

Machian gravity and the giant galactic forces

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One of the main motivations behind formulating the general theory of relativity was to provide a mathematical description to the Mach's principle. However, soon after its formulation, it was realized that the theory does not follow Mach's principle. As the theoretical predictions were matching with the observations, Einstein believed that the theory was correct and did not make any farther attempt to reformulate the theory to explain Mach's principle. Later on, several attempts were made by different researchers to formulate the theory of gravity based on Mach's principle. However most of these theories remain unsuccessful to explain different physical phenomena. In this report I have proposed a new theory of gravity which is completely based on the Mach's principle. The theory can explain the galactic velocity profiles up to a high degree of accuracy, without demanding the existence of any dark matter component in the universe. The theory can also explain the accelerated expansion of the universe without Dark Energy [17]. It is a metric theory and can be derived from the action principle, which guarantees energy or momentum conservation. Modern theories like TeVeS or Modified gravity give some mathematical modification of the Einstein's equation to explain different observational phenomenon like the galactic velocity profile, accelerated expansion of the universe or CMBR power spectrum. Most of these theories don't have any underlying logic. However, the new theory explained in this paper, is a mathematical formulation of the Mach's principle. Therefore, the theory is not just a mathematical jugglery to explain some observed facts, but arises from a deep physical argument.

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I. INTRODUCTION

Newtonian gravity, can provide a very accurate description of gravity, provided the gravitational field is weak, not time varying and the concerned velocity is much less than c . It can describe the motions of planets around the sun and the motions of satellites around the planets up to a very high accuracy. The general theory of relativity, which is the most sophisticated theory of gravity is designed to follow Newtonian gravity at large scale. As Newtonian gravity has been tested very accurately at the solar system scale, therefore there is a common belief that it should provide the correct explanation for the motions of stars revolving round the galaxies. However, astronomical observations show that Newtonian gravity fails reproduce the galactic velocity profiles, provided the velocity profiles are calculated from the visible matters in the galaxy. Therefore, considering Newtonian law of gravity as the correct description of gravity on the large scale, researchers have postulated that some new form of invisible matter, called dark matter, is responsible for the velocity profiles of the stars in a galaxy. It has also been postulated that SUSY particles are responsible for dark matter. However, present studies show that the possibility of existence of SUSY particles is almost negligible. Therefore, it will be more realistic to change the form of the Newtonian law of gravity instead of postulating the existence of such unknown matter.

Several new theories are postulated which can explain the galactic rotation curves without any form of dark matter. However, most of these theories are postulated just to explain the galactic rotation curves and have no physical basis. Some of these theories like MOND, [13, 14] explain the galactic rotation curves well, but violates many other physical laws including most basic laws, like the law of momentum conservation. On the other hand, theories like Modified gravity [9–12] and TeVeS [8] theory need extra scalar fields, vector fields and more than one random form of potential along with the general relativistic tensor field to explain the galactic velocity profiles. Therefore, even though these theories explain many observational phenomena without demanding any new form of matter, they can not be taken seriously.

A detailed analyses of these new theories show that all these have something in common. All these theories use additional vector fields and scalar fields along with the Einstein's equation. In Modified Gravity [9], Moffat modifies the GR Lagrangian with the addition of an extra massive vector field. He also considered that all the coupling terms and the vector field mass are varying with space-time. Therefore, he uses scalar fields to describe these variations. On the other hand, in Bekenstein's AQUAL and TeVeS [8] he adds extra scalar field and vector field to modify the

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metric curvature that can explain the galactic velocity profiles. Therefore, if our logic can provide us with a scalar fields and a vector fields similar to the above theories, then we can attempt to explain all the observational results without violating any well-established physical principles.

Therefore, in this report, I have proposed a theory that can explain the galactic velocity profile data and the accelerated expansion of the universe without demanding any form of dark matter or dark energy. The theory is completely based on the Mach's principle. While postulating the theory one should keep in mind the following principles

- Action principle : The theory proposed in this paper can be derived from an action principle, because that is the simplest way to guarantee all the conservation laws, such as the energy, momentum and the angular momentum conservations.
- Equivalence principle : In the previous papers [16, 17], it has been discussed that the Weak Equivalence Principle can not be the correct form of equivalence principle, as it does not consider the background contribution on mass of a particle. If it is considered that the inertial properties, especially the inertial mass of a particle, depend on the background, then depending on the position of the particle in the universe, the inertial mass of the particle will change, therefore, the postulate of the WEP cannot be true. A detailed logical description of this point is given in a later section.
- Special Relativity : In the previous papers [16, 17] it was discussed that that special relativity can come from a more general 5D theory in the limit $\hbar \rightarrow 0$. The 5D theory that was proposed can describe quantum mechanics from a completely classical point of view. As this paper is an extension of the previous work, we should not demand that the special theory of relativity would hold correct.
- Departure from Newtonian gravity : Newtonian gravity has been tested several times at the solar system scale. Therefore, the proposed theory should follow Newtonian gravity in a scale small enough compared to the Galactic scale, such as the solar system scale. Even at such scales, if the gravitational field is too strong or time varying, the general theory of relativity is needed. However, this paper considers that both these theories, i.e. Newtonian gravity and GR will break down at a scale of the order of few Kpc.
- Mach's principle : The theory has been designed to follow the Mach's principle considering that Mach's principle gives the correct logical explanation of inertia. It has been considered that the inertial mass of an object is not an intrinsic property of the object and gets its contribution from all other particles in the universe. A brief mathematical explanation can be given as follows. Let us consider that A and B are two different locations in space. Also, consider that a, b are two particles in the universe. Now according to our postulate m_i can be given by

$$m_{ia}(A) = \sum_{b \neq a} m_i^{(b)}(A) \quad (1)$$

Here $m_i^{(b)}(A)$ is some scalar contribution to the inertial mass of the particle a , which is located at A , from the particle b which is located at the position B .

