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Path Integral for the Relativistic Particle and Harmonic Oscillators

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The action for a massive particle in special relativity can be expressed as the invariant proper length between the end points. In principle, one should be able to construct the quantum theory for such a system by the path integral approach using this action. On the other hand, it is well known that the dynamics of a free, relativistic, spinless massive particle is best described by a scalar field which is equivalent to an infinite number of harmonic oscillators. We clarify the connection between these two—apparently dissimilar—approaches by obtaining the Green function for the system of oscillators from that of the relativistic particle. This is achieved through defining the path integral for a relativistic particle rigorously by two separate approaches. This analysis also shows a connection between square root Lagrangians and the system of harmonic oscillators which is likely to be of value in more general context.

1. INTRODUCTION

The action principle provides the most elegant approach to the study of the dynamics of a point particle in classical mechanics. In this approach, we postulate that the particle propagates from the event $\mathcal{P}_1(t_1, \mathbf{x}_1)$ to the event $\mathcal{P}_2(t_2, \mathbf{x}_2)$ via the path $\mathbf{x}(t)$, which extremizes the action functional:

$$A[\mathbf{x}(t)] = \int_{t_1}^{t_2} dt \, (\tfrac{1}{2} m \dot{\mathbf{x}}^2 - V(\mathbf{x}, t)) \equiv \int_{t_1}^{t_2} dt \, L \tag{1}$$

In the quantum theory of such a nonrelativistic particle, the action assumes far greater significance. Nonrelativistic quantum mechanics of point particles

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can be constructed by postulating that the probability amplitude for the particle to propagate from $\mathscr{P}_1$ to $\mathscr{P}_2$ is given by the "sum over paths":

$$K(t_2, \mathbf{x}_2; t_1, \mathbf{x}_1) = \sum_{\text{paths}} \exp(iA/\hbar) \tag{2}$$

For the action which is of the form in (1), it is possible to define the sum over paths rigorously and thus construct the kernel K; it can also be shown that this approach to quantum mechanics is identical to that based on, say, the Schrödinger equation (see, e.g., Feynman and Hibbs[1]).

All known classical laws of physics can be derived from suitably defined action principles. It, therefore, appears attractive to construct the quantum theory corresponding to a given classical theory by the "sum over path" prescription. One may, therefore, feel that the quantum theory of a free *relativistic* particle could be constructed from the sum of paths prescription (2) which uses the action for a *relativistic* free particle instead of (1). Such a prescription should directly give the amplitude for a free, relativistic particle to propagate from an event $\mathscr{P}_1$ to event $\mathscr{P}_2$, and will combine the principles of special relativity and quantum mechanics in a simple manner.

It is, however, often stated that when the principles of quantum theory are combined with those of special relativity, a fundamental change in the description of physical systems is necessitated. Relativistic quantum theory runs into inconsistencies when interpreted as a quantum version of the theory for a single relativistic particle. A much richer structure (and good agreement with experiment) is obtained by treating the relativistic quantum theory as a quantum theory of classical fields having an infinite number of degrees of freedom. In such a decription, the quantum theory for the relativistic free particle is mapped to a theory describing an infinite number of harmonic oscillators. The quantum states of this system describe states containing a different number of particles.

This fact raises an interesting question of principle. To begin with, notice that it is indeed possible to calculate the amplitude for a particle to propagate from $\mathscr{P}_1$ to $\mathscr{P}_2$ using standard prescriptions of quantum field theory. It is important to ascertain whether this amplitude is the same as the one which could be obtained from the path integral by using the relativistic action. If this is indeed the case, then there must exist an interesting relation between the path integral based on relativistic action and the Green functions describing an infinite number of harmonic oscillators. Establishing the equivalence between these two will allow us to understand clearly why it is possible to describe a free relativistic particle in terms of a infinite number of oscillators—or, equivalently, by a field.

The major difficulty in performing the above task lies in calculating the path integral for the relativistic particle, using the *nonquadratic* action

$$A_R = -m_0 \int ds = -m_0 \int \sqrt{\dot{x}^i \dot{x}_i}\, ds \tag{3}$$

where $x^i(s)$ is the world line of the relativistic particle. In particular, how should one choose the measure for the path integral in this context? Should we include all possible paths connecting x_i and y_i? Or should we confine our attention, say, to only those paths which are inside the future light cone of the initial point? And, lastly, how does one show that the above description is identical to that in terms of an infinite number of harmonic oscillators which constitute a field? It is interesting to spell out this relationship in order to convince oneself that the field and the particle descriptions have a fundamental connection with each other. This connection is far from obvious at the superficial level of description of the relativistic point particle. As far as the author knows, standard textbooks in quantum field theory do not describe the evaluation of the path integral for a relativistic particle or its connection with harmonic oscillators. (In recent years, these issues have drawn attention in the context of quantum cosmology. There is an extensive literature on the subject of path integrals applied to quantum cosmology. The question of choice of paths and the analogy with a relativistic particle are often discussed in this relation. See, e.g., Teitelbaum,[4,5] Hartle,[2] and references therein.)

We shall address these questions in this paper. We shall first show that it is possible to interpret the path integral with the square-root Lagrangian in terms of the Jacobi action in classical mechanics. This fact can be exploited to relate the relativistic path integral to another quadratic path integral which can be explicitly evaluated. This procedure also allows us to define a natural choice for the class of paths which should be included in the path integral with square-root action. (It turns out that one should include *all* paths in the Euclidean section.) Having done this, we shall show that there is a mathematical equivalence between: (a) the amplitude for a free particle to propagate from $\mathbf{x}_1$ to $\mathbf{x}_2$ *with a fixed energy* E in N dimensions and (b) the Green function for a system with an infinite number of harmonic oscillations labeled by an $(N-1)$ dimensional vector $\mathbf{k}$. This equivalence exists provided the dispersion relation for the oscillators is quadratic, $w_k^2 = k^2 + m^2$. One can then map the relativistic particle of mass m moving in 4 dimensions to an infinite number of harmonic oscillators each labeled by a 3-dimensional vector $\mathbf{k}$. Using this mapping, it is fairly easy to actually construct Lorentz invariant fields which are equivalent to relativistic particles at appropriate limits. This connection is clarified in the sequel.

The paper is organized as follows. In the next two sections we discuss the path integral prescription applied to the Jacobi form of action and prove the formal equivalence stated in the last paragraph. To do this, we shall use a fairly naive (but physically clear) definition of the path integral in Section 2. Section 3 is devoted to demonstrating rigorously the measure for the path integral used in Section 2. The conclusions are summarized in Section 4.

