IUCAA-10/92

Cylindrically Symmetric Viscous Universes

L.K. Patel* and Naresh Dadhich**

Inter-University Centre for Astronomy and Astrophysics

Post Bag 4, Ganeshkhind, Pune 411 007, INDIA

Abstract

We have considered here cylindrically symmetric cosmological models with viscous fluid having the Kasnerian time evolution. It turns out that "no matter" limit of our geodetic viscous universes, which can as well be looked upon as the string dust universes, is the non-flat empty Kasner spacetime. We thus propose that the universe may be born with the Kasner geometry at $t = 0$ with matter in the form of viscous fluid, then it may follow the path; geodetic to non-geodetic and finally to the radiation phase when viscosity is fully worn out. We have solutions to describe each phase separately.

Subject headings : General relativity, Cosmology, Cylindrical universes, Viscous fluids, String dust

Short title : Cylindrical viscous universes

PACS No. 04.20, 98.80.

---

* Permanent address : Department of Mathematics, Gujarat University, Ahmedabad 380 009, India

** e-mail address : naresh@iucaa.ernet.in

1. Introduction

Very close to the big-bang singularlity the only thing we know about the universe is that it is in highly dense state and the time evolution is Kasnerian[1]. The matter distribution is anisotropic and inhomogeneous. In a super dense state matter may have exotic form and behaviour to allow for viscous effects, heat and null radiation flow, string dust, incoherent radiation, etc. It is therefore imperative to construct models incorporating these features for description of the universe in the early stages.

It is clear that we have to trade off spherical symmetry of the Friedman-Robertson-Walker model that describes the present day universe for a lesser symmetry. For this reason, the Bianchi spacetimes, that incorporate easily anisotropy, are being applied extensively for both classical and quantum evolution of the universe in the early stages[2,3]. Another symmetry that may be relevant in this context is the cylindrical symmetry that can very comfortably incorporate the Kasnerian time evolution alongwith anisotropy and inhomogeneity as well as the string dust. The string fibres are alligned in a direction which is naturally provided by the axis of the cylinder. The Kasnerian character follows in general by simply considering the metric coefficients as separable functions of r and t[4,5]. Thus it is appropriate to study the cylindrical models with the Kasnerian time evolution for description of the early universe very close to the big-bang singularity.

In a series of two papers, we propose to study the Kasnerian cylindrical universes having viscous fluid, heat and null radiation flow. In the present paper I, we obtain some viscous fluid models having geodetic and non geodetic fluid flow.

One of the remarkable features of these models is that viscous fluid with geodetic flow can always be interpreted as a string dust distribution. Further the matter-free limit of these models is the non-flat empty Kasner spacetime. Usually in cosmology spacetime becomes flat when matter is switched off, here in contrast it goes over to the Kasner metric that characterises time evolution near the big-bang singularity. In the following paper II, we study models with heat and null radiation flow. It turns out that the same metric for different parameter ranges can describe a viscous fluid or a fluid with null radiation flow.

Our cylindrical models have the in-built Kasnerian time evolution and the same spacetime can, in some cases, represent more than one kind of matter distribution. This is very important because matter may occur in variety of exotic forms with very unusual properties. Hence spacetime models with flexibility in this regard will be quite relevant and interesting. Viscous effects in fluid flow, that are ordinarily negligible, will become important for such a dense state of matter. The role of viscosity in the evolution of the universe has been considered by Weinberg, and Padmanabhan and Chitre[6,7]. In the early stages of evolution, viscosity can arise due to : superconducting strings moving in cosmic magnetic field[8], particle creation generates a viscosity like term in an effective energy-momentum tensor[9] and photon and neutrino drag[10,11]. It is therefore desirable to construct fluid models with viscosity to describe evolution of the universe in early stages. Viscous fluids in cosmological models have been considered by several authors[12−19].

Davidson[20] has found a Kasnerian cylindrically symmetric universe filled with incoherent radiation ($p = \frac{1}{3}\rho$). It has the big bang singularity at $t = 0$ where ρ

diverges but it is subsequently well-behaved. We shall take the Davidson framework to include viscosity in the universe. We obtain viscous fluid models with geodetic and non-geodetic flow. The latter kind is the generalisation of the Davidson radiation model. It turns out that viscous effects are the strongest in the geodetic case where they dominate over the hydrodynamical pressure, they weaken as flow turns non-geodetic and totally vanish for the radiative universe.