The theory of gravitation, which is proposed in this paper, is based on the above postulates. The logical and the mathematical descriptions of the theory have been discussed in the following sections and are arranged as follows. The next section provides the logical description of the theory. In the third section, some of the mathematical tools, which are used for formulating the theory, are discussed. The fourth section describes the source free field equations for the theory. In the fifth section, the field equations are calculated in presence of source terms. The stress energy tensor for the perfect fluids in this new theory has also been proposed there. An approximate solution for the spherically symmetric static vacuum is presented in section six. The seventh section describes the galactic velocity profiles briefly. Finally, the last section gives the conclusions.

II. LOGICAL DESCRIPTION OF THE THEORY

This section presents the logical description of the theory, proposed in this paper. As this paper is a continuation of the previous papers [16, 17], the discussions of a previous papers have not been repeated here. A brief summary of the last paper can be presented as follows. According to the Mach's principle, the inertial properties of matter

depend on the background. Therefore, if two identical masses are kept at two different locations in the universe such that the backgrounds of those two different points of the universe are not the same, then the inertial masses of those two particles will show different properties depending on the background. In such a case, to preserve the energy conservation principle valid, an extra energy term is needed to be introduced to save the energy conservation principle, this extra energy should be coming from the background of the particle. As this energy is fully independent from space and time, and depends on the position of all the particles in the universe, we have assumed that this energy is related to one other coordinate dimension which somehow quantifies the background. This coordinate dimension is labelled as ζ . While traveling from one place in the universe to another, the value of ζ will change depending on the background. Hence, this extra energy should be added to the expressions for the particles energy if the energy conservation principle has to be kept valid.

A. Mach's principle

This section of the paper gives a brief description of the Mach's principle. A detailed discussion of Mach's principle is given in [2]. Mach's principle is somehow related to the measurement of the velocity or the acceleration of a particle. While measuring the velocity or acceleration of a particle, it is measured with respect to some other object, i.e. these measurements are always relative. Therefore, when someone says that a train is running with a velocity v and an acceleration a then we mean that the velocity or the acceleration of the train is measured with respect to the surface of the earth. Now, the earth is moving around the sun and the sun is circling our galaxy. The galaxy also has some random velocity inside the galaxy cluster and so on and so forth. Therefore, if the origin of the coordinate system for measuring the velocity and the acceleration of the train is chosen to be at the center of the galaxy then the velocity and the acceleration of the train will be completely different. Therefore, it is always important to choose a perfect coordinate system for measuring the velocity or the acceleration of a particle.

Now, for getting into the details of the Mach's principle, the following example can be considered [2]. Suppose a stone is tied with a string and is whirled around in a circle. Now, we can define two reference frames, one with origin at the center of the circle, which will be fixed with respect to us. The other reference frame fixed at the stone. We can analyze the forces on the stone, using the Newton's law, in the reference frame that is fixed at the center of the circle. If m_i is the inertial mass, v is the velocity of the stone, r is the radius of the circle and T is the tension on the string then using Newton's law, we can write

$$m_i \frac{v^2}{r} = T \quad (2)$$

Now, the same analysis can be done in the second reference frame, i.e. the frame that has its origin fixed at the stone. In that frame, the stone is at rest and therefore $v = 0$. Therefore, the left hand side of Eq.(2) will be zero. However, the right hand side which is actually representing the force on the string towards the center, is remaining as T . Therefore the equality of the Eq.(2) will not hold in this frame. Hence, Newton's laws are valid only in a reference frames, which have no acceleration. In accordance with the Newton's postulate, this reference frame is called the absolute frame or the inertial frame. Newton was well aware of this fact and he had postulated some fictitious forces that will be generated in any non inertial frame to balance the equations. These forces are usually called the inertial forces, and these have no existence outside mathematics. In this example, the fictitious force, which is generated, is called the centrifugal force. Its magnitude is equal and opposite to the force T , i.e. $-T$, and it helps us to balance the equations.

Therefore, the Newton's laws are only applicable to the inertial reference frames, i.e. the frame with no acceleration. But as it has been discussed, acceleration is not an absolute quantity and it has to be defined with respect to some other frame. So, while defining the inertial reference frame there is no way to say that the frame is somehow special because there is no other frame based on which we should calculate its acceleration. In the above example while defining the inertial reference frame, I have measured the acceleration with respect to the surface of the Earth. But the earth is rotating around its own axis and is moving around the sun that is again moving around our galaxy. Thus, the reference frame, which I have considered as an inertial frame, is not acceleration free, and so it is not an inertial frame in the true sense. Newton's law is not thus fully applicable in that frame and the expression used in Eq.(2) is not accurate as different type of inertial forces will act on this reference frame. However, quantitatively these inertial forces are so negligible that even without considering these forces one can come up to a fairly good approximation for the motion of the stone.

But question about fixing the inertial coordinate system remains. Mach [7] proposed to fix the inertial reference frame by looking at acceleration of the frame with respect to the distant stars. Therefore, the distant stars or galaxies

etc somehow define the inertial properties of matter. A detailed look at Mach's principle shows that it opposes the General theory of Relativity.

In the following section a brief discussion of the logic behind General theory of relativity is given. Then I shall show that Mach's principle gives some phenomenon that the General theory of Relativity cannot explain.