2. PATH INTEGRALS WITH JACOBI ACTION

The basic difference between the actions for a free relativistic particle and a free nonrelativistic particle is the following: The action for the nonrelativistic particle is quadratic in velocities while that of the relativistic particle has a square root in it. In order to understand the procedure to be followed in the case of a Lagrangian with a square root, it is probably best to recast the action for the nonrelativistic particle in a form similar to that of a relativistic particle. Since the final result for the nonrelativistic particle is known, this approach allows us to *define* the nonquadratic path integral in a simple way. (We will obtain the same definition from first principles in the next section.) More explicitly, we shall write the path integral for a nonrelativistic particle in two equivalent ways, one involving a square root in the action and the other having only a quadratic action. Knowing the latter, we can interpret—and evaluate—the former. This can be done as follows:

Let us begin by observing that the action in classical mechanics can also be written in the form

$$A[\mathbf{x}(t), \mathbf{p}(t)] = \int_0^T dt[\mathbf{p}\cdot\dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{x})]$$

$$= \int_0^T dt\left[\mathbf{p}\cdot\dot{\mathbf{x}} - \frac{\mathbf{p}^2}{2m} - V(\mathbf{x})\right] \tag{4}$$

To facilitate future use, we consider a theory in an N-dimensional *space*; the position $\mathbf{x}$ and momentum $\mathbf{p}$ are N-dimensional vectors. If we vary $\mathbf{p}$ and $\mathbf{x}$ independently in the above expression, we get

$$\delta A = \int_0^T dt\left[\left(\dot{\mathbf{x}} - \frac{\mathbf{p}}{m}\right)\cdot\delta\mathbf{p} - (\dot{\mathbf{p}} + \nabla V)\cdot\delta\mathbf{x}\right] + \mathbf{p}\cdot\delta\mathbf{x}\Big|_0^T \tag{5}$$

We confine our attention to variations for which $\delta \mathbf{x} = 0$ at $t = 0, T$ but $\delta \mathbf{p}$ is entirely arbitrary. This gives

$$\mathbf{p} = m\dot{\mathbf{x}}; \qquad \dot{\mathbf{p}} = -\nabla V \tag{6}$$

which are the Hamilton equations. We can now attempt to write the path integral using (4) in the form

$$K(\mathbf{x}_2, \mathbf{x}_1; T) = \sum_{\text{all } \mathbf{x}} \sum_{\text{all } \mathbf{p}} \exp \frac{i}{\hbar} \int_0^T dt(\mathbf{p} \cdot \dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{x})) \tag{7}$$

where the sum is over *all* functions $\mathbf{p}(t)$ but only those $\mathbf{x}(t)$ which have the correct boundary values. The sum over all $\mathbf{p}$ can be performed by slicing the time axis into small intervals, in a standard manner. This will lead to an expression for the kernel as the sum over all $\mathbf{x}$ of some amplitude; this amplitude turns out to be the usual one with the Lagrangian form of the action. Hence, $K(\mathbf{x}_2, \mathbf{x}_1, T)$ is the standard kernel.

We will now proceed to calculate (7) in a different way. Instead of summing over $\mathbf{x}$ and $\mathbf{p}$, we will: (i) first sum over all $\mathbf{x}(t)$ (satisfying the boundary conditions) for a *given value* of $H(\mathbf{p}, \mathbf{x}) \equiv E$ and then (ii) sum over all values of E. (In the original integral, we sum over all $\mathbf{p}(t)$ with a given $\mathbf{x}(t)$ and then sum over all $\mathbf{x}(t)$. We can, equivalently, sum over all $E = (p^2/2m) + V(\mathbf{x})$ at a fixed $\mathbf{x}(t)$ followed by a sum over all $\mathbf{x}(t)$, since—at fixed $\mathbf{x}(t)$—the energy is a given function of $\mathbf{p}$.) Assuming that the order of summation over $\mathbf{x}(t)$ and E can be interchanged, and absorbing various Jacobian factors into the definition of measure, we can replace (7) by

$$\begin{aligned} K(\mathbf{x}_2, \mathbf{x}_1; T) &= \sum_{\text{all } E} \sum_{\text{all } \mathbf{x}} \exp \frac{i}{\hbar} \left[\int_0^T dt(\mathbf{p} \cdot \dot{\mathbf{x}} - E) \right] \\ &= \sum_{\text{all } E} \sum_{\text{all } \mathbf{x}} \exp \frac{i}{\hbar} \left[\int_0^T dt \{2m(E - V(\mathbf{x}))\}^{1/2} |\dot{\mathbf{x}}| - ET \right] \\ &= \sum_{\text{all } E} \exp\left(-\frac{iET}{\hbar} \right) \left[\sum_{\text{all } \mathbf{x}} \exp \frac{i}{\hbar} \int_0^T dt \sqrt{2m(E - V(\mathbf{x}))\, \dot{\mathbf{x}}^2} \right] \end{aligned} \tag{8}$$

(We have effectively eliminated $\mathbf{p}$ in favor of E; a more formal, rigorous way of doing this manipulation is demonstrated in Appendix 1.) The "sum over all E" is just an integration over the parameter E with some weightage factor $g(E)$ to take into account the density of states. For any Hamiltonian

which is bounded from below we can normalize the Hamiltonian so that $E \geqslant 0$. Assuming this is done for convenience, we can write

$$K(\mathbf{x}_2, \mathbf{x}_1; T) = \int_0^\infty dE\, g(E)\, e^{-(i/\hbar)ET} G(\mathbf{x}_2, \mathbf{x}_1; E)$$

$$\equiv \int_0^\infty dE\, e^{-(i/\hbar)ET} B(\mathbf{x}_2, \mathbf{x}_1; E) \tag{9}$$

where

$$G(\mathbf{x}_2, \mathbf{x}_1; E) = \sum_{\text{all x}} \exp \frac{i}{\hbar} \int_0^T dt \sqrt{2m(E - V\mathbf{x})}\, \dot{\mathbf{x}}^2 \tag{10}$$

and $B = gG$. Relabeling t by $s = (t/T)$, we get

$$G(\mathbf{x}_2, \mathbf{x}_1; E) = \sum_{\text{all x}} \exp \frac{i}{\hbar} \int_0^1 ds \sqrt{2m(E - V) \left| \frac{d\mathbf{x}}{ds} \right|^2} \tag{11}$$

In this form, the paths are parametrized so that $\mathbf{x}(0) = \mathbf{x}_1$ and $\mathbf{x}(1) = \mathbf{x}_2$. By inverting the Fourier transform (9), we can write

$$B = g(E)\, G(\mathbf{x}_2, \mathbf{x}_1; E) = \int_0^\infty dT\, e^{(i/\hbar)ET} K(\mathbf{x}_2, \mathbf{x}_1; T) \tag{12}$$

(Strictly speaking, the Fourier integrals go over the range $(-\infty, +\infty)$. However, note that the kernel $K(T)$ is usually chosen to vanish for $T < 0$, and $B(E)$ vanishes for $E < 0$. We will not bother to indicate this fact explicitly everytime.) This is a result worthy of some discussion.