In §2, we set up Einstein's field equations for viscous fluid and their solutions for geodetic and non-geodetic viscous fluids are considered in §3. In some particular situations, a viscous fluid can as well be interpreted as a string-dust distribution. Our models with geodetic flow are shown to accord to this description in §4. In §5, we conclude with the discussion.

2. Field equations

The metric for a non-static cylindrically symmetric spacetime in the absence of rotation will be taken in the general form :

$$ds^2 = D^2 dt^2 - A^2 dr^2 - B^2 dz^2 - c^2 d\phi^2 \qquad (2.1)$$

where A, B, C and D are functions of the radial co-ordinate r and the time co-ordinate t.

We introduce the following tetrads

$$\theta^1 = A dr, \quad \theta^2 = B dz, \quad \theta^3 = c d\phi, \quad \theta^4 = D dt \qquad (2.2)$$

to write the metric as

$$ds^2 = (\theta^4)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)}\theta^a\theta^b \tag{2.3}$$

Here and in what follows the bracketed indices indicate tetrad components. Thus $g_{(ab)}$ = diag $(-1,-1,-1,1)$ are the tetrad components of the metric tensor g_{ik}. Using Cartan's equations of structure, one can now easily compute the connection 1-forms $\omega^a{}_b$, the tetrad components of the curvature tensor and the tetrad components $R_{(ab)}$ of the Ricci tensor. The surviving $R_{(ab)}$ are as follows :

$$\begin{aligned}
R_{(14)} =& \frac{1}{AD}\left[\frac{\dot{B}'}{B} + \frac{\dot{C}'}{C} - \frac{\dot{A}}{A}\left(\frac{B'}{B} + \frac{C'}{C}\right) - \frac{D'}{D}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right] \\
R_{(44)} =& \frac{1}{D^2}\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{D}}{D}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right] \\
& - \frac{1}{A^2}\left[\frac{D''}{D} - \frac{D'}{D}\left(\frac{A'}{A} - \frac{B'}{B} - \frac{C'}{C}\right)\right] \\
R_{(11)} =& \frac{1}{A^2}\left[\frac{B''}{B} + \frac{C''}{C} + \frac{D''}{D} - \frac{A'}{A}\left(\frac{B'}{B} + \frac{C'}{C} + \frac{D'}{D}\right)\right] \\
& - \frac{1}{D^2}\left[\frac{\ddot{A}}{A} + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{D}}{D}\right)\right] \\
R_{(22)} =& \frac{1}{A^2}\left[\frac{B''}{B} + \frac{B'}{B}\left(\frac{C'}{C} + \frac{D'}{D} - \frac{A'}{A}\right)\right] \\
& - \frac{1}{D^2}\left[\frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} - \frac{\dot{D}}{D}\right)\right] \\
R_{(33)} =& \frac{1}{A^2}\left[\frac{C''}{C} + \frac{C'}{C}\left(\frac{B'}{B} + \frac{D'}{D} - \frac{A'}{A}\right)\right] \\
& - \frac{1}{D^2}\left[\frac{\ddot{C}}{C} + \frac{\dot{C}}{C}\left(\frac{\dot{B}}{B} + \frac{\dot{A}}{A} - \frac{\dot{D}}{D}\right)\right]
\end{aligned} \tag{2.4}$$

Here and in what follows a prime and a dot indicate differentiation with respect to r and t respectively.