B. General theory of relativity and space-time curvature

In this section, the General theory of relativity and the space-time curvature have been explained by a thought experiment as described by Einstein[1, 15]. It can provide the basic understanding of General theory of relativity.

The main logic behind Einstein's General theory of Relativity is as follows. According to the Newton's law of gravity, it is known that if we place a particle in a gravitational field then

The inertial mass of the particle $\times$ acceleration of the mass = passive gravitational mass $\times$ strength of the gravitational field at the place

Then considering the inertial mass to be equal to the passive gravitational mass the conclusion was that in a small region of space-time it is not possible to distinguish between the acceleration and the gravitational field. This makes the statement of the Einstein's Equivalence principle. Next for deriving the general theory of relativity, he had shown how the acceleration can be realized by the curvature of space-time. The following experiment could be considered to show that the acceleration can be described by the space-time curvature.

Let us consider the special theory of relativity in a non-inertial frame, i.e. the frame which has some acceleration relative to some inertial frame (probably Mach's definition for the inertial frame can be used for choosing the inertial frame). Let us consider that we have a coordinate system, say K . Also assume that all the masses are far away from the origin of the coordinate system K and therefore no gravitation is acting on the coordinate system K . Hence the coordinate system K can be considered as an inertial system. Take another coordinate system K' whose z axis is aligned with that of K . The coordinate system K' is rotating about its z axis with a constant angular velocity. Now, as K' has some acceleration, we can say that K' is not an inertial reference frame. Therefore, the laws of rigid bodies defined by the special theory of relativity will not hold in this reference frame. Now let us consider a circle drawn on the $x'y'$ plane of the K' reference frame. Let us think that a large number of identical rigid rods are placed along the perimeter and a diameter to cover the perimeter and the diameter of the circle. Let the length of all these rods be l and the number of rods along the diameter be D and the number of rods along the perimeter be P in frame K . Now if K' was not rotating relatively to K then the ratio

$$\frac{Pl}{Dl} = \frac{P}{D} = \pi \quad (3)$$

But as K' is rotating, we will get a different result. Now suppose, at a definite time we determine the ends of all the rods. Now as the rods are moving with respect to the K frame, the rods along the perimeter would have undergone length contraction. Therefore the length of the rods along the perimeter will be say l' in the K' reference frame in which the rods are at rest. But the rods along the diameter will not face any such length contraction. Therefore, now the ratio of the perimeter vs. diameter will be

$$\frac{Pl'}{Dl} = \frac{P}{D} \frac{l'}{l} = \pi \frac{l'}{l} > \pi \quad (4)$$

It shows that the geometry in the K' reference frame is not the Euclidean geometry.

Similar thing will happen if we place two clocks, one at the center of the circle and another at some point in the periphery. In the K frame, the center of the circle is fixed, but the clock at the periphery of the circle moves at some constant speed. Therefore, in the K frame the clock at the center of the circle moves faster than that of the clock at the periphery of the circle, due to time dilation. Therefore, the conclusion is that the space-time cannot be defined in the same way in the non inertial frame K' , as they are defined in the inertial frame K . In other words, the geometry of space-time in a non-inertial frame is not a Euclidean geometry. Einstein had shown that in such cases the space-time structure could be explained using Riemannian geometry.

Now as it is argued that acceleration and gravitational field cannot be distinguished within a small enough region of space-time, Einstein, in General theory of relativity relates the space-time curvature with the stress energy tensor to formulate the theory of gravitation.

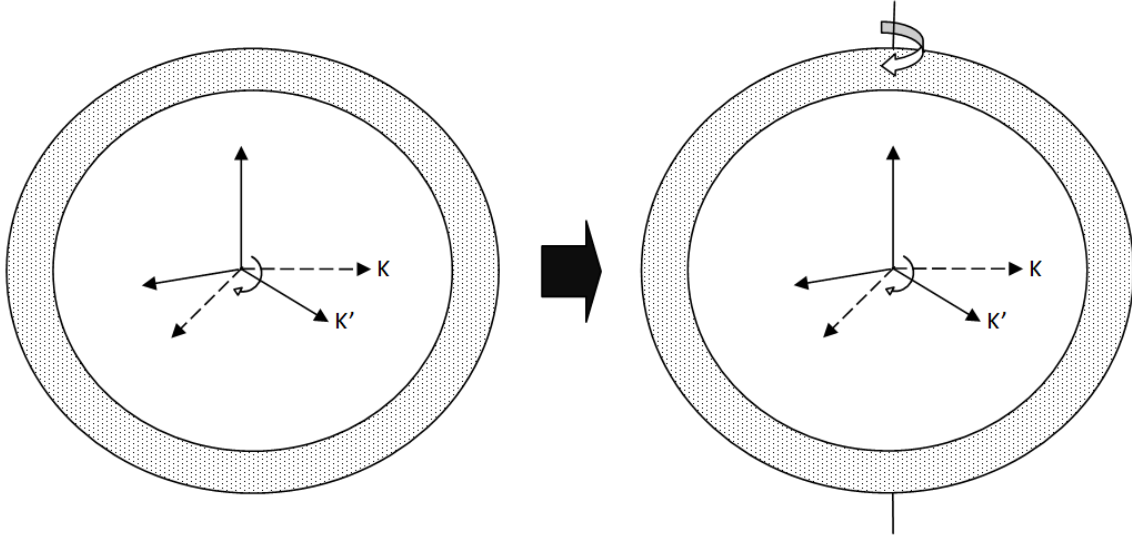


FIG. 1: Left: K is fixed and K' is rotating along z -axis. The background is fixed in K reference frame. So K is an inertial reference frame and K' is non-inertial reference frame. Right: The background starts rotating such that it is fixed in K' reference frame. In this case K' will become an inertial reference frame and K becomes non-inertial reference frame.