Since $K(\mathbf{x}_2, \mathbf{x}_1; T)$ gives the amplitude for propagation from $\mathbf{x}_1$ to $\mathbf{x}_2$ in a time interval T, it is natural to interpret B as the *amplitude for propagation between these two points with a fixed energy E*. This interpretation is also borne out by the following fact: The argument of the exponent in (11) is what is called the Jacobi action. A variational principle based on this action will lead to paths of particles with definite energy (see, e.g., Landau and Lifshitz[3]). Thus, both sides of this equation describe the probability amplitude for a particle to propagate from $\mathbf{x}_1$ to $\mathbf{x}_2$ with a fixed energy E.

What is more, we have now related a path integral with a square-root action [viz. $G(\mathbf{x}_2, \mathbf{x}_1; E)$] to the Fourier transform of the standard propagator $K(\mathbf{x}_2, \mathbf{x}_1; T)$. The latter quantity can be evaluated by standard procedures, thereby allowing us to compute G. More formally, we have provided an *interpretation* for the "sum over paths" appearing in (11)! Since the Jacobi action has a square root in it, the standard definition of

"sum over paths" (based on time slicing) will not work in this case. We can now take (12) as *defining* (11). That is, we take

$$\sum_{\text{all x}} g(E) \exp \frac{i}{\hbar} \int_0^1 ds \sqrt{2m(E-V) \left|\frac{d\mathbf{x}}{ds}\right|^2}$$

$$= \sum_{\text{all x}} \exp \frac{i}{\hbar} \int_0^1 ds \sqrt{2m(E-V)\,|\dot{\mathbf{x}}|^2}$$

$$\equiv \int_0^\infty dT\, e^{(i/\hbar)ET} \int \mathscr{D}x(t) \exp \frac{i}{\hbar} \int_0^T dt \left[\frac{1}{2} m\dot{\mathbf{x}}^2 - V\right] \tag{13}$$

In arriving at the first equality we have redefined the measure in "sum over paths" by absorbing the $g(E)$ factor.

This result can be easily generalized to the Euclidean sector, thereby avoiding oscillatory exponents. Elementary algebra, similar to the above, will show that the Euclidean analogue of (13) is

$$\sum_{\text{all x}} \exp -\frac{1}{\hbar} \int_0^1 d\eta \sqrt{2m(|E|+V)\,|\dot{\mathbf{x}}|^2}$$

$$= \int_0^\infty d\tau\, e^{-|E|\tau/\hbar} \int \mathscr{D}x \exp -\frac{1}{\hbar} \int_0^\tau \left(\frac{1}{2} m\dot{\mathbf{x}}^2 + V\right) d\tau \tag{14}$$

There is one crucial point in the above definitions which needs to be mentioned. The Jacobi action, defined as

$$A_{\text{Jacobi}} = \int_0^T dt \sqrt{2m(E-V(\mathbf{x}))\,|\dot{\mathbf{x}}|^2} \tag{15}$$

has the property of being "reparametrization invariant" under "time relabeling." This is clear from the fact that, on transforming the transformation $t \to \tau = \tau(t)$ where

$$t = \int_0^T N(\tau')\, d\tau' \tag{16}$$

the integrand does not change:

$$\int dt \sqrt{2m(E-V)\,|\dot{\mathbf{x}}|^2} = \int N\, d\tau \sqrt{2m(E-V)\,N^{-2}\,|\mathbf{x}_\tau|^2}$$

$$= \int d\tau \sqrt{2m(E-V)\left|\frac{d\mathbf{x}}{d\tau}\right|^2} \tag{17}$$

Therefore two different paths $\mathbf{x}(t)$ and $\mathbf{x}(t(\tau))=\mathbf{x}'(\tau)$ will give the same value to the action, even though the functional form of $\mathbf{x}(t)$ and $\mathbf{x}'(\tau)$ may be quite different. Obviously, we will get an infinite value for G, if we count all these paths are different. Therefore, if the sum over paths in (11) is to be well defined, we must remove this "reparametrization invariance" by choosing a definite parametrization. This is partially achieved by the rescaling $t \to \tau = t/T$ in the definition of (11). A more formal analysis which uses a "shift factor" $N(t)$ and gauge-fixing terms will lead to the same result.

We shall define this path integral more rigorously in the next section, using a discretized lattice. For the moment, let us accept the above analysis and prodeed to see what it implies for the relativistic free particle. Since we have a definite prescription for evaluating the path integral with a square root, we can explicitly compute the probability amplitude for the particle to propagate from $\mathscr{P}_1$ to $\mathscr{P}_2$. This can now be done exactly in the spirit of nonrelativistic quantum mechanics, without any reference to any quantum field. We shall perform this computation and then see what correspondence exists with respect to conventional field theory.

The calculation of the propagation amplitude is straightforward. The Euclidean free particle Kernel in N-dimension is

$$\begin{aligned} K(\mathbf{x};\tau) &= \int \mathscr{D}x \exp -\frac{1}{\hbar}\int_0^\tau \frac{1}{2} m\,|\dot{\mathbf{x}}|^2\, d\tau \\ &= \left(\frac{m}{2\pi\hbar\tau}\right)^{N/2} \exp -\frac{1}{\hbar}\frac{m\,|\mathbf{x}|^2}{\tau}, \qquad \mathbf{x}=\mathbf{x}_2-\mathbf{x}_1 \end{aligned} \tag{18}$$

Therefore, calling $|E|=\lambda$, we find

$$B(\mathbf{x};\lambda) = \int_0^\infty d\tau\, e^{-\lambda\tau/\hbar}\left(\frac{m}{2\pi\hbar\tau}\right)^{N/2} \exp -\frac{1}{\hbar}\frac{m\,|\mathbf{x}|^2}{2\tau} \tag{19}$$

Writing

$$\left(\frac{m}{2\pi\hbar\tau}\right)^{N/2} \exp -\frac{1}{\hbar}\frac{m\,|\mathbf{x}|^2}{2\tau} = \int \frac{d\mathbf{k}}{(2\pi)^N}\, e^{i\mathbf{k}\cdot\mathbf{x}-(\hbar k^2/2m)\tau} \tag{20}$$

and doing the τ-integration first, we get

$$\begin{aligned} B(\mathbf{x};\lambda) &= \int_0^\infty d\tau\, e^{-\lambda\tau/\hbar}\int \frac{d\mathbf{k}}{(2\pi)^N}\, e^{i\mathbf{k}\cdot\mathbf{x}-(\hbar k^2/2m)\tau} \\ &= \int \frac{d\mathbf{k}}{(2\pi)^N}\, e^{i\mathbf{k}\cdot\mathbf{x}}\int_0^\infty d\tau\, e^{-(\tau/\hbar)(\lambda+\hbar^2k^2/2m)} \\ &= \int \frac{d\mathbf{k}}{(2\pi)^N}\, e^{i\mathbf{k}\cdot\mathbf{x}}\left[\frac{1}{\hbar}\left(\lambda+\frac{\hbar^2k^2}{2m}\right)\right]^{-1} \end{aligned} \tag{21}$$

