

Uncertainty Principle and the Horizon Size of Our Universe¹

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Quantum uncertainties prevent simultaneous measurement of the expansion factor $S(t)$ and its time derivative $\dot{S}(t)$. Consequently the "Hubble size" $(\dot{S}/S)^{-1}$ has an inherent uncertainty in the quantum state that describes the semiclassical evolution of the universe. We show that the quantum uncertainty in the Hubble size of the universe is amplified to unacceptably large values in any inflationary process.

1. INTRODUCTION

Consider a closed FRW universe described by the "expansion factor" $S(t)$ and populated by a scalar field $\phi(t)$. Classical physics allows one to measure $S(t)$, $\dot{S}(t)$, $\phi(t)$, and $\dot{\phi}(t)$ simultaneously to arbitrary accuracy. In a full quantum cosmological model, of course, there will be commutation relation between "coordinates" $S(t)$, $\phi(t)$ and their conjugate momenta. Since $S(t)$ and $\dot{S}(t)$ cannot be measured simultaneously, variables like the Hubble ("horizon") size

$$l_H(t) \equiv (\dot{S}/S)^{-1} \quad (1)$$

will have an inherent uncertainty (Δl_H) in many quantum states describing the universe. This raises the questions: How can one compute these uncertainties in some suitable approximation? How significant are they?

We show in this essay that: (i) These uncertainties can be computed in a systematic manner in the semiclassical limit, (ii) they can be very large in

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inflationary models. Inflation amplifies the small scale uncertainties by a large factor and leads to

$$(\Delta l_H/l_H) \gtrsim 3 \times 10^{60} \quad (2)$$

even in the post-inflationary epochs. As far as we can see no fine tuning or remodeling will help in circumventing this result. The analysis presented here unleashes yet another serious difficulty for inflationary models.

We describe the approach in Section 2 and derive the main conclusions in Section 3. Here we also compare the results for inflationary and noninflationary models of the universe to make sure that the large uncertainties are, indeed, due to inflation. Section 4 summarizes the results.

2. UNCERTAINTIES IN THE SEMICLASSICAL UNIVERSE

The minisuperspace of a FRW universe containing a scalar field $\phi(t)$ can be described by the action

$$\begin{aligned} A &= (1/16\pi G) \int R(-g)^{1/2} d^4x + \int [\frac{1}{2}\dot{\phi}^i\phi_i - V(\phi)](-g)^{1/2} d^4x \\ &= (3\pi/4G) \int dt \{kS - \dot{S}^2 + (8\pi G/3) S^3 [\frac{1}{2}\dot{\phi}^2 - V(\phi)]\} \\ &\equiv \int L dt \end{aligned} \quad (3)$$

where $V(\phi)$ is the potential for the scalar field. (The spatial volume is normalized to the volume of the $k = +1$ case.) The dynamics of $S(t)$, $\phi(t)$ is described by the equations

$$2(\dot{S}/S) + [(\dot{S}^2 + k)/S^2] = -8\pi G[\frac{1}{2}\dot{\phi}^2 - V(\phi)] \quad (4)$$

$$(l/S^3)d/dt(\dot{\phi}S^3) = -\partial V/\partial\phi \quad (5)$$

while the invariance under time relabeling leads to the constraint equation

$$(\dot{S}^2 + k)/S^2 = (8\pi G/3)[\frac{1}{2}\dot{\phi}^2 + V(\phi)] \quad (6)$$

The momenta conjugate to (S, ϕ) are (p_S, p_ϕ) where

$$p_S = (\partial L/\partial\dot{S}) = -(3\pi/2G) S\dot{S} \quad (7)$$

and

$$p_\phi = (\partial L/\partial\dot{\phi}) = 2\pi^2 S^3 \dot{\phi} \quad (8)$$

Classically, (S, p_S) and (ϕ, p_ϕ) can all be measured simultaneously to arbitrary accuracy. Quantum mechanics, however, does not permit this luxury. From the study of quantum fields [1] in curved space-time, we know that

$$\Delta p_\phi \Delta\phi \gtrsim \hbar \quad (9)$$

We similarly expect an uncertainty principle between ΔS and Δp_S . In the naivest possible approximation, one may treat S and p_S to be c numbers and the source of gravity to be the expectation value $\langle T_{ik} \rangle$. In this limit there is no uncertainty in S, p_S . We therefore have to proceed one level higher than this approximation. This can be done as follows:

