International Journal of Modern Physics D
Vol. 18, No. 14 (2009) 2189–2193

ENTROPY DENSITY OF SPACE–TIME AND GRAVITY: A CONCEPTUAL SYNTHESIS*

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Communicated by D. V. Ahluwalia

I show that combining the principle of equivalence and the principle of general covariance with the known properties of local Rindler horizons, perceived by accelerated observers, leads to the following inescapable conclusion: *The field equations describing gravity in any diffeomorphism-invariant theory must have a thermodynamic interpretation.* This synthesis of quantum theory, thermodynamics and gravity shows that the gravitational dynamics can be interpreted completely in terms of entropy balance between matter and space–time. This idea has far-reaching implications for the microstructure of space–time and quantum gravity.

The principle of equivalence has two key consequences: (i) gravity has a geometrical description in terms of the metric tensor g_{ab} and the effect of gravity on matter can be understood by using the laws of special relativity in the local inertial frames; (ii) in flat space–time, we can choose a special coordinate system with the metric η_{ab} so that we can always attribute the part $g_{ab} - \eta_{ab}$ to the choice of noninertial coordinates. We cannot do this in a curved space–time. So it no longer makes sense to ask "how much of g_{ab}" is due to our using noninertial coordinates and "how much" is due to genuine gravity. No coordinate system or observer is special. The laws of physics should not select out any special class of observers.

These facts, familiar to all of us from textbooks, actually have some deep implications (see Fig. 1 for the flow of logic) when we consider some judicious thought experiments with noninertial observers.

We begin by recalling that, in any space–time, there will exist observers who do not have access to part of the space–time because of the existence of horizons they perceive. This is a direct consequence of the fact that gravity affects the light rays and hence the causal structure. The classic example is the Rindler horizon in flat space–time, which is as effective in blocking information for an accelerated observer

*This essay received an honorable mention in the 2009 Essay Competition of the Gravity Research Foundation. It was refereed, not as a regular IJMPD research paper, but as an essay.

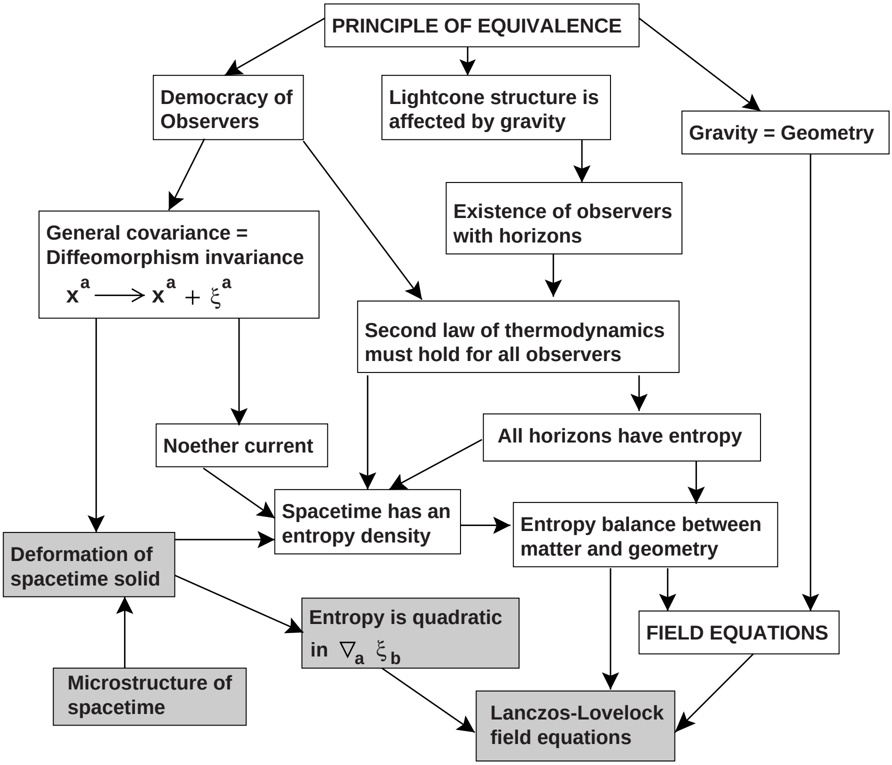

Fig. 1. The logical structure of the synthesis between entropy density of space–time and gravitational dynamics.

as the Schwarschild horizon at $r = 2M$ is for an observer at $r > 2M$. The Rindler horizon does not block information for an inertial observer *just as* the Schwarschild horizon does *not* block information for someone plunging into the black hole. The state of motion of the observer is crucial in deciding the physical effects of a horizon *in all cases.*

Next, it is easy to see that all horizons must have entropy *vis-à-vis* the observers who perceive the horizons. If they do not, an observer can pour some hot tea with entropy across the horizon *à la Wheeler*,[1] thereby violating the second law of thermodynamics in the region accessible to him and his friends who perceive the horizon. The only way to avoid this paradox is to attribute an entropy to any horizon which must change when energy flows across it.