This can be rewritten in the form

$$B(\mathbf{x};\lambda)=\left(\frac{2m}{\hbar}\right)\int\frac{d\mathbf{k}}{(2\pi)^N}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(\mu^2+|\mathbf{k}|^2)} \tag{22}$$

where $\mu^2=(2m\lambda/\hbar^2)$. We thus see that the amplitude for a free particle to propagate from $\mathbf{x}_1$ to $\mathbf{x}_2$ in N dimensions with energy E can be expressed as an N-dimensional fourier transform of $(k^2+\mu^2)^{-1}$. This is a direct consequence of the fact that the Lagrangian is quadratic in the velocities.

In particular, consider the 4-dimensional Euclidean space corresponding to the standard $(3+1)$-dimensional spacetime. Taking $\hbar=1$, $m=(m_0/2)$, $V=0$, and $|E|=E=m_0$, Eq. (14) can be interpreted as

$$\begin{aligned}\sum_{\text{all x}}\exp -m_0\int_0^1 d\eta(\dot{x}_i\dot{x}^i)^{1/2}&\\ =\sum_{\text{all x}}\exp[-A_{R,\text{Euclidean}}]&\\ =\int_0^\infty d\tau\, e^{-m_0\tau}\int\mathscr{D}x\exp-\tfrac{1}{4}m_0\int_0^\tau\dot{x}^i\dot{x}_i\,d\eta&\end{aligned} \tag{23}$$

thereby explicitly giving meaning to the path integral with the square-root action. In this case, the propagator $B(x;m_0)$ becomes

$$B(x^i;m_0)=m_0\int\frac{d^4k}{(2\pi)^4}\frac{e^{ikx}}{(k^2+m_0^2)},\qquad x^i=x_1^i-x_2^i \tag{24}$$

This is the standard Feynman propagator for a scalar particle, except for an (unimportant) overall multiplicative constant m_0. (This can be eliminated by rescaling $\tau\to m_0\tau$ and dropping the overall constant.) *It must be emphasized that we had obtained the propagator directly by providing a meaning to the square-root path integral.* We have thus achieved our first goal: viz. to calculate the propagator for a relativistic free particle from the path integral without introducing any extra concepts (like that of quantum fields). We shall now see how the standard procedure in terms of fields leads to the same propagator. To do this, we shall show that the same expressions can be interpreted as a Green function for a system with an infinite number of oscillators.

Consider a D-dimensional abstract space, labeled by a D-dimensional vector $\mathbf{k}=(k_1,k_2,...,k_D)$. (It will turn out later that a relativistic particle in N-dimension is related to harmonic oscillators labeled by an $(N-1)$-dimensional vector $\mathbf{k}$; i.e., $N=D+1$. For the sake of easy identification

later on, we will consider the dimension to be D here and distinguish it from N.) To each point in this space, we associate a harmonic oscillator $q_{\mathbf{k}}$ with frequency $\omega_{\mathbf{k}}$. The Green function for each oscillator, of course, can be expressed as the time-ordered expectation value of coordinate operators, i.e.,

$$G_{\mathbf{k}}(t'', t') = \langle E_0 |\, T(q_{\mathbf{k}}(t'')\, q_{\mathbf{k}}(t'))\, | E_0 \rangle$$
$$= \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{e^{ivt}}{v^2 - \omega_{\mathbf{k}}^2 + i\varepsilon} = \int_{-\infty}^{\infty} \frac{dv}{2\pi} G_{\mathbf{k}}(v)\, e^{ivt} \tag{25}$$

where $|E_0\rangle$ is the ground state of the system and $t = t'' - t'$. It is convenient to write this in a somewhat more general form

$$G_{\mathbf{kp}}(t'', t') = \langle E_0 |\, T(q_{\mathbf{k}}(t'')\, q_{\mathbf{p}}(t'))\, | E_0 \rangle = G_{\mathbf{k}}(t'', t')\, \delta(\mathbf{k} - \mathbf{p}) \tag{26}$$

Fourier-transforming this expression in $\mathbf{k}$ and $-\mathbf{p}$, we get

$$G(t; \mathbf{x}) \equiv \int \frac{d\mathbf{k}}{(2\pi)^D} \int \frac{d\mathbf{p}}{(2\pi)^D} G_{\mathbf{kp}} e^{i(\mathbf{k}\cdot\mathbf{x}_1 - \mathbf{p}\cdot\mathbf{x}_2)}$$
$$= \int_{-\infty}^{+\infty} \frac{dv\, d\mathbf{k}}{(2\pi)^{D+1}} \frac{e^{-ivt + i\mathbf{k}\cdot\mathbf{x}}}{v^2 - \omega_{\mathbf{k}}^2 + i\varepsilon} \tag{27}$$

with $t = t_1 - t_2$; $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$. The Euclidean version of this expression will be

$$G_{\mathrm{Eu}}(\mathbf{x}; \tau) = \int \frac{dv\, d\mathbf{k}}{(2\pi)^{D+1}} \frac{e^{i\mathbf{k}\cdot\mathbf{x} + iv\tau}}{v^2 + \omega^2(\mathbf{k})} \tag{28}$$

This expression is valid for a set of harmonic oscillators with an arbitrary dispersion relation $\omega = \omega(\mathbf{k})$. However, this form is strikingly similar to that obtained for $B(\mathbf{x}, \lambda)$. The identification between the two propagators B and G_{Eu} becomes exact if we choose the frequencies of the harmonic oscillators to be

$$\omega^2(\mathbf{k}) = \mathbf{k}^2 + \mu^2 \tag{29}$$

and $N = D + 1$. Then

$$G_{\mathrm{Eu}}(\mathbf{x}, v) = \int \frac{dv\, d\mathbf{k}}{(2\pi)^{D+1}} \frac{e^{i\mathbf{k}\cdot\mathbf{x} + iv\tau}}{v^2 + \mathbf{k}^2 + \mu^2}$$
$$= \int \frac{d\mathbf{p}}{(2\pi)^N} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2 + \mu^2}, \qquad N = D + 1 \tag{30}$$

where the $(D+1)$-dimensional vectors $\mathbf{p}$ and $\mathbf{x}$ have the components $(\mathbf{k}, \nu)$ and $(\mathbf{x}, \tau)$. Comparing (30) and (22), we see that