The energy momentum tensor for a viscous fluid is given by

$$T_{ik} = (\bar{p} + \rho)\nu_i\nu_k - \bar{p}g_{ik} - \eta\mu_{ik} \tag{2.5}$$

where

$$\nu_i\nu^i = 1, \quad \mu_{ik} = \nu_{i;k} + \nu_{k;i} - (\nu_i f_k + \nu_k f_i), \quad f_i = \nu^k\nu_{i;k} \tag{2.6}$$

and

$$\bar{p} = p - (\xi - \frac{2}{3}\eta)\theta, \quad \theta = \nu^i_{;i} \tag{2.7}$$

The symbol θ without any index denotes the expansion scalar. Here a semicolon indicates covariant differentiation and p, ρ, ξ, and η are respectively the fluid pressure, the material density, the coefficient of bulk viscosity and the coefficient of shear viscosity. The shear scalar σ for the velocity field ν_i is defined by

$$\sigma^2 = \sigma_{ik}\sigma^{ik}, \quad \sigma_{ik} = \frac{1}{2}\mu_{ik} - \frac{1}{3}\theta(g_{ik} - \nu_i\nu_k).$$

The Einstein's field equations

$$R_{ik} - \frac{1}{2}g_{ik}R = -8\pi T_{ik}$$

can be expressed in the tetrad form as

$$R_{(ab)} = -8\pi[(\bar{p} + \rho)\nu_{(a)}\nu_{(b)} - \frac{1}{2}(\rho - \bar{p} - 2\eta\theta)g_{(ab)}] + 8\pi\eta\mu_{(ab)} \tag{2.8}$$

We use co-moving co-ordinates to write

$$\nu_{(a)} = (0,0,0,1) \tag{2.9}$$

The surviving $\mu_{(ab)}$ are given by

$$\mu_{(11)} = -\frac{2\dot{A}}{AD},\ \mu_{(22)} = -\frac{2\dot{B}}{BD},\ \mu_{(33)} = -\frac{2\dot{C}}{CD}. \tag{2.10}$$

The velocity field ν_i given by (2.9) has non-vanishing expansion θ, acceleration f_i and the shear σ. The expressions for the Kinematical parameters θ, f_i, and σ are given by

$$\theta = \frac{1}{D}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right),\quad f_i = \left(-\frac{D'}{D}, 0, 0, 0\right)$$

$$\sigma^2 = \frac{1}{9D^2}\left[\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{2\dot{A}}{A}\right)^2 + \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} - \frac{2\dot{B}}{B}\right)^2 + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{2\dot{C}}{C}\right)^2\right] \tag{2.11}$$

where $x^1 = r$, $x^2 = z$, $x^3 = \phi$, $x^4 = t$.

In view of equations (2.8) - (2.10), we get the following relations :

$$R_{(14)} = 0 \tag{2.12}$$

$$8\pi\eta[\mu_{(11)} - \mu_{(22)}] = R_{(11)} - R_{(22)} \tag{2.13}$$

$$8\pi\eta[\mu_{(11)} - \mu_{(33)}] = R_{(11)} - R_{(33)} \tag{2.14}$$

$$8\pi\rho = -\frac{1}{2}[R_{(44)} + 3R_{(22)}] + 4\pi\eta[2\theta + 3\mu_{(22)}] \tag{2.15}$$

$$8\pi\bar{p} = -\frac{1}{2}[R_{(44)} - R_{(22)}] - 4\pi\eta[2\theta + \mu_{(22)}] \tag{2.16}$$

where $R_{(ab)}$ are given by (2.4). Thus we have a system of five equations for determining three physical parameters, $\rho, \bar{p}$ and η and four metric potentials A, B, C and D.

Putting $\eta = 0$ in the above analysis, it can be seen that we can always incorporate the bulk viscosity ξ in a perfect fluid solution replacing p by $p - \xi\theta$ with the same equation of state. That is a perfect fluid solution can as well be interpreted as fluid with bulk viscosity.

Here it should also be noted that the coefficient ξ of the bulk viscosity does not appear explicitly in the above five equations. For the specification of ξ we assume the equation of state

$$p = K\rho \tag{2.17}$$

where K is a constant, $0 < K < 1$.

3. Viscous models

Here we present some explicit solutions of the above field equations (2.12) to (2.16) which are separable. Therefore we shall assume the following forms of the metric functions A, B, C and D[20] :

$$A = t^{\alpha}(1+r^2)^a, B = t^{\beta}(1+r^2)^b, C = r(1+r^2)^c t^{\gamma}, D = (1+r^2)^d \tag{3.1}$$

where $a, b, c, d, \alpha, \beta$ and γ are real constants.