This idea can be made sharper. Take any event in any space–time and choose a local inertial frame at that event. By boosting along one of the axes with an acceleration κ, we can introduce a local Rindler observer who perceives a horizon with temperature proportional to κ. If he sees a matter energy flux crossing the horizon, he will associate a flow of entropy with it. If ξ^a is a local, approximate, timelike Killing vector, then the energy flux through a local patch of (timelike) surface with normal r_a will be $T_{ab}\xi^a r^b$ and the associated entropy flux will be $\beta_{\rm loc}T_{ab}\xi^a r^b$, where $\beta_{\rm loc}^{-1} = \beta^{-1}/N$ is the local temperature, with N being the standard lapse function giving the redshift factor. To maintain the second law of thermodynamics, this entropy flux must match the entropy change of the locally perceived horizon. For all these to hold locally at every event, there *must exist a space–time entropy current* $\beta_{\rm loc}J^a$, built out of the metric and its derivatives, such that $\beta_{\rm loc}(r_a J^a)$ gives the corresponding entropy flux. So we expect the relation $\beta_{\rm loc}r_a J^a = \beta_{\rm loc}T^{ab}r_a\xi_b$ to hold at all events, once we introduce a local Killing vector ξ^a and a local temperature giving $\beta_{\rm loc}$. J^a must be conserved, since we do not expect irreversible entropy production in the space–time.

To find such a conserved current J^a we have only to note that it was the diffeomorphism invariance of the theory which forced us to treat all observers on an equal footing and make sure the horizons perceived by the observers cannot be used to violate the second law of thermodynamics. So the diffeomorphism invariance should also provide us with the conserved current J^a and indeed it does, in the form of the Noether current. Consider a theory of gravity, obtained from a generally covariant action principle involving a gravitational Lagrangian $L_{\rm grav}(R^a_{bcd}, g^{ab})$ which is a scalar made from the metric and curvature tensor. The total Lagrangian is the sum of $L_{\rm grav}$ and the matter Lagrangian L_m. The variation of the gravitational Lagrangian density generically leads to a surface term and hence can be expressed in the form

$$\delta(L_{\rm grav}\sqrt{-g}) = \sqrt{-g}(E_{ab}\delta g^{ab} + \nabla_a \delta v^a). \tag{1}$$

Under suitable boundary conditions the theory will lead to the field equation $2E_{ab} = T_{ab}$, where T_{ab} is defined through the usual relation $(1/2)T_{ab}\sqrt{-g} = -(\delta A_m/\delta g^{ab})$. We also know that, for any Lagrangian $L_{\rm grav}$, the functional derivative E_{ab} satisfies the generalized off-shell Bianchi identity: $\nabla_a E^{ab} = 0$. Consider now the variations in δg_{ab} which arise through the diffeomorphism $x^a \to x^a + \xi^a$. In this case, $\delta(L_{\rm grav}\sqrt{-g}) = -\sqrt{-g}\nabla_a(L_{\rm grav}\xi^a)$, with $\delta g^{ab} = \nabla^a\xi^b + \nabla^b\xi^a$. Substituting these in Eq. (1) and using $\nabla_a E^{ab} = 0$, we obtain the conservation law $\nabla_a J^a = 0$, for the current,

$$J^a \equiv L_{\rm grav}\xi^a + \delta_\xi v^a + 2E^{ab}\xi_b, \tag{2}$$

where $\delta_\xi v^a$ represents the boundary term which arises for the specific variation of the metric in the form $\delta g^{ab} = \nabla^a\xi^b + \nabla^b\xi^a$. As we shall see we will not require the explicit form of $\delta_\xi v^a$ except for one easily proven result,

$$\delta_\xi v^a = 0, \tag{3}$$

at an event when ξ^a is an approximate Killing vector which satisfies the conditions

$$\nabla_{(a}\xi_{b)} = 0, \quad \nabla_a\nabla_b\xi_c = R_{cbad}\xi^d \tag{4}$$

at that event.

Let us now consider the form of $J^a(x)$ at any event $\mathcal{P}$ around which we introduce the notion of a local Rindler horizon. Choose a local inertial frame (LIF) around some event $\mathcal{P}$ with coordinates X^a such that $\mathcal{P}$ has the coordinates $X^a = 0$ in the LIF. Pick a future-directed null vector k^a at $\mathcal{P}$ and align the coordinates of the LIF such that k^a lies in the $X-T$ plane at $\mathcal{P}$. Construct, in the neighborhood of $\mathcal{P}$, a local Rindler frame (LRF) with coordinates x^a by the usual coordinate transformations from the inertial frame to Rindler framelines, with acceleration κ. Let ξ^a be the (approximate) Killing vector corresponding to translation in the Rindler time such that the vanishing of $\xi^a\xi_a \equiv -N^2$ characterizes the location of the local horizon $\mathcal{H}$ in the LRF. As usual, we shall do all of the computation on a timelike surface ("stretched horizon") infinitesimally away from $\mathcal{H}$. On this surface $N = \epsilon$, where ϵ is an infinitesimal constant. The local temperature on the stretched horizon will be $\kappa/2\pi N$, so that $\beta_{\rm loc} = \beta N$. The approximate, local, Killing vector ξ^a satisfies two conditions in Eq. (4) at $\mathcal{P}$ which any true Killing vector will, of course, satisfy everywhere. Let r_a be the spacelike unit normal to Σ, pointing in the direction of increasing N. We know that as $N \to 0$ the stretched horizon approaches the local horizon and Nr^i approaches ξ^i.