$$B(\mathbf{x}; \lambda) = \frac{2m}{\hbar} G_{\text{Eu}}(\mathbf{x}) \tag{31}$$

In other words, it is possible to interpret the propagator B either in terms of the path integral for a single relativistic particle or in terms of a system with an infinite number of harmonic oscillators. To make the identification complete, we will choose $m = (m_0/2)$, $\lambda = m_0$ and set $\hbar = 1$. (This will make $\mu^2 = (2m\lambda/\hbar^2) = m_0^2$.) Then we can write

$$\begin{aligned} B(\mathbf{x}; m_0) &= \sum_{\text{all } \mathbf{x}(t)} \exp -m_0 \int_0^1 d\eta (\dot{x}_i \dot{x}^i)^{1/2} \\ &= \int_0^\infty d\tau\, e^{-m_0 \tau} K(\mathbf{x}; \tau) \\ &= \int_0^\infty d\tau\, e^{-m_0 \tau} \sum_{\mathbf{x}(t)} \exp\left[-\frac{m_0}{4} \int_0^\tau ds\, |\dot{\mathbf{x}}|^2 \right] \\ &= m_0 \int \frac{d\mathbf{k}}{(2\pi)^N} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2 + m_0^2} = m_0 G_{\text{Eu}}(\mathbf{x}) \end{aligned} \tag{32}$$

This is a remarkable result! The first line expresses the propagator for a free relativistic particle in terms of the path integral containing the action for the relativistic particle. The second line expresses the equality of this propagator to (the Laplace transform of) another path integral containing only the quadratic action. It is this crucial equality—which we obtained in (13)—that allowed us to calculate the propagator explicitly [see (22)]. Finally, the last line shows that the same propagator can be interpreted in terms of a system with an infinite number of oscillators (labeled by an $(N-1)$-dimensional vector $\mathbf{k}$), with a particular dispersion relation $\omega^2(\mathbf{k}) = k^2 + m_0^2$. Hence there is an equivalence between the propagation amplitude for a free particle with mass m_0 in N dimensions and the Green's function for an infinite number of harmonic oscillators labeled by an $(N-1)$-dimensional vector $\mathbf{k}$.

Two crucial factors have gone into this identification. First is the fact that the Lagrangian for a free particle contains velocities only up to second power. It is this feature which allowed us to obtain the form $(|\mathbf{k}|^2 + \mu^2)$ in the denominator in (22), which is crucial for the matching between (22) and (28). The second fact is the dispersion relation for the oscillators which

we were forced to assume: $\omega^2 = \mathbf{k}^2 + \mu^2$. This dispersion relation—with $(\omega^2 - k^2) = \text{const}$—carries the information about the Lorentz invariance of the relativistic particle. The only quadratic expression in ω and k which is Lorentz invariant must have this form. In other words, the system with an infinite number of oscillators is capable of maintaining the Lorentz invariance of the system. This can be seen much more explicitly by the following procedure: The original Green function for the oscillators could be expressed as the expectation value $\langle E_0 | (q_{\mathbf{k}}(t)\, q_{\mathbf{p}}(t')) | E_0 \rangle$. Fourier transforming this expression in $\mathbf{k}$ and $-\mathbf{p}$, we get, for $t_1 > t_2$,

$$
\begin{aligned}
&G(t_1 - t_2; \mathbf{x}_1 - \mathbf{x}_2) \\
&\quad = \int \frac{d\mathbf{k}}{(2\pi)^D} \int \frac{d\mathbf{p}}{(2\pi)^D}\, G_{\mathbf{kp}}(t_1 - t_2)\, e^{i\mathbf{k}\cdot\mathbf{x}_1 - i\mathbf{p}\cdot\mathbf{x}_2} \\
&\quad = \int \frac{d\mathbf{k}}{(2\pi)^D} \int \frac{d\mathbf{p}}{(2\pi)^D} \langle E_0 |\, q_{\mathbf{k}}(t_1)\, q_{\mathbf{p}}(t_2)\, | E_0 \rangle\, e^{i\mathbf{k}\cdot\mathbf{x}_1 - i\mathbf{p}\cdot\mathbf{x}_2} \\
&\quad = \int \frac{d\mathbf{k}}{(2\pi)^D} \int \frac{d\mathbf{p}}{(2\pi)^D} \langle E_0 |\, (q_k(t_1)\, e^{i\mathbf{k}\cdot\mathbf{x}_1})(q_p(t_2)\, e^{i\mathbf{p}\cdot\mathbf{x}_2})^{\dagger}\, | E_0 \rangle \qquad (33)
\end{aligned}
$$

In arriving at the last expression we have used the fact that the $q_{\mathbf{k}}(t)$'s are Hermitian; i.e., $q_k^{\dagger} = q_k$. Let us now define an operator

$$
\hat{\phi}(t, \mathbf{x}) \equiv \hat{\phi}_{\mathbf{x}}(t) \equiv \int \frac{d\mathbf{k}}{(2\pi)^D}\, \hat{q}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}
$$

which satisfies the equation

$$
\begin{aligned}
\frac{\partial^2 \phi}{\partial t^2} &= \int \frac{d\mathbf{k}}{(2\pi)^D}\, \ddot{q}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = -\int \frac{d\mathbf{k}}{(2\pi)^D}\, (\omega_{\mathbf{k}}^2 q_{\mathbf{k}})\, e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \nabla^2 \phi + m^2 \phi \qquad (35)
\end{aligned}
$$

or

$$
[\partial_i \partial^i + m^2]\, \phi = 0 \qquad (36)
$$

which is manifestly Lorentz invariant. Note that the dispersion relation $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2$ has played a crucial role in this.

Then we can write

$$
G(t_1 - t_2; \mathbf{x}_1 - \mathbf{x}_2) = \langle E_0 |\, \hat{\phi}(t_1, \mathbf{x}_1)\, \hat{\phi}^{\dagger}(t_2, \mathbf{x}_2)\, | E_0 \rangle \qquad (t_1 > t_2) \qquad (37)
$$

We can take care of the condition $t_1 > t_2$ by putting a time-ordering operator. Then we finally get

$$G(t_1 - t_2; \mathbf{x}_1 - \mathbf{x}_2) = \langle E_0 | \, T(\hat{\phi}(t_1, \mathbf{x}_1) \, \hat{\phi}^\dagger(t_2, \mathbf{x}_2)) \, | E_0 \rangle \tag{38}$$

We thus see that the propagation amplitude for a particle can be expressed in two different (but equivalent) ways. It can be interpreted directly in terms of a sum over paths prescription [Eq. (24)] or it can be thought of as the ground state expectation value of a scalar field at two locations [Eq. (38)]. The entire dynamics of the field follows directly from the Lorentz-invariant field equation. Thus, a relativistic particle can be studied using a scalar field in a mathematically equivalent fashion. This equivalence is at the cornerstone of quantum field theory since it facilitates the description in terms of a field. (Of course, the approach based on the fields is far more general and can accommodate, for example, different spins and interactions, in the most natural fashion possible.)