C. Rotation of background and Mach's principle

The discussions in the previous sections confirm that it is difficult to fix an inertial reference frame. As the acceleration is a relative measurement, it is not possible to fix a point with respect to which the acceleration of an inertial reference is measured. Therefore, Mach's logic for defining the inertial coordinate system can be taken into consideration. However, this logic creates some problems in the previous thought experiment. In the thought experiment the frame K has been defined as inertial frame. So if there is an observer in the K frame she will see that the background created by all the distant stars and galaxies is static or moves with a constant velocity.

Let us consider that the all the background is static in the K reference frame. The observer, who is sitting in the K' reference frame, will see that the background stars and galaxies etc are rotating in her reference frame. Therefore, according to Mach's principle, just by observing the rotational motion of the background objects she can conclude that Newton's laws or Euclidean geometry cannot be applied to her coordinate frame because she has some acceleration.

Now let the entire background start rotating in such a way that the observer in the K' reference frame finds herself in a static position with respect to the distant stars. Obviously, the observer who is sitting in the K reference frame will see that the background stars are rotating. Therefore, even though the observers at the K or K' reference frame have done nothing they will find that their definition for the inertial reference frame gets flipped. According to Machian concept about an inertial frame, in this present situation the K' reference frame will be an inertial frame and K will become a non inertial reference frame. Hence, the Newton's laws and Euclidean Geometry will be applicable in the K' frame and the K frame will behave in a completely different way.

Now, in accordance with the example, the coordinate frame K is fixed and K' is rotating with respect to the background, which causes the space-time to curve in K' reference frame and the Euclidean geometry cannot be applied in the K' frame but can be applied in the K frame. Then the background starts rotating which makes the K' reference frame fixed with respect to the background and K reference frame moving. Hence, the K' reference frame becomes an inertial reference frame. But the space-time in the K' reference frame was curved previously as it was a non inertial frame. Therefore, there should be some mathematical mechanism, which can make a curved space-time flat and a flat space-time curved, when the background rotates.

The curved geometry of the four-dimensional space-time does not have any term that can measure the effect from the background. Hence, Einstein's General theory of Relativity cannot explain the phenomenon discussed above. In the previous paper it was discussed that the contribution from the background can be incorporated by adding a fifth dimension (ζ) which can quantify the background. In the following sections, it will be discussed that if the five-dimensional coordinate system is curved then all the effects mentioned above can be explained. This also shows that the theory presented in this paper is more feasible than the General Theory of Relativity. The theory can also

explain the galactic velocity profiles up to a higher degree of accuracy. The accelerated expansion of the universe without any dark energy can also be explained with the same theory.

III. DEFORMATION OF SPACE-TIME FROM THE BACKGROUND

In this section some of the properties of the metric and Einstein's tensor (G_{AB}) in 5 dimension have been explored. These properties will be used while describing the theory. The 5 dimensional coordinate system is composed of one time, 3 spatial and one background component, i.e. (t, x^i, ζ) . The line element can be written as

$$ds^2 = \tilde{g}_{AB} dx^A dx^B \quad (5)$$

where $\tilde{g}_{AB}$ is the 5 dimensional metric. The following convention has been used for the tensor indices. The indices $A, B, C, \dots$ run from 0 to 4. The indices $\alpha, \beta, \gamma, \dots$ run from 0 to 3, and the indices $i, j, k, \dots$, i.e. the spatial indices run from 1 to 3. The fifth dimension i.e. ζ is a space-like coordinate [16, 17]. As the 5th dimension is nothing special compared to the other four dimensions, one should demand that the line element will remain invariant under any deformation of $\tilde{g}_{AB}$.

If there is no deformation of the metric from the background dimension, i.e. if all the $\tilde{g}_{A4}$ are constants then one can expect to get back the 4 dimensional General theory of Relativity. Along the fifth dimension, the metric will remain flat. However, if the fifth dimension somehow deforms space-time, i.e. if $\tilde{g}_{A4}$ are not constant and are functions of (t, x^i, ζ) then there will be some contribution in the 4 dimension at curvature tensor from the fifth dimension. Therefore, the entire theory may be different.

For checking the effect of the fifth dimension on the general theory of relativity, the five dimensional metric can be written in the 4+1 dimensional form, using Kaluza-Klein technique and its effects can be analyzed. The 5 dimensional metric $\tilde{g}_{AB}$ can be written in 4+1 dimensional format as

$$\tilde{g}_{AB} = \begin{pmatrix} g_{\alpha\beta} + \kappa^2 \phi^2 A_\alpha A_\beta & \kappa \phi^2 A_\alpha \\ \kappa \phi^2 A_\beta & \phi^2 \end{pmatrix} \quad (6)$$

and

$$\tilde{g}^{AB} = \begin{pmatrix} g^{\alpha\beta} & -\kappa A^\alpha \\ -\kappa A^\beta & \frac{1}{\phi^2} + \kappa^2 A^\mu A_\mu \end{pmatrix} \quad (7)$$

Here $g_{\alpha\beta}$ is a 4 dimensional metric, A_α is a 4 dimensional vector and ϕ is a scalar.