With the assumed separability, the equation (2.12) gives the relations .

$$\gamma = \alpha, \quad b(\beta - \alpha) = d(\beta + \alpha).$$

But $\gamma = \alpha$ implies $\mu_{(11)} = \mu_{(33)}$. Hence (2.14) now becomes $R_{(11)} = R_{(33)}$ that gives one more relation connecting a, b, c and d. This relation is

$$d^2 + b^2 = (d+b)(a+c-1).$$

The remaining three (2.13), (2.15) and (2.16) give us values of the physical parameters $\bar{p}, \rho$ and η. The expressions for these parameters are quite lengthy and complicated. Therefore we shall not report them here.

In general fluid stream lines are non-geodetic because of pressure gradient. Since we have viscosity, the pressure gradient can be cancelled out by it to render the fluid motion geodetic. In the following we shall consider the two kinds of viscous fluid universes : (a) non-geodetic and (b) geodetic, which are in view of eqn. (2.11), characterised by $D' \neq 0$ and $D' = 0$.

(a) Non-geodetic case

In this case $d \neq 0$. We have verified that in this case there is only one physically plausible solution, for which we get

$$\alpha=\gamma=\frac{2}{3},\beta=-\frac{1}{3},a=\frac{1}{10}(2+7\in),b=-\frac{1}{5}(1+\in),c=\frac{1}{10}(-2+3\in),d=\frac{3}{5}(1+\in) \tag{3.2}$$

where $\in$ is an arbitrary constant.

It should be noted that determination of $\alpha=\gamma=\frac{2}{3},\beta=-\frac{1}{3}$ is more generally true. Instead of (3.1) if we write $A=t^{\alpha}f_1(r), B=t^{\beta}f_2(r), C=t^{\gamma}f_3(r)$ and $D=D(r)$, we shall obtain the above values of α,β,γ. It is a general property of cylindrical symmetry and separability of the metric potentials[4,5].

Therefore the geometry of our universe in this case is described by the line-element

$$\begin{aligned} ds^2 =&(1+r^2)^{6(1+\in)/5}dt^2-t^{4/3}(1+r^2)^{(2+7\in)/5}dr^2\\ &-r^2t^{4/3}(1+r^2)^{(-2+3\in)/5}d\phi^2-t^{-2/3}(1+r^2)^{-2(1+\in)/5}dz^2 \end{aligned} \tag{3.3}$$

The physical parameters ρ, $\bar{p}$, η and ξ, for this case, are given by

$$8\pi\rho=\frac{2}{5}(6+\in)t^{-4/3}(1+r^2)^{-(12+7\in)/5} \tag{3.4}$$

$$8\pi\bar{p}=\frac{4}{15}(3+13\in)t^{-4/3}(1+r^2)^{-(12+7\in)/5} \tag{3.5}$$

$$8\pi\eta=-\in t^{-1/3}(1+r^2)^{-(9+4\in)/5} \tag{3.6}$$

$$8\pi\xi = \left[\frac{4}{5}(3K-1) + \frac{2\in}{15}(3K-31)\right] t^{-1/3}(1+r^2)^{-(9+4\in)/5} \tag{3.7}$$

If $\in \leq 0$ and $K \geq \frac{1}{3}$, then it is easy to see that η and ξ are positive and $\rho \geq \bar{p}$. For positive ρ, we must have $\in > -6$. All the physical parameters are positive for $\in < 0$.

Here it is interesting to note that ξ and η do not vanish for $K = \frac{1}{3}$, i.e. the radiation field may also be viscous. The drag in radiation field is known to cause the photon viscosity[10].

¿From the above expressions we obtain

$$\frac{\bar{p}}{\rho} = \frac{2(3+13\in)}{3(6+\in)}, \quad \frac{\eta}{\xi} = \frac{-15\in}{12(3K-1)+2\in(3K-31)}$$

Thus $\bar{p}/\rho$ and η/ξ are constants. This is a noteworthy feature of the solution.