3. RIGOROUS EVALUATION OF THE JACOBI PATH INTEGRAL

In the discussion above, we define the path integral over the Jacobi action by connecting it up with the more conventional path integral which can be defined rigorously. In doing so, we have bypassed the need for defining the measure for the Jacobi integral from first principles. It is, however, interesting to consider the question of defining the Jacobi path integral from the first principles. This analysis will allow us to explore an alternative means of defining path integrals.

In defining the path integral in nonrelativistic quantum mechanics, we discretize the time axis, define the path integral with a nonzero spacing ε, and finally take the limit of ε going to zero. To define the path integral in 4-dimensions we will use a similar procedure. We will work in the Euclidean space and introduce a cubic lattice with spacing ε. The path integral will be defined on the lattice and then we will take the limit of $\varepsilon \to 0$. To obtain a finite value in the limit of $\varepsilon \to 0$ we have to choose the measure and the mass parameter, m to scale in a specific fashion with ε. This can be done fairly easily in this case, and the final expression will agree with the path integral obtained earlier. The calculation proceeds as follows:

We will work directly in the Euclidean sector of D-dimensions. Since we are primarily interested in the issues of principle, regarding the measure

for the path integral, it is enough for us to consider the path integral for a free particle. We have, therefore to evaluate

$$\mathcal{G}(\mathbf{x}_2, \mathbf{x}_1; E) = \sum_{\text{all } \mathbf{x}(t)} \exp -\frac{1}{\hbar}\int_0^1 ds \sqrt{2mE}\left|\left(\frac{d\mathbf{x}}{ds}\right)^2\right|^{1/2}$$

$$= \sum_{\text{all } \mathbf{x}(t)} \exp -\mu_0 \int_0^1 ds \left|\left(\frac{d\mathbf{x}}{ds}\right)^2\right|^{1/2} \tag{39}$$

where we have defined $\mu_0^2 = (2mE/\hbar^2)$ as before. We notice that the quantity

$$l(\mathbf{x}_2, \mathbf{x}_1) = \int_0^1 ds \left|\left(\frac{d\mathbf{x}}{ds}\right)^2\right|^{1/2} \tag{40}$$

is just the length of the curve $\mathbf{x}(s)$, connecting $\mathbf{x}(0) = \mathbf{x}_1$ and $\mathbf{x}(1) = \mathbf{x}_2$. We will, therefore, consider the path integral

$$\mathcal{G}_E(\mathbf{x}_2, \mathbf{x}_1; \mu_0) = \sum_{\text{all } \mathbf{x}(t)} \exp -\mu_0 l[\mathbf{x}(t)] \tag{41}$$

in the Euclidean sector, where l is the length of the path.

This quantity can be defined through the following limiting procedure: Consider a lattice of points in a D-dimensional lattice with a uniform lattice spacing of ε. We will work out $\mathcal{G}_E$ in the lattice and will then take the limit of $\varepsilon \to 0$ with suitable measure $M(\varepsilon)$. To obtain a finite result, it is also necessary to treat μ_0 (which is the only parameter in the problem) as a function of lattice spacings; i.e., $\mu = \mu(\varepsilon)$, where the functional form is to be chosen in such a way to ensure finiteness. We will reserve the symbol μ_0 with a zero subscript, for the value of this parameter in the continuum limit. Thus, we will define the path integral result as a limit:

$$\mathcal{G}(\mathbf{x}_2, \mathbf{x}_1; \mu_0) = \lim_{\varepsilon \to 0} [M(\varepsilon)\, \mathcal{G}(\mathbf{x}_2, \mathbf{x}_1; \mu(\varepsilon))] \tag{42}$$

where the functions $M(\varepsilon)$ and $\mu(\varepsilon)$ are to be chosen so as to ensure finiteness. The rationale for this expression arises from the following point of view: We treat the continuum space as a limit of a lattice with the lattice spacing ε going to zero. We now construct a sequence of path integrals parametrized by the spacing ε by choosing certain functions $\mu(\varepsilon)$ and $M(\varepsilon)$ and *define* the continuum path integral as the limit of this sequence. We shall show in a moment that this limit exists only if $\mu(\varepsilon) \simeq (\ln 2D)/\varepsilon$ and $M(\varepsilon) \simeq (2D)^{-1}\, \varepsilon^{-(D-2)}$ near $\varepsilon \simeq 0$. The form of $\mu(\varepsilon)$, $M(\varepsilon)$ for ε far away from zero, of course, makes no difference in the result we are after.

In a lattice with spacing of ε, (41) can be evaluated in a straightforward manner. Because of the translation invariance of the problem, $\mathscr{G}_E$ can only depend on $\mathbf{x}_2 - \mathbf{x}_1$; so we can set $\mathbf{x}_1 = 0$ and call $\mathbf{x}_2 = \varepsilon \mathbf{R}$, where $\mathbf{R}$ is a D-dimensional vector with integral components: $\mathbf{R} = (n_1, n_2, n_3, ..., n_D)$. Let $C(N, \mathbf{R})$ be the number of paths of length $N\varepsilon$ connecting the origin to the lattice point $\varepsilon \mathbf{R}$. (This N should not be confused with the symbol N used to describe the dimension of space earlier.) Since all the paths contribute a term $[\exp - \mu(\varepsilon)(N\varepsilon)]$ to (41), we get

$$\mathscr{G}_E(\mathbf{R}; \varepsilon) = \sum_{N=0}^{\infty} C(N; \mathbf{R}) \exp[-\mu(\varepsilon)\, N\varepsilon] \tag{43}$$

The generating function determining $C(N; \mathbf{R}) = C(N; n_1, n_2, ..., n_D)$ can be calculated by the following arguments: Consider any particular path connecting the origin to the lattice point $\mathbf{R}$. Suppose that this path take r_1 steps toward the positive direction ("right") in the first axis and l_1 steps toward the negative direction ("left") in the first axis. Then $n_1 = r_1 - l_1$; similarly $n_i = r_i - l_i$. The number of paths with a specified number of (r_i, l_i) for $i = 1, ..., D$ is just the number of ways of choosing the integers $(r_1, ..., r_D, l_1, ..., l_D)$ with $\sum r_i + \sum l_i = N$. This is given by the coefficient of the polynomial expansion

$$\begin{aligned}(x_1 + x_2 + \cdots + x_D + y_1 + y_2 + \cdots + y_D)^N \\ = \sum Q(N; r_i, l_i)\, x_1^{r_1} \cdots x_D^{r_D}\, y_1^{l_1} \cdots y_D^{l_D}\end{aligned}$$