Einstein's tensor is an important component in the Einstein's field equation. Therefore to check how the 5 dimensional Einstein's tensor projects itself in the 4 dimensional space, I can take 5 dimensional $\tilde{G}_{AB} = 0$, and using Eq.(6) and Eq.(7) break it apart into the 4 dimensional $G_{\mu\nu}$, in the same way as that of Kaluza Klein technique but without making the fifth dimension compactified. This will give the following results

$$G_{\alpha\beta} = \frac{\kappa^2 \phi^2}{2} (g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} / 4 - F_\alpha^\gamma F_{\beta\gamma}) - \frac{1}{\phi} [\nabla_\alpha (\partial_\beta \phi) - g_{\alpha\beta} \square \phi] + P_{\alpha\beta} \quad (8)$$

$$\nabla^\alpha F_{\alpha\beta} = -3 \frac{\partial^\alpha \phi}{\phi} F_{\alpha\beta} + Q_\beta \quad (9)$$

$$\square \phi = \frac{k^2 \phi^3}{4} F_{\alpha\beta} F^{\alpha\beta} + U \quad (10)$$

Here $F_{\alpha\beta} = A_{\alpha;\beta} - A_{\beta;\alpha}$ is the field tensor. $P_{\alpha\beta}$, Q_β and U are the terms containing the derivatives of the metric elements with respect to the fifth dimension i.e. ζ .

In Eq.(8), as the left hand side is a tensor, the entire right hand side is also a tensor in 4 dimension, though the individual terms in the right hand side may not follow the tensorial properties. Eq.(8) shows a nice property. The left hand side of the equation is same as that of the Einstein's tensor used in General theory of Relativity. But as the calculations are done in 5 dimension, therefore the 4 dimensional Einstein's tensor comes up with some terms in the

right hand side, even in absence of any matter in the space-time. It can be shown that these extra terms can behave in exactly the same way as matter behaves and curve the space time [5]. Therefore, the fifth dimension can curve the space-time and give the effects, which are necessary to follow the Mach's principle.

If the inertial reference frame is actually defined by looking at the background, i.e. if the acceleration of a particle is actually measured with respect to the background then if the background is fixed with respect to some reference frame then no object will feel any force in the reference frame, provided there is no other means to apply force on the object. In such a case the terms in the right hand side of Eq.(8) will become zero. However, if in some coordinate system the background objects start accelerating then the terms on the right hand side of Eq.(8) will become nonzero. Therefore, the objects in that reference frame will feel a force. Such reference frames are called non-inertial reference frames and the forces that an object feels in these frames are called the inertial forces. Therefore, the Mach's principle is fully satisfied by our equations.

IV. MACH'S PRINCIPLE AND THE FIELD EQUATIONS IN ABSENCE OF SOURCE TERMS

According to the General theory of relativity, there is no way to distinguish between the gravitational force and the acceleration. However, the acceleration here is measured in an inertial reference frame. Discussions in the previous sections show that defining an inertial reference frame is difficult because that depends on the background. Problem with GR is that there is no term that can measure the contribution from the background. But, in the 5 dimensional theory, as there is an extra dimension which can measure the effect of the background, no such choice of the inertial frame is needed. The theory can be applied to any reference frame.

Here we have calculated the field equations in absence of any gravitating mass. In such a case, we should expect the five dimensional flat line element that is discussed in the previous paper to be valid in some coordinate system. So, in that case

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - \frac{\hbar^2}{4} d\zeta^2 \quad (11)$$

Here all the elements of $\tilde{g}_{AB}$ are constants. Therefore, if the Riemann tensor is calculated in this coordinate system then all the components of the five dimensional Riemann Tensor, $\tilde{R}_{BCD}^A$ will vanish. Therefore, the required field equations in source free (gravitating matter free) case will anyway be satisfied if all the components of $\tilde{R}_{BCD}^A$ vanish. In this case there is no point in choosing any inertial coordinate reference frame because vanishing of $\tilde{R}_{BCD}^A$ does not imply the vanish of $R_{\nu\alpha\beta}^\mu$. In a non-inertial reference frame $R_{\nu\alpha\beta}^\mu$ will not vanish even though $\tilde{R}_{BCD}^A$ vanishes, and the non-vanishing $R_{\nu\alpha\beta}^\mu$ take care of all the inertial forces.

It is also clear from the equations that if there is any gravitating object then it can curve the five dimensional coordinate system. Hence, that will give a nonzero $\tilde{R}_{BCD}^A$, and that non-vanishing geometry cannot be taken away using any choice of coordinate system. This implies that in presence of gravitating matter no choice of coordinate system can make all the components of $\tilde{g}_{AB}$ constant.

The Einstein's tensor $\tilde{G}_{AB}$ which is a derived quantity from $\tilde{R}_{BCD}^A$ is used in the General theory of relativity. Therefore, same quantity can be used in the field equations of our theory. In a source-free condition, the vanishing Riemann curvature tensor implies that the Einstein tensor also vanish, i.e.

$$\tilde{G}_{AB} = 0 \quad (12)$$

This gives the source free field equations for the theory. The motivation behind taking this particular condition as the field equation can be describe as follows. For calculating the actual field equations one should find some 2^{nd} order tensor which should follow the following criteria

1. It should not contain any differential coefficients of $\tilde{g}_{AB}$ higher than two.
2. It must be linear in these second order differential coefficients.
3. Its divergence must vanish identically.

Ricci tensor is the only 2nd rank nonzero tensor which can be formed from the Riemann tensor $\tilde{R}_{BCD}^A$. $\tilde{G}_{AB}$ is a quantity which is derived from the Ricci tensor and follows all the above properties. Therefore, $\tilde{G}_{AB}$ is used to represent the field equation. It is also trivial to show that the $\tilde{G}_{AB} = 0$ implies $\tilde{R}_{AB} = 0$.

V. MACHIAN GRAVITY IN PRESENCE OF THE SOURCE TERMS

The source free field equations from the Machian gravity theory is discussed in the previous section. In this section, the field equations have been derived in presence of source terms. Then, we have discussed how the theory follows all the conservation equations.

