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Cylindrically symmetric cosmological models in the Kaluza-Klein spacetime

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Abstract

We consider a non-diagonal cylindrically symmetric metric in the Kaluza-Klein spacetime. We obtain a number of homogeneous and inhomogeneous perfect fluid cosmological models, which include the 5-dimensional analogue of the recently found 4-dimensional non-singular stiff fluid model. The dimensional reduction is however admitted only in the diagonal case. Amongst the homogeneous models, which are all as expected big-bang singular, there is the 5-dimensional version of the Friedman-Robertson-Walker flat model.

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1 Introduction

There has been considerable interest in solutions of Einstein's equations in higher dimensions in the context of physics of the early Universe both from cosmological as well as particle physics points of view. It is argued that the extra dimensions are not observable at the present time, owing to their size being assumed to be of the order of the Planck length, but they may perhaps be relevant for the very early Universe [1]. It is expected that as t increases extra dimensions shrink rapidly to leave us with 4-dimensional (4-d) Universe. Chodos and Detweiler [2] considered the 5-d Kasner vacuum solutions in which the extra dimension shrank while the usual 3-space expanded with t. Recently 5-d spacetimes have been studied in the context of inhomogeneous cosmologies [3-6].

Inhomogeneous cosmologies are relevant for several reasons, principally to have general generic initial conditions and to facilitate formation of large scale structures in the Universe. In higher dimensions, several Kaluza-Klein (KK) extensions of the Friedman-Robertson-Walker (FRW) model have been considered [7-9] but they are all big-bang singular. The remarkable feature of inhomogeneous spacetimes is that they admit non-singular family of models [10-11] satisfying the strong energy condition $\rho + 3p > 0$ with equations of state $\rho = 3p$ and $\rho = p$. The models have cylindrical symmetry. Banerjee et al [12] have considered the 5-d generalizations of the non-singular family in the KK spacetime.

There has been an attempt [13] to study the non-singular models in the general diagonal metric which is separable in space and time in the comoving coordinates. Very recently Mars [14] has obtained a non-diagonal singularity-free model with the equation of state $\rho = p$. The spacetime is described by the metric,

$$\begin{aligned} ds^2 &= e^{ca^2r^2}\cosh(2at)(dt^2 - dr^2) - r^2\cosh(2at)d\theta^2 \\ &\quad - \cosh^{-1}(2at)(dz + ar^2 d\theta)^2 \end{aligned} \tag{1.1}$$

where a and c are constants.

In this paper we wish to obtain perfect fluid models in 5-d KK spacetime. We take the metric in non-diagonal form (1.1) keeping time dependence free to be determined. It turns out that non-singular character of (1.1) is maintained in 5-d as well. Inhomogeneous models will always have equation of state $\rho = p$ while the homogeneous ones can admit general perfect fluid. The coordinate reduction is admitted in some models only in the diagonal case. In Section 2 we set up the field equations which separate out inhomogeneous and homogeneous models by the parameter b. They are respectively considered in Sections 3, 4 and 5. We conclude with a discussion.

2 Field equations

We consider the 5-d analogue of the Mars metric [14] and write

$$ds^2 = T^{2\alpha} e^{2br^2}(dt^2 - dr^2) - r^2 T^{2\gamma} d\varphi^2 - T^{2\beta}(dz + ar^2 d\varphi^2) - T^{2\delta} d\psi^2 \qquad (2.1)$$

where $T(t)$ and all other parameters are constants. Here ψ is the Kaluza-Klein (KK) parameter with $0 \leq \psi \leq 2\pi R_5$, where R_5 is the radius of the KK circle. The metric is globally regular for the entire range of the other four coordinates, $0 \leq r < \infty$, $-\infty < t$, $z < \infty$, $0 \leq \varphi \leq 2\pi$.

We introduce the orthonormal tetrad,

$$\begin{aligned} \theta^1 &= T^\alpha e^{br^2} dr, \qquad \theta^2 = T^\beta (dz + ar^2 d\varphi) \\ \theta^3 &= rT^\gamma d\varphi, \qquad \theta^4 = T^\delta d\psi, \qquad \theta^5 = T^\alpha e^{br^2} dt \end{aligned}$$

which makes $g_{ab} = (-1, -1, -1, -1, 1)$ and hence forth all the quantities will be referred to this frame.