It is clear that the metric (3.3) satisfies the regularity conditions at the axis $r = 0$. From the axis the proper radial distance to the co-moving co-ordinate r_1 at the epoch t is

$$R(r_1, t) = t^{2/3} \int_0^{r_1} (1+r^2)^{(2+7\in)/10} dr \tag{3.8}$$

which is monotomically increasing with t, so that the fluid steadily expands radially. Also for any fixed $t > 0$ and $\in > -\frac{2}{7}$, $R(r_1, t) \to \infty$ as $r_1 \to \infty$, that implies that the fluid cylinder is radially infinite. There is evidently a singularity at $t = 0$ but the spacetime is well behaved for $t > 0$.

The fluid velocity is

$$\nu_i = \left(0, 0, 0, (1+r^2)^{3(1+\epsilon)/5}\right) \tag{3.9}$$

and hence the acceleration vector is

$$f_i = \left(-\frac{6}{5}(1+\epsilon)r(1+r^2)^{-1}, 0, 0, 0\right) \tag{3.10}$$

Thus the stream lines of the viscous fluid are not geodetic. The expansion and shear scalars θ and σ have the expressions

$$\theta = t^{-1}(1+r^2)^{-3(1+\epsilon)/5}, \quad \sigma = \theta\sqrt{2/3} \tag{3.11}$$

and they diverge at $t = 0$ but monotomically decreasing to zero as $t \to \infty$. The fluid motion is however irrotational.

For our case Raychaudhuri equation[21] becomes

$$\theta_{;i}\nu^i - \frac{1}{3}\theta^2 - f^i_{;i} + 2\sigma^2 = R_{ik}\nu^i\nu^k.$$

Equations (2.8) give

$$R_{ik}\nu^i\nu^k = -4\pi(\rho + 3\bar{p} + 2\eta\theta). \tag{3.12}$$

Therefore the energy conditions[22] require $R_{ik}\nu^i\nu^k \leq 0$ implying $\rho + 3\bar{p} + 2\eta\theta \geq 0$. Using the expressions for $\rho, \bar{p}, \eta$ and θ we have verified that these energy conditions are satisfied provided $\epsilon \geq -\frac{6}{11}$. Thus physical considerations constrain the parameter ϵ to the range $-\frac{6}{11} \leq \epsilon \leq 0$.

If $\epsilon = 0$, we have

$$8\pi\rho = 24\pi\bar{p} = \frac{12}{5} t^{-4/3}(1+r^2)^{-12/5}, \quad \eta = 0$$

$$8\pi\xi = \frac{4}{3}(3K-1)t^{-1/3}(1+r^2)^{-9/5}. \tag{3.13}$$

In this case, we obtain a generalization of Davidson's model with fluid having only the bulk viscosity. Further if $K = \frac{1}{3}$ we get $\xi = 0$ and we recover Davidson's perfect-fluid radiation universe.

(b) Geodetic case

For this case $f_i = 0$, which from equation (2.11) implies $D' = 0$. Hence we put $d = 0$ from the begining. Now $d = 0$ implies $R_{(44)} = 0$ and consequently we have $\rho + 3\bar{p} + 2\eta\theta = 0$. As ρ, η and θ are positive, $\bar{p}$ has to be negative in this case. Thus geodetic nature of stream lines of viscous fluid is intimately linked with negativity of $\bar{p}$. This is a novel feature of this geodetic case. Furthermore $d = 0$ requires $b = 0$ (from the consistency condition following (3.1)) which in turn implies $R_{(22)} = 0$. Then there are only two physically plausible solutions and they are :

case (bi) $\alpha = \gamma = \frac{2}{3}, \beta = -\frac{1}{3}, a = c, a$ arbitrary.

case (bii) $\alpha = \gamma = \frac{2}{3}, \beta = -\frac{1}{3}, c = -\frac{1}{2}, a$ arbitrary.

The physical parameters are related as follows :

$$\bar{p} = -\frac{2}{3}\rho, \quad 2\eta\theta = \rho, \quad \xi\theta = (K+1)\rho.$$

For the case (bi) the physical parameters $\rho, \bar{p}, \eta$ and ξ are given by

$$8\pi\rho = -4at^{-4/3}(1+r^2)^{-2(a+1)}. \tag{3.14}$$

The physical requirements $\rho \geq 0, \eta > 0, \xi > 0$ imply $a < 0$. Evidently $t = 0$ is a singularity. But the model remains singularity free for all $t > 0$. Clearly $\rho, \bar{p}, \eta$ and ξ tend to zero as $t \to \infty$.