A. Equation of motion and the field equation

If there is no external force acting on a particle, then the particle will follow a geodesic path. As it is considered that gravitation is not a force and can be represented by space-time curvature, any particle in a gravitational field will follow a geodesic path. If some coordinate system is considered, then the geodesic equation in that coordinate system can be written as

$$\frac{d^2 x^A}{ds^2} - \tilde{\Gamma}_{BC}^A \frac{dx^B}{ds} \frac{dx^C}{ds} = 0 \quad (13)$$

here $A, B, \dots$ varies from 0(1)4.

We should consider the following assumptions. Firstly we will consider that the velocity of the point at which we are measuring the gravitational field is very small compared to c . Therefore, we shall put $\frac{dx^i}{ds} \approx 0$. We will also consider that the rate of change of the background dimension is very small, i.e. $\frac{d\zeta}{ds} \approx 0$. Under these assumptions it can be considered that $ds \approx dt$.

As $\frac{dx^0}{ds} \approx 1$ under the above conditions, our equation of motion for a particle in the a gravitational field will be

$$\frac{d^2 x^A}{dt^2} = \tilde{\Gamma}_{0,0}^A \quad (14)$$

Now the above equations shows that the above $\tilde{\Gamma}_{00}^A$ somehow plays the role of the Newton's gravitational field at least for the spatial components i.e. for $A = i$. We can consider that this equation is valid for all the components i.e. for 0^{th} component as of Einstein and also that of 4^{th} component, which is coming from the background.

The above equation expresses the influence gravitation upon a material particle.

We will consider an almost flat coordinate system. Therefore, the metric can be written as

$$\tilde{g}_{AB} = \text{diag}(1, -1, -1, -1, -1) + \tilde{\gamma}_{AB} \quad (15)$$

Now

$$\tilde{\Gamma}_{00}^A = \frac{1}{2} \tilde{g}^{AB} (2\tilde{g}_{B0,0} - \tilde{g}_{00,B}) \quad (16)$$

Considering the variation of $\tilde{g}_{B0}$ with respect to time to be very small, we can take the $\tilde{g}_{B0,0} \approx 0$. This gives

$$\tilde{\Gamma}_{00}^A = \frac{1}{2} \tilde{\gamma}_{00,A} \quad (17)$$

All these approximations are taken to relate the first order approximated term from our theory with the Newton's law. Then the field equation can be derived from the approximated value. Eq.(14) and Eq.(17) leads us to

$$\frac{d^2 x^A}{dt^2} = \frac{1}{2} \frac{\partial \tilde{\gamma}_{00}}{\partial x^A} \quad (18)$$

Newton's law gives us

$$\frac{d^2 x^A}{dt^2} = \frac{\partial \varphi}{\partial x^A} \quad (19)$$

where φ is the Newtonian potential. Now under the above approximations, the equation from our theory should reduce to Newton's gravitational equations, in the solar system scale. Therefore, comparing Eq.(18) and Eq.(19) we can get

$$\tilde{\gamma}_{00} = 2\varphi \quad (20)$$

In the source free case we find the field equation as $\tilde{G}_{AB} = 0$, which gives us $\tilde{R}_{AB} = 0$. A few trivial calculations can show that under the above approximations $\tilde{R}_{00} = \partial_A \partial^A \tilde{\gamma}_{00}$. As we are considering the case of time independent gravitational field, therefore the time derivative will become zero. In addition, as it has been considered that our equation should follow Newton's gravitational law in small distance, in small distance the derivative with respect to ζ can be neglected because the variation of ζ will be very small there. Therefore it will give $\tilde{R}_{00} = \nabla^2 \tilde{\gamma}_{00}$. From Eq.(20) we also know that under such approximations $\tilde{\gamma}_{00} = 2\varphi$. Also, Poisson equation for Newton's gravitational law is $\nabla^2 \varphi = 4\pi\rho$. Therefore to get a good theory we should relate $\tilde{R}_{00}$ to $8\pi\rho$. Einstein tensor $\tilde{G}^{AB}$ can be calculated from $\tilde{R}^{AB}$, and it has some beautiful properties. Therefore, it will be convenient to use $\tilde{G}^{AB}$ in the field equations in the same way as in GR. We will relate $\tilde{G}^{AB}$ with a tensor $\tilde{T}^{AB}$, where $\tilde{T}^{AB}$ can be called the five dimensional stress energy tensor. Now for deriving the stress energy tensor, it should be kept in mind that the field equation should give $\tilde{R}_{00} = 8\pi G\rho$ under the approximations made in this section.

We have related $\tilde{G}^{AB}$ with the stress energy tensor by the similar equation as that of Einstein.

$$\tilde{G}_{AB} = 8\pi G \tilde{T}_{AB} \quad (21)$$

B. Lagrangian function and the Energy momentum conservation

In the previous section, the field equations from the theory are derived. In this present section it is shown that the field equations can be derived from the Lagrangian principle. This is the simplest way to show that none of the conservation principle will be violated by the field equations.