In the tetrad frame the non-zero R_{ab} read as follows:

$$\begin{aligned}
A^2R_{15} &= -\mu\left[2br(\beta+\gamma+\delta)+\frac{\alpha-\gamma}{r}\right] \\
A^2R_{11} &= -\alpha\dot{\mu}-\alpha(\beta+\gamma+\delta)\mu^2+2a^2T^{2(\beta-\gamma)} \\
A^2R_{22} &= -\beta\dot{\mu}-\beta(\beta+\gamma+\delta)\mu^2-2a^2T^{2(\beta-\gamma)} \\
A^2R_{33} &= -\gamma\dot{\mu}-\gamma(\beta+\gamma+\delta)\mu^2+2a^2T^{2(\beta-\gamma)} \\
A^2R_{44} &= -\delta[\dot{\mu}+(\beta+\gamma+\delta)\mu^2] \\
A^2R_{55} &= -4b+(\alpha+\beta+\gamma+\delta)\dot{\mu}+[\beta(\beta-\alpha)+\gamma(\gamma-\alpha)+\delta(\delta-\alpha)]\mu^2
\end{aligned} \tag{2.2}$$

where $A^2 = T^{2\alpha}e^{2br^2}$, $\mu = \dot{T}/T, \dot{T} = dT/dt$.

We take the matter content of spacetime to be perfect fluid,

$$T_{ik} = (\rho+p)u_iu_k - pg_{ik}. \tag{2.3}$$

The Einstein equation will then read as

$$R_{ik} = -8\pi[(\rho+p)u_iu_k - \frac{1}{3}(\rho-p)g_{ik}]. \tag{2.4}$$

We employ the comoving coordinates so we write

$$u_i = (0,0,0,0,1). \tag{2.5}$$

Then equation (2.4) will yield

$$R_{15} = 0,\ R_{11} = R_{22} = R_{33} = R_{44} \tag{2.6}$$

$$8\pi\rho = -\frac{1}{2}(R_{55}+4R_{22}) \tag{2.7}$$

$$8\pi p = \frac{1}{2}(-R_{55}+2R_{22}) \tag{2.8}$$

From (2.2) and (2.6) it follows that when $b \neq 0$ only stiff fluid $\rho = p$ is admitted. This means inhomogeneous perfect fluid with the equation of state different from $\rho = p$

cannot be sustained by the metric (2.1). However, when $b = 0$, i.e. the spacetime is homogeneous, perfect fluid with $\rho = p$ as well as for $\rho \neq p$ is admissible.

The condition $\rho = p$ from (2.7) and (2.8) implies that $R_{22} = 0$ which means all but R_{55} are zero. We then write, from (2.2)

$$\alpha = \gamma \text{ and } b = 0 \text{ or } \beta + \gamma + \delta = 0 \qquad (2.9)$$

$$\alpha\dot{\mu} + \alpha(\beta + \gamma + \delta)\mu^2 - 2a^2T^{2(\beta-\gamma)} = 0 \qquad (2.10)$$

$$\beta\dot{\mu} + \beta(\beta + \gamma + \delta)\mu^2 + 2a^2T^{2(\beta-\gamma)} = 0 \qquad (2.11)$$

$$\delta(\dot{\mu} + (\beta + \gamma + \delta)\mu^2) = 0 \qquad (2.12)$$

In view of $\alpha = \gamma, R_{11} = R_{33}$. Now we shall consider the two cases $b \neq 0$ and $b = 0$ separately and they will respectively represent inhomogeneous and homogeneous models.

The expansion scalar θ associated with the 4-velocity u_i is given by

$$A\theta = \mu(\alpha + \beta + \gamma + \delta), \quad A = T^{\alpha}e^{br^2} \qquad (2.13)$$

3 Inhomogeneous models

Here $b \neq 0$ and hence $\alpha + \beta + \delta = 0$ from (2.9). The other equations (2.10)- (2.12) lead to

$$(\alpha + \beta)\dot{\mu} = 0, \ \delta\dot{\mu} = 0, \ \alpha\dot{\mu} = 2a^2T^{2(\beta-\alpha)}. \qquad (3.1)$$

This gives rise to the only following two cases:

(i) $\alpha = -\beta = 1/2, \quad \delta = 0$ and $T = \cosh 2at$

(ii) $\dot{\mu} = 0, \ a = 0$ and $T = e^{kt}$

Case (i): The density ρ is given by

$$8\pi\rho = 8\pi p = (2b - a^2)e^{-2br^2}\cosh^{-1}(2at) \qquad (3.2)$$

which will be positive for $2b \geq a^2$. The equality will give the 5-d singularity-free non-diagonal vacuum spacetime.