We shall assume that—in the semi-classical limit that we are considering—the wave function for the universe is sharply peaked around the classical solution. This classical solution $S = S(t)$, $\phi = \phi(t)$ is determined by (4)–(6). In such a semi-classical state $|\psi\rangle$, the uncertainties are connected by simple differential relations like

$$\Delta S = \left| \frac{\partial S}{\partial\phi} \right| \Delta\phi + \left| \frac{\partial S}{\partial p_\phi} \right| \Delta p_\phi \quad (10)$$

etc., where $S(\phi, p_\phi)$ is a classical solution [2]. (The necessity and consistency of this level of approximation was illustrated in blackhole spacetimes earlier [2] and will not be repeated here.) Eliminating t between the solutions $\phi = \phi(t)$, $S = S(t)$ we can obtain the relations $S = S(\phi)$ and $\dot{S} = S'(\phi)\dot{\phi}$. Or, equivalently

$$S = S(\phi), \quad p_S = -(3/4\pi G)[S'(\phi)/S^2] p_\phi \quad (11)$$

Straightforward calculation will now show that

$$\Delta S \Delta p_S = (3/4\pi G)(\dot{S}/S\dot{\phi})^2 \Delta\phi \Delta p_\phi + O\{[\Delta\phi]^2\} \quad (12)$$

We neglect the last term in (12), which is of order $(\Delta\phi)^2$. (Our results will not change even if this term is retained.)

Thus, in the semiclassical limit, the constraint

$$\Delta\phi \Delta p_\phi \gtrsim \hbar \quad (13)$$

implies

$$\Delta S \Delta p_S \gtrsim (3\hbar/4\pi G)(\dot{S}/\dot{\phi}S)^2 \quad (14)$$

It is clear that (14) represents a purely quantum mechanical constraint on the simultaneous measurement of (S, p_S) given a corresponding constraint on (ϕ, p_ϕ) .

The above result implies that S and p_S , or equivalently S and \dot{S} , cannot have definite values simultaneously. Consider now variables involving both S and \dot{S} like the "Hubble distance"

$$l_H = H^{-1} = (\dot{S}/S)^{-1} = -(3\pi/2G)(S^2/p_S) \quad (15)$$

It is not difficult to compute the uncertainty Δl_H in l_H due to the constraint (14): We can show that, in a quantum state that is peaked around the classical evolution

$$\left(\frac{\Delta l_H}{l_H}\right)^2 \geq \frac{4\hbar |\dot{S}|}{\pi^2 \dot{\phi}^2 S^4} \quad (16)$$

Clearly, the actual value of this expression is going to depend on the particular cosmological model chosen. We now study (16) in two specific cosmological models.

(It should be noted that we are treating S and ϕ as "coordinates" and p_S, p_ϕ as "momenta." This choice is crucial and is to be treated as an issue of principle. Because of this choice, quantities like (\dot{S}/S) cannot be measured with arbitrary accuracy in our formalism. Instead, suppose we use $H(t) \equiv (\dot{S}/S)$ as a basic "coordinate." Then, using the classical solutions $H(t), \phi(t)$, one can express $H = H(\phi)$ and ΔH as $H' \Delta\phi$. Such an approach will, in general, lead to different conclusions regarding the uncertainty in H . We feel that the choice of S as a basic variable is most natural. We thank the referee who pointed out the importance of our choice).

3. HORIZON FLUCTUATIONS IN SPECIFIC COSMOLOGIES

We first demonstrate that the uncertainty $(\Delta l_H/l_H)$ in (16) is completely negligible in a conventional, noninflationary cosmology. Later, we show that inflation amplifies the value of $(\Delta l_H/l_H)$ tremendously.

A noninflationary model with $V(\phi) = 0$ is described by the following solution

$$\dot{\phi} = (a/S^3), \quad S^3(t) = (12\pi)^{1/2} a L_p t \quad (17)$$

where a is a constant and $L_p = (G\hbar/c^3)^{1/2} \approx 10^{-33}$ cm. (For simplicity we have set $k=0$; the results do not depend crucially on this choice.) We determine, by assuming that at present ($t = t_0 \approx 10^{10}$ y), $S(t_0) \gtrsim 10^{28}$ cm. We get

$$a = 10^{89} \text{ cm} \quad (18)$$

Using (18) and (17) in (16), we get

$$\begin{aligned} (\Delta l_H/l_H)^2 &\gtrsim (L_p/a)(64/3\pi^3)^{1/2} \\ &\approx 0.83(L_p/a) \approx 0.83 \times 10^{-122} \end{aligned} \quad (19)$$

We note that $(\Delta l_H/l_H)$ is completely insignificant and is independent of time.