For the theory, the Lagrangian density can be defined as $\mathcal{L} = \sqrt{-\tilde{g}}\tilde{R}$. Now any volume element at any point in a space remains same under any choice of coordinate system because the volume element is a property of geometry of space. Now, $\int \int \int \int \sqrt{-\tilde{g}} dx^5$ gives volume element at any point of space. Also the Ricci scalar $\tilde{R}$ is a property of the geometry of space. As we have defined the Lagrangian as $\int \int \int \int \sqrt{-\tilde{g}} \tilde{R} dx^5$, the Lagrangian density should remain invariant under any choice of coordinate system. Therefore, to get the field equations from the theory, we should vary the Lagrangian with respect to the metric tensor, i.e. $\tilde{g}_{AB}$. The variation will give us the field equation

$$\tilde{R}_{AB} - \frac{1}{2} \tilde{R} \tilde{g}_{AB} = \tilde{T}_{AB} \quad (22)$$

The right hand side of the equation i.e. $\tilde{T}_{AB}$ gives the source term which we have defined as the energy momentum tensor in the previous section. Therefore, the theory can be derived from a Lagrangian and hence will follow all the conservation principles. However, in the present case the notion of the conservation principle will not be the same as that of general relativity.

The left hand side of Eq.(22) is the Einstein's tensor which gives zero on covariant differentiation with respect to any coordinate system. This shows $T_{;B}^{AB} = 0$. This is same as $T_{;\mu}^{A\mu} + T_{;4}^{A4} = 0$. This equation represents energy-momentum conservation. It is seen that the equation has a contribution from the background. Therefore, when we talk energy and momentum conservation principle, we need to take into account the energy from the curvature of the space-time-background and the particle's own internal energy, momentum and the contribution in its energy from the background.

C. Stress energy tensor for fluid

The properties of a fluid is measured by its density (ρ) and pressure (p). Going in the same way as of the General theory of relativity, the stress energy tensor of a fluid can be defined as

$$\tilde{T}_{AB} = (\rho + p) \tilde{u}_A \tilde{u}_B + p \tilde{g}_{AB} \quad (23)$$

Here $\tilde{u}_A$ is the 5-velocity of the fluid.

It can be seen that, the $\tilde{T}_{4\alpha}$ components of the stress energy tensor are of order $O(\hbar)$ and $\tilde{T}_{44}$ is of the order $O(\hbar^2)$. Therefore, in the limit $\hbar \rightarrow 0$, these T_{4A} terms can be approximated as 0. In that case the stress energy tensor becomes equivalent to the general relativistic stress energy tensor.

According to the discussion in the previous section $\tilde{T}_{;B}^{AB} = 0$. Replacing the value of $\tilde{T}_{AB}$ from Eq.(23), we can derive the Euclidean fluid dynamics equation in the context of Machian Gravity theory.

The 5 conservation equations i.e. $\tilde{T}_{;B}^{AB} = 0$ along with the relation between p and ρ and the equation for line element i.e. $\tilde{g}_{AB}d\tilde{x}^Ad\tilde{x}^B = 0$, provides a complete solution to the motion of the fluid, provided g_{AB} is given. This is because there are total 7 equations to satisfy 7 unknowns, which are $p, \rho, u_0, \dots, u_4$.

If g_{AB} are also unknown then the field equations (Eq.(21)) along with some normalization condition $\sqrt{-\tilde{g}} = 1$, can be brought in. This will provide 16 equations for fixing 15 independent components of $\tilde{g}_{AB}$. Therefore, the equations may appear over determined. However, it should also be noted that there are 5 equations, $\tilde{G}_{;B}^{AB} = 0$, which $\tilde{G}^{AB}$ should satisfy. These equations have already been considered while considering the conservation principle. Therefore, there are essentially 11 independent equations in the field equation i.e. Eq.(21). Therefore, from the 11 equations we need to determine the 15 independent component of $\tilde{g}_{AB}$. This is always good because it shows that we have 4 degrees of freedom to choose the five dimensional coordinate system.

VI. STATIC, SPHERICALLY SYMMETRIC, VACUUM SOLUTION

Here we are interested in the static spherically symmetric vacuum solution near a gravitating body. The solution has been calculated in a weak gravitational field limit. Moreover, velocity of the objects on which the effect of the gravitational fields are been calculated, are far less than c . Later this solution will be used to show that the galactic velocity profile can be explain without requiring the presence of additional dark matters.

According to the field equation of Machian Gravity

$$\tilde{R}_{AB} = \tilde{T}_{AB} - \frac{1}{2}\tilde{g}_{AB}\tilde{T} \quad (24)$$

In the previous section, the stress energy tensor, i.e. $\tilde{T}_{AB}$ was defined as $\tilde{T}_{AB} = (\rho + p)\tilde{u}_A\tilde{u}_B + p\tilde{g}_{AB}$. As the stress energy tensor is defined in the same way as in GR, therefore one can expect everything to work in the same way as GR with very little modification. As the calculations in this section are done for a weak gravitational field, only important component of the field equation here is the 00 component. This component should give an equation similar to the Newtonian gravitational law in the small scale (solar system scale). In the galactic scale, though one may expect some departure from the Newton's law due to the background component of the equation. (Of course the constant term i.e. $8\pi G$ in the field equation has been taken as 1.)

The 00 component of the field Eq.(24), which will reduce to give us the Newton's law is

$$\tilde{R}_{00} = \tilde{T}_{00} - \frac{1}{2}\tilde{g}_{00}\tilde{T} \quad (25)$$

Of course, for the vacuum solution, the stress energy tensor will vanish and hence $\tilde{T}_{00} = 0$ and $\tilde{T} = 0$, which gives $\tilde{R}_{00} = 0$.