The metric for this can be written as

$$ds^2 = dt^2 - t^{4/3}(1+r^2)^{2a}(dr^2 + r^2 d\phi^2) - t^{-2/3}dz^2 \tag{3.15}$$

When $a = -1$, the universe becomes homogeneous as all physical parameters depend only on t. This represents a homogeneous cylindrical viscous universe.

When $a = 0, \rho, \bar{p}, \eta$ and ξ vanish and the metric goes over to

$$ds^2 = dt^2 - t^{4/3}(dr^2 + r^2 d\phi^2) - t^{-2/3}dz^2 \tag{3.16}$$

which is the famous Kasner vacuum solution.

For the case (bii), we have the following expressions for $\rho, \bar{p}, \eta$ and ξ :

$$8\pi\rho = 2(a + \frac{3}{2})t^{-4/3}(1+r^2)^{-2(a+1)}. \tag{3.17}$$

For a physically viable model, we must have $a + \frac{3}{2} \geq 0$. The metric is given by

$$ds^2 = dt^2 - t^{4/3}(1+r^2)^{2a}dr^2 - t^{4/3}r^2(1+r^2)^{-1}d\phi^2 - t^{-2/3}dz^2. \tag{3.18}$$

For $a = -1$, all the physical parameters become functions of time t only and we again have homogeneous cylindrical universe.

When $a = -3/2$, we again get an empty space-time :

$$ds^2 = dt^2 - \frac{t^{4/3}dr^2}{(1+r^2)^3} - \frac{t^{4/3}r^2d\phi^2}{1+r^2} - t^{-2/3}dz^2 \tag{3.19}$$

which is a transform of Kasner solution. This can be seen by introducing a new radial coordinate R defined by $R = r(1+r^2)^{-1/2}$ which will take (3.19) to (3.16).

Since the geodetic character of the fluid implies $\rho + 3\bar{p} + 2\eta\theta = 0$, the energy conditions are marginally satisfied. For both the solutions the expansion θ and the shear σ are given by

$$\theta = \frac{1}{t}, \qquad \sigma = \theta\sqrt{2/3} \tag{3.20}$$

which tend to zero as $t \to \infty$. We have also verified that the fluid motion is irrotational.

For all the viscous fluid solutions discussed in this section $\frac{\sigma}{\theta} = \sqrt{\frac{2}{3}}$. The present upper limit for the ratio $\frac{\sigma}{\theta}$ is 10^{-3}, obtained from indirect arguments concerning the isotropy of primordial black body radiation[23]. For our models, this ratio is considerably larger than 10^{-3}. This shows that our solutions may be appropriate only for the early stages of evolution of the universe.

4. String-dust interpretation

Letelier[24], Matravers[25] and Nevin[26] have described some exact solutions of Einstein field equations with string-dust source. Stachel[27] has given the energy momentum tensor for string dust as

$$T_{ik} = \mu(\nu_i \nu_k - \omega_i \omega_k) \tag{4.1}$$

where

$$\nu^i \nu_i = 1, \quad \omega^i \omega_k = -1, \quad \nu^i \omega_i = 0. \tag{4.2}$$

and μ is the string tension density.

In general it is clear from the expressions for energy momentum tensors for perfect fluid and string dust that the solution representing one cannot be interpreted for the other. But it turns out that our models with geodetic flow can be interpreted as string dust distributions.

The Einstein field equations for T_{ik} given by (4.1) can be written as

$$R_{(ab)} = -8\pi\mu[\nu_{(a)}\nu_{(b)} - \omega_{(a)}\omega_{(b)} - g_{(ab)}] \tag{4.3}$$

We use co-moving coordinates and assume that string fibres are lying parallel to the axis of symmetry (i.e. z-axis). So we have

$$\nu_{(a)} = (0,0,0,1), \quad \omega_{(a)} = (0,1,0,0) \tag{4.4}$$

Using (4.4) in (4.3), we get the relations

$$R_{(14)} = R_{(44)} = R_{(22)} = 0 \tag{4.5}$$

and

$$R_{(11)} = R_{(33)} = -8\pi\mu. \qquad (4.6)$$

Thus it is clear that a viscous-fluid solution of Einstein's equations having the properties (4.5) and (4.6) can only be interpreted as a string-dust solution.