The metric will read as

$$\begin{aligned} ds^2 &= e^{2br^2}\cosh(2at)(dt^2 - dr^2) - r^2\cosh(2at)d\varphi^2 \\ &\quad - \cosh^{-1}(2at)(dz + ar^2 d\varphi)^2 - d\psi^2 \end{aligned} \tag{3.3}$$

This is the 5-d generalization of the 4-d non-singular Mars [14] stiff fluid model (1.1) which follows when ψ =const. The behaviour of the model is typical of the non-singular models [10-11]. That is $\rho \to 0$ as $t \to \pm\infty$ or $r \to \infty$, as t increases ρ increases with the contraction, reaching the maximum value indicated by the parameter $2b - a^2$ at $t = 0$ and $r = 0$. At $t = 0$, contraction turns into expansion and $\rho \to 0$ as $t \to \infty$.

When $a = 0$, the metric becomes diagonal and static representing a static stiff fluid model with ρ being maximum at $r = 0$ and exponentially going to zero as $r \to \infty$. It is the 5-d analogue of the 4d stiff fluid solution [15-16] which reads as follows:

$$ds^2 = e^{2br^2}(dt^2 - dr^2) - r^2 d\varphi^2 - dz^2 - d\psi^2. \tag{3.4}$$

Case (ii): Here we have $a = 0, T = e^{kt}$ and

$$8\pi\rho = 8\pi p = \frac{1}{2}[4b - k^2(\alpha^2 + \delta^2 + \alpha\delta)]e^{-2\alpha kt}e^{-2br^2}. \tag{3.5}$$

For $\rho \geq 0$ we must have

$$4b \geq k^2(\alpha^2 + \delta^2 + \alpha\delta) \geq 0 \tag{3.6}$$

where the equality will again imply a 5-d vacuum solution. The constants α and δ are undetermined. For $\alpha k > 0$, ρ decreases exponentially with increasing t and r.

The metric reads as

$$
\begin{aligned}
ds^2 &= e^{2br^2} e^{2\alpha kt}(dt^2 - dr^2) - r^2 e^{2\alpha kt} d\varphi^2 \\
&\quad - e^{-2(\alpha+\delta)kt} dz^2 - e^{2\delta kt} d\psi^2.
\end{aligned} \tag{3.7}
$$

We still have 4 parameters b, k, α and δ free. If we choose $k\delta < 0$, ψ-dimension will reduce exponentially leaving the 4-d spacetime.

The interesting particular case is $\alpha = 0$, which gives

$$
8\pi\rho = 8\pi p = \frac{1}{2}(4b - k^2\delta^2)e^{-2br^2} \tag{3.8}
$$

and the metric (3.7) reduces to

$$
ds^2 = e^{2br^2}(dt^2 - dr^2) - r^2 d\varphi^2 - e^{-2\delta kt} dz^2 - e^{2\delta kt} d\psi^2. \tag{3.9}
$$

It is noteworthy that though the metric is time dependent yet ρ is independent of t. This is the non-static generalization of the metric (3.4), which results when $\delta = 0$. The matter-free limit of (3.9) is a 5-d empty space for $4b = k^2\delta^2$ and its metric is given by

$$
ds^2 = e^{\frac{1}{2}\lambda^2 r^2}(dt^2 - dr^2) - r^2 d\varphi^2 - e^{-2\lambda t} dz^2 - e^{2\lambda t} d\psi^2 \tag{3.10}
$$

where $\lambda = \delta k$. It is an inhomogeneous and anisotropic vacuum spacetime which is everywhere regular. The dimensional reduction will occur for $\lambda < 0$. $\lambda = 0$ implies a 5-d flat spacetime.