In our problem, we allow (r_i, l_i) also to vary keeping $r_i - l_i = n_i$ fixed for each i. The number of paths with this property is clearly given by using the above expression with $y_i = (1/x_i)$. Then we get

$$\begin{aligned}\left(x_1 + x_2 + \cdots + x_D + \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_D}\right)^N \\ = \sum C(N; n_1, n_2, ..., n_D)\, x_1^{n_1} \cdots x_D^{n_D}\end{aligned}$$

The expansion of the left-hand side gives the generating function for $C(N; \mathbf{R})$. For further manipulation, it is convenient to set $x_1 = e^{ik_1}, ..., x_D = e^{ik_D}$. Then we can write

$$\begin{aligned}F^N \equiv [e^{ik_1} + e^{ik_2} + \cdots + e^{ik_D} + e^{-ik_1} + \cdots + e^{-ik_D}]^N \\ = \sum_{\mathbf{R}} C(N; \mathbf{R})\, e^{i\mathbf{k}\cdot\mathbf{R}}\end{aligned} \tag{44}$$

Therefore,

$$\sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}}\,\mathcal{G}_E(\mathbf{R};\varepsilon) = \sum_{N=0}^{\infty}\sum_{\mathbf{R}} C(N;\mathbf{R})\,e^{i\mathbf{k}\cdot\mathbf{R}}\exp[-\mu(\varepsilon)\,N\varepsilon]$$
$$= \sum_{N=0}^{\infty} e^{-\mu(\varepsilon)\varepsilon N}F^N = \sum_{N=0}^{\infty}[Fe^{-\mu(\varepsilon)\varepsilon}]^N$$
$$= [1-Fe^{-\mu(\varepsilon)\varepsilon}]^{-1} \tag{45}$$

Inverting the Fourier transform, we get

$$\mathcal{G}_E(\mathbf{R};\varepsilon) = \int \frac{d^D\mathbf{k}}{(2\pi)^D}\frac{e^{-i\mathbf{k}\cdot\mathbf{R}}}{(1-e^{-\mu(\varepsilon)\varepsilon}F)}$$
$$= \int \frac{d^D\mathbf{k}}{(2\pi)^D}\frac{e^{-i\mathbf{k}\cdot\mathbf{R}}}{(1-2e^{-\mu(\varepsilon)\varepsilon}\sum_{j=1}^{D}\cos k_j)} \tag{46}$$

Converting to the physical length scales $\mathbf{x}=\varepsilon\mathbf{R}$ and $\mathbf{p}=\varepsilon^{-1}\mathbf{k}$, we get

$$\mathcal{G}_E(\mathbf{x};\varepsilon) = \int \frac{\varepsilon^D d^D\mathbf{p}}{(2\pi)^D}\frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{(1-2e^{-\mu(\varepsilon)\varepsilon}\sum_{j=1}^{D}\cos p_j\varepsilon)} \tag{47}$$

We are now ready to take the limit of zero lattice spacing. As $\varepsilon\to 0$, the denominator of the integrand becomes

$$1-2e^{-\varepsilon\mu(\varepsilon)}\left(D-\frac{1}{2}\varepsilon^2|\mathbf{p}|^2\right) = 1-2D\,e^{-\varepsilon\mu(\varepsilon)}+\varepsilon^2\,e^{-\varepsilon\mu(\varepsilon)}|\mathbf{p}|^2$$
$$= \varepsilon^2\,e^{-\varepsilon\mu(\varepsilon)}\left[|\mathbf{p}|^2+\frac{1-2D\,e^{-\varepsilon\mu(\varepsilon)}}{\varepsilon^2\,e^{-\varepsilon\mu(\varepsilon)}}\right] \tag{48}$$

so that we will get, for small ε,

$$\mathcal{G}_E(\mathbf{x};\varepsilon) \simeq \int \frac{d^D\mathbf{p}}{(2\pi)^D}\frac{A(\varepsilon)\,e^{-i\mathbf{p}\cdot\mathbf{x}}}{|\mathbf{p}|^2+B(\varepsilon)} \tag{49}$$

where

$$A(\varepsilon) = \varepsilon^{D-2}e^{\varepsilon\mu(\varepsilon)}$$
$$B(\varepsilon) = \frac{1}{\varepsilon^2}[e^{\varepsilon\mu(\varepsilon)}-2D] \tag{50}$$

The continuum theory has to be defined in the limit of $\varepsilon \to 0$ with some measure $M(\varepsilon)$; that is, we want to obtain

$$\mathcal{G}_E(\mathbf{x}; \mu_0)|_{\text{continuum}} = \lim_{\varepsilon \to 0} \{M(\varepsilon)\, \mathcal{G}_E(\mathbf{x}; \varepsilon)\} \tag{51}$$

The choice of the measure is dictated by the requirement that the right-hand side should be finite in this limit. It is easy to see that we only need to demand

$$\lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon^2}\,(e^{\varepsilon\mu(\varepsilon)} - 2D)\right] = \mu_0^2 \tag{52}$$

and

$$\lim_{\varepsilon \to 0} [M(\varepsilon)\, \varepsilon^{D-2} e^{\varepsilon\mu(\varepsilon)}] = 1 \tag{53}$$

The first condition implies that, near $\varepsilon \approx 0$,

$$\mu(\varepsilon) \approx \frac{\ln 2D}{\varepsilon} + \frac{\mu_0^2}{2D}\,\varepsilon^2 \approx \frac{\ln 2D}{\varepsilon} \tag{54}$$

Using this in the second condition (53), we can determine the measure as

$$M(\varepsilon) = \frac{1}{2D}\,\frac{1}{\varepsilon^{D-2}} \tag{55}$$

With this choice, we get

$$\lim_{\varepsilon \to 0} \mathcal{G}_E(\mathbf{x}; \varepsilon)\, M(\varepsilon) = \int \frac{d^D\mathbf{p}}{(2\pi)^D}\,\frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{|\mathbf{p}|^2 + \mu_0^2} \tag{56}$$

which is the continuum Green function we have computed earlier. This analysis gives a rigorous meaning to the Jacobi path integral.

The measure in (55) has a simple interpretation. Since μ scales with energy E appearing in the Jacobi action as $\mu \propto E^{1/2}$, (54) shows that ε scales as $\varepsilon \propto E^{-1/2}$; combining this with (55), we get $M(\varepsilon) \propto \varepsilon^{-(D-2)} \propto E^{D/2-1}$. In the continuum limit M plays the role of the density of states $g(E)$. For a D-dimensional space, the number of states with momentum between $|\mathbf{p}|$ and $|\mathbf{p}| + d|\mathbf{p}|$ is proportional to $p^{D-1}\,dp$. Since $E \propto p^2$, we have $p^{D-1}\,dp \propto E^{D/2-1}\,dE$; in other words, $g(E) \propto E^{D/2-1}\,dE$, which is precisely the scaling of M.