Let us now consider a universe with an inflationary phase (see, for example, Ref. 3). This requires a potential $V(\phi)$ which is almost flat for $0 < \phi < \phi_f$ (say) and drops steeply for $\phi > \phi_f$. Let $V(\phi) \approx V_0$ in the flat region. Then the complete history of the inflationary model can be adequately approximated as follows: (with $H^2 = (8\pi G V_0/3)$)

$$S^3(t) = [a/(2V_0)^{1/2}] \sinh(3Ht) \quad (\text{for } t < t_f) \quad (20)$$

and

$$\begin{aligned} S^3(t) &= [a/(2V_0)^{1/2}](t/t_f) \sinh(3Ht_f) \quad (\text{for } t > t_f) \\ &\approx [a/2(2V_0)^{1/2}](t/t_f) \exp(3Ht_f) \end{aligned} \quad (21)$$

It is assumed that the "slow roll over" and inflation ends at $t = t_f$ (i.e., $\phi(t_f) \approx \phi_f$). Adequate inflation [4] can be easily achieved with the following choice of parameters: $V_0 \approx (10^{14} \text{ GeV})^4$, $H = (2 \times 10^9 \text{ GeV})$, $a \approx L_p$ and

$$Ht_f \approx 103 \quad (22)$$

It is now straightforward to evaluate (16) for both ($t < t_f$) and ($t > t_f$). We get, for the inflationary phase ($t < t_f$)

$$(\Delta l_H/l_H)^2 \approx (64/3\pi^3)^{1/2} \cosh(3Ht) \quad (t < t_f) \quad (23)$$

In other words, inflation amplifies the uncertainty $(\Delta l_H/l_H)$. At the end of inflation ($t = t_f$)

$$(\Delta l_H/l_H)_{t=t_f}^2 \approx (64/3\pi^3)^{1/2} \cdot \frac{1}{2} \exp(3Ht_f) \approx 10^{123} \quad (24)$$

which is huge! It can be checked easily that this uncertainty, once produced, will not "go away." From (21) we find that, for $t > t_f$

$$(\Delta l_H/l_H)_{t>t_f}^2 \approx (1/Ht_f)(1/6) \exp(3Ht_f) \approx 10^{121} \quad (25)$$

Since we have not bothered to evolve the $S(t), \phi(t)$ across $t = t_f$ —taking into account the "fall" of ϕ into the potential— ϕ and \dot{S} are discontinuous across $t = t_f$. Because of this artificiality, (24) and (25) differ somewhat. However, there is no question that both are large.

In any exponential inflation, $(\dot{S}/S) \approx \text{const.}$ and $\phi^2 S^3 \propto S^{-3}$. Thus $[(\Delta l_H)/l_H]^2$ in (16) will increase as S^3 . Thus any inflation by a factor Z will be accompanied by an amplification of $(\Delta l_H/l_H)$ by a factor $Z^{3/2}$. This large uncertainty will persist in the post-inflationary era.

4. CONCLUSION

A comparison of (19) and (23) shows that inflation does amplify the inherent quantum uncertainties. [These uncertainties are, of course, different from the usual quantum fluctuations which are studied [5] as a source for density contrast $(\delta\rho/\rho)$.]

The currently observed universe with a size $\sim 10^{28}$ cm has originated from a region of size $\sim 10^8$ cm in a noninflationary model while the same size originates from a region $\sim 10^{-32}$ cm in an inflationary model. The quantum uncertainties present at a smaller scale leads to a correspondingly higher value for $(\Delta l_H/l_H)$ in the second model.

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General Relativistic Electromagnetic Mass Models of Neutral Spherically Symmetric Systems

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The recent work of Grøn [1] concerning charged analogues of Florides' class of solutions is discussed and generalized. The properties of this kind of model are investigated. In particular it is shown that the ratio m/r as well as the acceleration of gravity are maximum inside the body rather than at the boundary. Some exact solutions of the Einstein-Maxwell equations illustrating these properties are presented. The solutions are matched continuously to the exterior Schwarzschild solution and they represent electromagnetic mass models of neutral systems. All physical quantities are finite inside the distributions. The energy density is positive and decreases monotonically from its maximum value at the center to zero at the boundary.

1. INTRODUCTION

In a recent publication [1], Grøn considered charged generalizations of Florides' [2] interior Schwarzschild solution. Specifically, Grøn studied static, spherically symmetric distributions of charged matter under the assumptions that: (a) the matter is a perfect fluid; (b) the component T_r^r of the energy momentum tensor vanishes everywhere, and (c) the distribution has a finite extent and is surrounded by empty space. He showed that these assumptions lead to an interesting class of solutions of the Einstein-Maxwell equations representing electromagnetic mass models of neutral spherically symmetric systems, in the sense that all physical quantities vanish when the charge density vanishes everywhere.

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