We have seen that $R_{(14)} = 0$ for fluid distribution as well, $R_{(44)} = 0$ for geodetic flow determines $D = 1$ and physical plausibility of geodetic fluids in the case (b) fixes $B = B(t)$ that makes $R_{(22)} = 0$. This is how geodetic viscous fluids meet the constraints of string dust distribution. That is the requirement of geodecity of viscous fluid makes the transverse stresses ($T_{(11)} = T_{(33)} = 0$) vanish, and the energy density negative of the stress along the axis of the cylinder. This is exactly what is needed for strings alligned along the axis.

For the viscous fluid universes described by the metrics (3.15) and (3.18) we have

$$8\pi\mu = -4at^{-4/3}(1+r^2)^{-2(a+1)} = 8\pi\rho$$

and

$$8\pi\mu = 2(a+3/2)t^{-4/3}(1+r^2)^{-2(a+1)} = 8\pi\rho$$

respectively where a is an arbitrary constant.

Thus the geodetic viscous-fluid cosmological models can also be interpreted as the string-dust universes.

5. Discussion

The Kasnerian time evolution of the models is the consequence of separability of metric potentials into functions of t and r and cylindrical symmetry[4,5]. These two assumptions exactly determine $p_1 = p_3 = 2/3, p_2 = -1/3$. In all these models, the ratios of physical parameters $\bar{p}/\rho$, σ/θ and η/ξ turn out to be constants.

In ordinary fluids viscosity produces negligible effect on fluid flow. But it becomes nonignorable in fluids that permit an easy exchange of energy between translational and internal degrees of freedom as in the case of a gas of rough spheres. In cosmology viscous effects will be important for matter with very short mean free times, interacting with radiation quanta with a finite mean free time[28]. Such a situation is presumed to obtain in the early universe.

We have two cases : (a) viscous fluid with non-geodetic flow and (b) with geodetic flow. The former case is the generalization of Davidson's radiation universe having bulk and shear viscosities. When $\eta = 0$ and $p = \frac{1}{3}\rho$ bulk viscosity vanishes and we get back to the Davidson[20] universe. It is noted that viscosity coefficients ξ and η do not vanish for $p = \frac{1}{3}\rho$. That is the drag of the radiation field gives rise to photon viscosity[10].

In general fluid flow is nongeodetic because of pressure gradient. Presence of viscosity can make it geodetic (see the expressions (2.5) and (2.7)). That is acceleration caused by pressure gradient is eaten up by viscous effects. The geodecity of fluid flow also implies $R_{ik}\nu^i\nu^k = \rho + 3\bar{p} + 2\eta\theta = 0$, marginally satisfying the energy condition. Since $\rho, p, \xi, \eta, \theta$ are non-negative, $\bar{p} = p - \left(\xi - \frac{2}{3}\eta\right)\theta < 0$. That means viscous contribution should dominate over pressure. Thus material fluid should be of highly unusual kind. It is conceivable that in a highly compact state near the big-bang singularity, matter may satisfy such extreme constraints. In this context it is worth noting that the geodetic viscous fluid can as well be interpreted as a string dust matter.

Another note-worthy feature of the geodetic case is that the switching off of viscosity removes the matter as well and the universe goes over to the non-flat vacuum Kasner space-time. Generally removal of matter in cosmological models leads to flat space-time. In contrast here we have the Kasner empty universe as the "no-matter" limit. This is quite interesting from the point of view of evolving the universe from the big-bang.

It is well-known that the behaviour of the universe as $t \to 0$ is, correctly and in sufficient generality, described by the Kasner metric[1,29]. The approach to the singularity is oscillatory. This is the general feature that remains qualitatively undisturbed by the introduction of matter in the universe. In our model of goedetic viscous fluid, the Kasner metric arises naturally as the matter-free limit. We propose that the universe may be born with the Kasner geometry and matter in the form of viscous fluid that passes through geodetic to non-geodetic phases and finally viscosity wears out to lead to the Davidson's radiation universe. What we have done here is to start from $t \approx 0$ and then follow the evolution of the universe as described by a series of solutions of Einstein's equations. This is reverse of the usual considerations when the singularity is approached as $t \to 0$. Since Einstein's equations are time symmetric, and particularly the Kasner metric that describes the general behaviour close to the singularity does not distinguish between t and $-t$, hence the behaviour of the universe close to the singularity will be time symmetric. Although its ultimate evolution will be dependent on the initial conditions and hence non-equivalent in general.