The calculations in the previous section show that under the weak gravitational field limit, $\tilde{R}_{00}$ can be approximated as $\partial_C\partial^C\tilde{\gamma}_{00}$. In addition, for the static solution the time derivative will vanish and we will be left with the equation

$$\partial_\zeta^2\tilde{\gamma}_{00} + \partial_x^2\tilde{\gamma}_{00} + \partial_y^2\tilde{\gamma}_{00} + \partial_z^2\tilde{\gamma}_{00} = 0 \quad (26)$$

Under the assumption of spherical symmetry, the above equation (Eq.(26)) in spherical polar coordinate system gives

$$\partial_\zeta^2\tilde{\gamma}_{00} + \frac{1}{r}\partial_r^2(r\tilde{\gamma}_{00}) = 0 \quad (27)$$

and hence

$$\partial_\zeta^2(r\tilde{\gamma}_{00}) + \partial_r^2(r\tilde{\gamma}_{00}) = 0 \quad (28)$$

The ‘separation of variables’ technique can be used for solving the above equation. Considering $(r\tilde{\gamma}_{00}) = R(r)\chi(\zeta)$, where $R(r)$ is a function of only r and $\chi(\zeta)$ is a function of only ζ , and then replacing it in Eq.(28), one can get

$$\frac{1}{R} \frac{\partial^2 R}{\partial r^2} = -\frac{1}{\chi} \frac{\partial^2 \chi}{\partial \zeta^2} = \lambda^2 \dots (\text{say}) \quad (29)$$

Here λ is a constant. Its solution can be written as

$$R = P_1 e^{\lambda r} + P_2 e^{-\lambda r} \quad (30)$$

and

$$\chi = Q_1 \cos(\lambda \zeta) + Q_2 \sin(\lambda \zeta) \quad (31)$$

Here P_1 , P_2 , Q_1 and Q_2 are constants. As explained before, the term $\tilde{\gamma}_{00}$, under weak-field approximation, is the Newtonian potential and hence it cannot increase exponentially with distance. Therefore, taking $P_1 = 0$, we can get

$$(r\tilde{\gamma}_{00}) = S + P_2 e^{-\lambda r} (Q_1 \cos(\lambda \zeta) + Q_2 \sin(\lambda \zeta)) \quad (32)$$

Here S is constant, as adding a constant will not affect any of the calculations. Now if we consider that over a place (a scale of the order of a galaxy) the background is almost similar and hence the change in ζ is really small. Therefore, here we may just take $\lambda \zeta \sim 0$ (a constant term $\hbar$ is also multiplied and hence the full term is very small). In this limit $\cos(\lambda \zeta) \rightarrow 1$ and $\sin(\lambda \zeta) \rightarrow 0$.

Replacing these limiting values in Eq.(32) we can rewrite the equation as

$$\tilde{\gamma}_{00} = \frac{S}{r} - KM \frac{e^{-\lambda r}}{r} \quad (33)$$

where K and M are positive constants. Here, M can be related with the mass inside the sphere of radius r . The above equation will take a special form if the constant S is replaced with $(K+1)M$. In this case, $\tilde{\gamma}_{00}$ will become

$$\tilde{\gamma}_{00} = \frac{M}{r} [1 + K (1 - e^{-\lambda r})] \quad (34)$$

In accordance with the previous discussion, γ_{00} under weak field approximation becomes twice Newtonian potential i.e. 2Φ . Therefore, inside a galaxy the approximate Newtonian potential will take the form

$$\Phi = \frac{GM}{r} [1 + K (1 - e^{-\lambda r})] \quad (35)$$

Here, we have put back G , the Newtonian gravitational constant. This particular form of potential has been used by Moffat [9–12] to show that it can explain the galactic velocity profiles correctly.

In Eq.(35), λ and K are the background dependent quantities. Galactic velocity profiles shows that λ is of the order of few kpc^{-1} . When r is small, $e^{-\lambda r} \sim 1$. Therefore, Φ will take the form of the actual Newtonian potential i.e. $\Phi = \frac{GM}{r}$. This will give the Newtonian gravitational equation at the solar system scale. Again in the asymptotic limit of $r \rightarrow \infty$, the exponential term goes to 0. Hence, for large values of r , although the form of the potential will remain same, it will become $(1+K)$ times that of the Newtonian potential. Thus, we have larger forces near the edges of the galaxies and the galactic velocity profiles can be explained.

VII. GALACTIC VELOCITY CURVE

In the previous section, we have seen that the Newtonian potential from this theory is given by $\Phi = \frac{M}{r} [1 + K (1 - e^{-\lambda r})]$. Therefore, at a large scale, it will give the effect of having some extra matter in the scale, which can give an alternate explanation to the galaxy velocity profiles without requiring dark matter.

The gravitational field will be given by

$$\frac{\partial \Phi}{\partial r} = -\frac{M}{r^2} [1 + K (1 - e^{-\lambda r} (1 + \lambda r))] \quad (36)$$

Moffat and others [3, 10–12] use this above form of the gravitational field and have shown that this field can explain the galactic velocity profiles accurately. For calculating the velocity of the particles in this gravitational field, the field can be compared to the centrifugal force of the particle that will give

$$\frac{v^2}{r} = \frac{M}{r^2} [1 + K (1 - e^{-\lambda r} (1 + \lambda r))] \quad (37)$$

and hence

$$v = \sqrt{\frac{M}{r} [1 + K (1 - e^{-\lambda r} (1 + \lambda r))]} \quad (38)$$

The above equation shows us that at large r the velocity of the particles will be larger than what is expected from the Newton's laws by a factor of $\sqrt{1 + K}$.

VIII. CONCLUSION

A new theory of gravitation, based on Mach's principle, has been proposed. It is a metric theory and can be derived from the action principle, which guarantees all the conservation principles. The theory can explain the galactic velocity profiles and the accelerated expansion of the universe in absence of any kind of dark matter and the dark energy. The most important thing is that unlike General theory of Relativity or Newtonian gravity, the new theory does not need any inertial frame to apply the field equations. Therefore, the new theory is more realistic than General theory of Relativity.

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