4. CONCLUSIONS

Path integrals provide a very suggestive and appealing route to quantum theory. However, this approach is plagued with inherent ambiguities because of (i) the wide latitude available to us in choosing the class of paths over which summation has to be performed, (ii) the measure chosen over such a class, and (iii) the limiting procedure used to give meaning to oscillating exponentials. In this paper, we took specific choices to handle all the three ambiguities. These choices have been strongly motivated by first studying the well-known nonrelativistic path integral from the point of view of Jacobian. It is probably worthwhile to emphasize how the motivation from Jacobi action has helped us in adopting choices in a natural fashion.

To begin with, consider the issue of the class of paths over which the path integral for square-root action has to be performed. The question which is usually debated in the literature is whether we should include spacelike paths or confine ourselves to those paths which are only in the forward light cone of the initial event or to some other set of paths. It is indeed possible to define the relativistic path integral with any such choice of paths and obtain *a* propagator. If, for example, we have confined ourselves to paths which are within the forward light cone, then the propagator which is obtained will vanish outside the light cone. If we have no principle to guide us other than the action for the relativistic particle, it is quite difficult to decide whether such a restriction should be imposed or not. Here is where the path integral for the nonrelativistic particle *reexpressed in terms of the Jacobi action* provides a helpful guidance. We know that to obtain the usual kernel for the nonrelativistic quantum mechanics, we must use the sum over *all* paths $\mathbf{x}(t)$ in the Jacobi action. *Some of these paths will clearly have* $E < V$ *and hence will be classically forbidden.* They correspond to the tunnelling amplitude across a potential for a particle of energy E. This suggests that we must also sum over all the paths $x^i(s)$ in the case of a relativistic particle even if the paths may have spacelike behavior in parts. The spacelike trajectories can now be formally interpreted as analogous to tunnelling trajectories in nonrelativistic quantum mechanics.

In order to give meaning to oscillating exponentials, we have always worked with the Euclidean versions of the theory or have used an $i\varepsilon$-prescription which is equivalent to the Euclidean theory. This, of course, is standard and leads naturally to the Feynman propagator rather than any other propagator. (see, e.g., Ref. 6)

Working with the Euclidean sector also allows us to define the measure in a relatively straightforward manner. The choice of measure described rigorously in Section 3 is a perfectly natural generalization of the standard measure used in nonrelativistic path integral evaluated by time

slicing. In the spirit of relativity we now use slicing in both space and time to define path integral on a lattice. Since the sum over a class of paths is inherently ill defined, any prescription which leads to a finite result is, of course, acceptable.

APPENDIX

In the paper, we transformed the path integral over $\mathbf{p}(t)$ and $\mathbf{x}(t)$ to a path integral over E and $\mathbf{x}(t)$. This manipulation is justified in a somewhat more formal manner in this appendix. To do this we will need the following three results from functional integration:

$$\delta(f(t)) = \sum_{\lambda(t)} \exp i \int dt\, \lambda(t) f(t) \tag{1}$$

$$\sum_{\mathbf{p}} \exp i \int dt [\mathbf{p} \cdot \dot{\mathbf{x}} + a(t)\, p^2] = \exp i \int dt \frac{\dot{x}^2}{4a(t)} \tag{2}$$

$$\sum_{\lambda(t)} \exp i \int dt \left(\lambda(t)\, a(t) + \frac{b(t)}{\lambda(t)} \right) = \exp i \int dt [-4ab]^{1/2} \tag{3}$$

The first result is merely the definition of delta functional; the second and third can be obtained in the Euclidean Sector by standard time slicing techniques and can be analytically continued. They are direct generalizations of the corresponding results of ordinary integrals. The proofs of these identities can be found in any standard textbook on functional integration.

Let us now consider the path integral in the Hamiltonian form:

$$K(\mathbf{x}, t) = \sum_{\mathbf{x}(t)} \sum_{\mathbf{p}(t)} \exp \frac{i}{\hbar} \int dt (\mathbf{p} \cdot \dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{x})) \tag{4}$$

with $H \geqslant 0$. We will introduce inside this expression the factor unity in the form

$$1 = \int_0^\infty dE\, \delta(E - H(\mathbf{p}, \mathbf{x})) \tag{5}$$

thereby obtaining

$$\begin{aligned} K(\mathbf{x}, t) &= \int_0^\infty dE \sum_{\mathbf{x}} \sum_{\mathbf{p}} \delta(E - H(\mathbf{p}, \mathbf{x})) \exp \frac{i}{\hbar} \int dt (\mathbf{p} \cdot \dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{x})) \\ &= \int_0^\infty dE \sum_{\mathbf{x}} \sum_{\mathbf{p}} \delta(E - H)\, e^{-iEt} \exp i \int dt (\mathbf{p} \cdot \dot{\mathbf{x}}) \end{aligned} \tag{6}$$

So

$$\int_0^\infty K(\mathbf{x}, t)\, e^{iEt}\, dt \equiv B(\mathbf{x}, E) = \sum_{\mathbf{x}} \sum_{\mathbf{p}} \delta\left(\frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) - E\right) \times \exp i \int dt\, \mathbf{p} \cdot \dot{\mathbf{x}} \tag{7}$$

We now express the delta functional using (1):

$$\delta\left(\frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) - E\right) = \sum_{\lambda(t)} \exp i \int \lambda(t) \left[\frac{p^2}{2m} + V(\mathbf{x}) - E\right] dt \tag{8}$$

Then

$$\begin{aligned} B(\mathbf{x}, E) &= \sum_{\mathbf{x}} \sum_{\lambda(t)} \exp i \int dt\, \lambda(t)[V(\mathbf{x}) - E] \sum_{\mathbf{p}} \exp i \int dt \left[\mathbf{p} \cdot \dot{\mathbf{x}} + \frac{\lambda(t)}{2m} p^2\right] \\ &= \sum_{\mathbf{x}} \sum_{\lambda(t)} \exp i \int dt\, \lambda(t)[V(\mathbf{x}) - E] \exp i \int dt\, \frac{1}{2} \frac{m}{\lambda(t)} \dot{\mathbf{x}}^2 \\ &= \sum_{\mathbf{x}} \sum_{\lambda(t)} \exp i \int dt \left[\lambda(t)(V(\mathbf{x}) - E) + \frac{m}{2\lambda(t)} \dot{\mathbf{x}}^2\right] \\ &= \sum_{\mathbf{x}} \exp i \int dt \sqrt{2m(E - V)}\, |\dot{\mathbf{x}}|^2 \end{aligned} \tag{9}$$

In arriving at the second equality, we have used (2), and in arriving at the last equality we have used (3). This proves the result obtained in the text.

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