We do not however have a one single solution to describe all the phases but instead we do have a solution for each phase, starting with the Kasner to our geodetic and non-geodetic viscous and to the Davidson radiation phase. Fluid models with heat and null radiation flow are considered in a separate paper[30].

Acknowledgements

We thank J.V. Narlikar, T. Padmanabhan and V. Sahni for reading the manuscript. We also thank the referee for constructive criticism that has resulted in an improvement in presentation. One of us (LKP) wishes to thank Indian National Science Academy for a visiting fellowship and Inter-University Centre for Astronomy and Astrophysics for hospitality.

References

1. L. Landau and E.M. Lifshitsz, Classical Theory of Fields (Pergamon Press, New York, 1985).
2. W.L. Roque and G.F.R. Ellis, Galaxies, axiosymmetric systems and relativity, edited by M.A.H. MacCallum (Cambridge University Press, 1985).
3. A. Ashtekar and J. Samuel, Class. Quantum Grav., **8**, 2191 (1991).
4. R.A. Harris and J.D. Zund, Tensor, **30**, 255 (1976).
5. M. MacCallum, Wiss. Z. Jena, **39**, 102 (1990).
6. S. Weinberg, Ap. J. **168**, 175 (1971).
7. T. Padmanabhan and S.M. Chitre, Phys. Lett. **A120**, 433 (1987).
8. J.P. Ostriker, C. Thompson and E. Witten, Phys. Lett. **B180**, 231 (1986).
9. B.L. Hu, Advances in astrophysics, edited by L.Z. Fang and R. Ruffini (World Scientific, Singapore. 1983).
10. L.A. Thomas, Q.J. Math **1**, 239 (1930).
11. C.W. Misner, Nature, **214**, 40 (1967).
12. J.M. Murphy, Phys. Rev., **D8**, 4231 (1973).
13. Z. Klimck, Acta cosmologica, **2**, 49 (1974).
14. S.R. Roy and S. Prakash, J. Phys., **A9**, 261 (1976).
15. K.A. Dunn and B.O.J. Tupper, Ap. J., **222**, 405 (1978).
16. S.R. Roy and J.P. Singh, Acta Physica Austrica, **55**, 57 (1983).
17. A. Banerjee, S.R. Duttachaudhuri and A.K. Sanyal, Gen. Relativ. Grav., **18**, 461 (1986).

18. M.B. Ribeiro and A.K. Sanyal, J. Math. Phys., **28**, 657 (1987).

19. L.K. Patel and S.S. Koppar, J. Austral. Math. Soc. Ser. B., **33**, 77 (1991).

20. W. Davidson, J. Math. Phys., **32**, 1560 (1991).

21. A.K. Raychaudhuri, Phys. Rev., **98**, 1123 (1955).

22. S.W. Hawking and R. Penrose, Proc. Roy. Soc., A314, 529 (1969).

23. C.B. Collins, E.N. Glass and D.A. Wilkinson, Gen. Relativ. Grav., **12**, 805 (1980).

24. P.S. Letelier, Phys. Rev., **D20**, 1274 (1979).

25. D.R. Matravers, Gen. Relativ. Grav., **20**, 279 (1988).

26. J.M. Nevin, Gen. Relativ. Grav., **23**, 253 (1991).

27. J. Stachel, Phys. Rev., **D21**, 2171 (1980).

28. S. Weinberg, Gravitation and Cosmology (John Wiley & Sons, New York, 1972).

29. V.A. Belinskii, J.M. Khalatnikov and E.M. Lifshitz, Advanc. Phys., **19**, 525 (1982); **31**, 639 (1970).

30. L.K. Patel, and N. Dadhich, To appear in Ap. J. (1992).