With this background, we compute J^a for the ξ^a introduced above in the neighborhood of $\mathcal{P}$. Since it is a Killing vector locally, satisfying Eq. (4), it follows that $\delta_\xi v = 0$, giving the current as $J^a = L_{\rm grav}\xi^a + 2E^{ab}\xi_b$. The product r_aJ^a for the vector r^a, which satisfies $\xi^a r_a = 0$ on the stretched horizon, becomes quite simple: $r_aJ^a = 2E^{ab}r_a\xi_b$. This equation is valid around the local patch in which ξ^a is the approximate Killing vector. The quantity $\beta_{\rm loc}r_aJ^a$ has a natural interpretation as the local entropy flux density. On using the field equations $2E_{ab} = T_{ab}$, we find the entropy current to be

$$\beta_{\rm loc}r_aJ^a = \beta_{\rm loc}T^{ab}r_a\xi_b \tag{5}$$

and show that the matter entropy flux is precisely balanced by the gravitational entropy flux. (Note that, in the limit of $N \to 0$, this gives a *finite* result, $\beta\xi_aJ^a = \beta T^{ab}\xi_a\xi_b$, as it should.)

We never said what kind of theory of gravity we are dealing with. The field equations — whatever the theory may be, as long as it obeys the principle of equivalence and diffeomorphism invariance — always have an interpretation in terms of local entropy balance! *Different theories of gravity are characterized by different forms of entropy current J^a, just as different physical systems are characterized by different forms of entropy functionals.*

It is possible to interpret our result in very physical terms. Consider a process involving a virtual displacement of the horizon that makes "cups of tea" disappear to the outside observer (see Refs. 2 and 3 for more details). An infinitesimal displacement of a local patch of the stretched horizon in the direction of r_a, by an infinitesimal proper distance ϵ, will change the proper volume by $dV_{\rm prop} = \epsilon\sqrt{\sigma}d^{D-2}x$, where σ_{ab} is the metric in the transverse space. The flux of energy through the surface will be $T^a_b\xi^b r_a$ and the corresponding entropy flux can be obtained by multiplying the energy flux by β_{loc}. Hence the "loss" of matter entropy to the outside observer because the virtual displacement of the horizon has engulfed some hot tea is $\delta S_m = \beta_{\text{loc}}\delta E = \beta_{\text{loc}} T^{aj}\xi_a r_j dV_{\text{prop}}$. Using Eq. (5) we see that it matches the corresponding change in the gravitational entropy, thereby showing the validity of local entropy balance for any β. In this limit, ξ^i also goes to $\kappa\lambda k^i$, where λ is the affine parameter associated with the null vector k^a which we started with and all of the reference to the LRF goes away. It is clear that the properties of the LRF are relevant conceptually for defining the intermediate notions (local Killing vector, horizon temperature, etc.) but the essential result is independent of these notions. *Just as we introduce a local inertial frame to decide how gravity couples to matter, we use local Rindler frames to determine the physical content of the field equations.* The quantity, $\delta S = \beta_{\text{loc}} u_a J^a dV_{\text{prop}}$, can be interpreted as the entropy associated with a volume dV_{prop} as measured by an observer with four-velocity u^a. If we consider observers moving along the orbits of the Killing vector ξ^a, then $u^a = \xi^a/N$ and, as we approach the horizon at which $\xi^a\xi_a \to 0$, we get $\delta S_{\text{grav}} = \beta(\xi_j\xi_a T^{aj})\, dV_{\text{prop}}$. In the same limit ξ^j will become null (and proportional to the original null vector k^j which we started with). For any null vector, one can interpret the right-hand side, $\beta(\xi_j\xi_a T^{aj})\, dV_{\text{prop}}$, as the matter entropy present in a proper volume dV_{prop} (see e.g. Ref. 2 and 3). So we are once again led to an entropy balance equation.

Does this mean that we cannot really select out any theory of gravity from a thermodynamic perspective? Not quite (see the gray boxes in Fig. 1). Recall that thermodynamics relies entirely on the form of the entropy functional to make predictions. If we constrain the form of the entropy current J^a, we constrain the theory. The results strongly suggest that we should think of gravity as an emergent, long-wavelength phenomenon like elasticity and the diffeomorphism $x^a \to x^a + \xi^a$ giving the elastic deformations of the "space–time solid." It then makes sense to demand that the entropy density be quadratic in the gradients $\nabla_a\xi_b$. This puts a strong constraint on the Noether current and selects out a special class of theories which happen to Lanczos-Lovelock models in D dimensions.[4] In four dimensions, it uniquely selects Einstein gravity! *Einstein's field equations are the only ones which will maintain the entropy balance in local Rindler frames and will have an entropy density that is expressible as a quadratic function of the gradients of space–time deformations.* Any microscopic theory of gravity now has its task cut out: Obtain the entropy density of space–time in the long-wavelength limit from first principles.

References

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