 *Phys. Rev. D* **22** 1285  
 Padmanabhan T 1982 *Astrophys. Space Sci.* **83** 247  
 Padmanabhan T, Seshadri T R and Singh T P 1986 *J. Mod. Phys.* in press  
 Pauli W (ed) 1955 *Niels Bohr and The Development of Physics* (New York: McGraw-Hill)  
 Unruh W G 1984 *Quantum Theory of Gravity* ed S M Christensen (Bristol: Adam Hilger) p 234