4 Homogeneous stiff-fluid models

In this case $b = 0$ and $\alpha = \gamma$. From (2.12) it follows

$$
\delta = 0 \quad \text{or} \quad \dot{\mu} + (\alpha + \beta + \delta)\mu^2 = 0. \tag{4.1}
$$

Case (i): $\delta = 0$. From (2.10) and (2.11) we get

$$
(\alpha + \beta)\dot{\mu} + (\alpha + \beta)^2\mu^2 = 0. \tag{4.2}
$$

If $\alpha+\beta \neq 0$, equations (2.10) and (4.2) imply $a = 0$ which makes the metric diagonal. Equation (4.2) is solved to give

$$T = t^{\frac{1}{\alpha+\beta}} \tag{4.3}$$

and we have

$$8\pi\rho = 8\pi p = \frac{2\alpha}{\alpha+\beta} t^{-2(2\alpha+\beta)}. \tag{4.4}$$

The metric of spacetime is given by

$$ds^2 = t^{\frac{2\alpha}{\alpha+\beta}}(dt^2 - dr^2) - r^2 t^{\frac{2\alpha}{\alpha+\beta}} d\varphi^2 - t^{\frac{2\beta}{\alpha+\beta}}(dz - ar^2 d\varphi)^2 - d\psi^2. \tag{4.5}$$

When $\alpha = 0$ it reduces to a 5-d non-diagonal vacuum solution. The metric (4.5) has the big-bang singularity at $t = 0$.

On the other hand if $\alpha + \beta = 0$, then $T = \cosh^{1/2\alpha}(2at)$ but $\rho \sim -a^2$. Hence it is ruled out.

Case (ii): $\delta \neq 0$. In this case equations (2.10), (2.12) and (4.2) lead to the following two subcases:

When $\alpha+\beta \neq 0$, $T^{\alpha+\beta+\delta} = t$ and $a = 0$. The metric of the fluid model will be given by

$$ds^2 = t^{\frac{2\alpha}{n}}(dt^2 - dr^2 - r^2 d\varphi^2) - t^{\frac{2\beta}{n}} dz^2 - t^{\frac{2\delta}{n}} d\psi^2 \tag{4.6}$$

where $n = \alpha + \beta + \delta$. The density has the expression

$$8\pi\rho = 8\pi p = \frac{\alpha n + 2\alpha(n-\alpha) + 2\beta(n-\alpha-\beta)}{n^2 t^{2(1+\alpha/n)}}. \tag{4.7}$$

It is possible to choose α, β, δ such that $\rho \geq 0$ where the equality will represent a vacuum spacetime. The metric (4.6) reduces to (4.5) for $\delta = 0$. It admits coordinate reduction for $\delta/n < 0$. Vacuum spacetime will result for $\alpha(\alpha+3\beta+3\delta)+2\beta\delta = 0$.

When $\alpha + \beta = 0$, here again $a = 0$ and $T = t^{1/\delta}$. So we have

$$ds^2 = t^{\frac{2\alpha}{\delta}}(dt^2 - dr^2 - r^2 d\varphi^2) - t^{\frac{-2\alpha}{\delta}} dz^2 - t^2 d\psi^2 \tag{4.8}$$

and

$$8\pi\rho = 8\pi p = \frac{\alpha(\alpha+\delta)}{\delta^2 t^{2(1+\alpha)}}. \tag{4.9}$$

When $\alpha = -1$, ρ becomes constant > 0 provided $1 \geq \delta$. It is interesting that the metric potentials are in powers of t but the density is uniform. It is also remarkable that the metric (4.8) satisfies one of the Kasnerian constraints, $p_1 + p_2 + p_3 + p_4 - p_5 = 1$ but not the other. Hence it could be termed as the Kasnerlike stiff-fluid model.

Vacuum spacetimes, which will be Kasnerian, will follow when $\alpha = 0$ or $\alpha + \delta = 0$. When $\alpha = 0$, it is interesting to note that 4-d spacetime is flat and the curvature is produced solely by the extra dimensional metric potential.

5 Homogeneous perfect-fluid models

Here $b = 0$. From (2.2) we get for $R_{15} = 0, R_{11} = R_{22} = R_{33} = R_{44}$ the following set of equations:

$$(\beta - \alpha)\dot{\mu} + (\beta - \alpha)(\beta + \alpha + \delta)\mu^2 + 4a^2 T^{2(\beta-\alpha)} = 0 \tag{5.1}$$

$$(\beta - \delta)\dot{\mu} + (\beta - \delta)(\beta + \alpha + \delta)\mu^2 + 2a^2 T^{2\beta-\alpha)} = 0 \tag{5.2}$$

and $\alpha = \gamma$, $\mu = \dot{T}/T$. They lead to the following two cases:

$$\text{(i)} \quad \dot{\mu} + (\beta + \alpha + \delta)\mu^2 = 0, \qquad 2\delta \neq \alpha + \beta \tag{5.3}$$

$$\text{(ii)} \quad (\beta - \alpha)(\dot{\mu} + 3\delta\mu^2) + 4a^2 T^{2(\beta-\alpha)} = 0, \qquad 2\delta = \alpha + \beta. \tag{5.4}$$

Case (i) Here the general solution is $T = t$ with $\alpha = \frac{8a^2+5}{6}$, $\beta = \frac{8a^2-1}{6}$, $\delta = \frac{4a^2+1}{3}$.

The density and pressure are then given by

$$8\pi\rho = \frac{1}{12}(4a^2+1)(32a^2+11)t^{-\frac{8a^2+11}{3}} \tag{5.5}$$

$$8\pi p = \frac{1}{12}(4a^2+1)(11-8a^2)t^{-\frac{8a^2+11}{3}} \tag{5.6}$$

Clearly the positivity of ρ and p requires $a^2 \leq 11/8$ where equality will give a dust distribution.

For $a^2 \neq 11/8$, we have the equation of state

$$\rho = \frac{11+32a^2}{11-8a^2}p \tag{5.7}$$

from which the radiation Universe, $\rho = 3p$, will result for $a^2 = 11/28$ and $a = 0$ will give the stiff fluid $\rho = p$.

Case (ii): If $\alpha = \beta$, $a = 0$ keeping $T(t)$ arbitrary and we have the 5-d version of FRW with flat spatial section. Here $T(t)$ will behave as the scale factor. ρ and p are as follows:

$$\begin{aligned} 8\pi\rho &= \left(\frac{3\alpha-1}{\alpha}\right)\frac{\dot{S}^2}{S^4} \\ 8\pi p &= -3\frac{\ddot{S}}{S^3} \end{aligned}$$

where we have defined $T^\alpha(t) = S(t)$. Clearly $\ddot{S} \leq 0$ and $\alpha \geq 1/3$ or $\alpha < 0$. Note that $S = t$ will give the dust distribution.

When $\alpha \neq \beta$, equation (5.4) can admit $T = t$ as a solution. In that case it reduces to the case (i) considered above.

6 Discussion

¿From (2.4) we compute

$$R_{ik}u^i u^k = -\frac{16\pi}{3}(\rho + 2p) \leq 0 \tag{6.1}$$

and hence according to the Raychandhuri equation [17], singularity can be avoided only when acceleration is non-zero for a vorticity free spacetime. That is non-singular character of spacetime requires it to be inhomogeneous. In the case of diagonal non-singular models, it turns out that 5-d non-singular analogue exists only for the stiff fluid $\rho = p$ [12]. Here we have obtained the 5-d analogue of the non-singular non-diagonal stiff fluid model [14]. It cannot suffer the dimensional reduction. The parameter a is the measure of non-diagonality of the metric while b of inhomogeneity. When $a = 0$, we have the metric (3.7) with exponential r and t dependence in the inhomogeneous case. It admits the dimensional reduction for $k\delta < 0$ yielding the usual 3-space as t increases.

The perfect fluid without $\rho = p$ is admissible only in the homogeneous case. Here we obtain a family of 5-d models which includes a 5-d version of FRW flat model.

All homogeneous models are as expected big-bang singular. The metric (4.6) will admit the dimensional reduction when $\delta k < 0$ as before. It turns out that dimensional reduction is possible only when $a = 0$, i.e. the metric turns diagonal.

The metric (4.8) is interesting in the sense that it is Kasnerlike satisfying one of the Kasnerian constraints. It can hence be called a Kasnerlike stiff fluid model which can have uniform density for $\alpha = -1$. It cannot suffer the dimensional reduciton. It has the 4-d analogue in the form $g_{rr} = -t, g_{\varphi\varphi} = -rt, g_{zz} = -1, g_{tt} = t$ and $8\pi\rho = t^{-4}$.

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