Plane waves viewed from an accelerated frame: Quantum physics in a classical setting

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(Received 14 April 1997)

We report here an analogue for the vacuum state in *classical field theory* and its Planckian nature with respect to uniformly accelerated observers. We find that when a real, monochromatic mode of a classical field is Fourier transformed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum has three separate terms none of which have a simple *classical* meaning. But they bear a striking resemblance to the *quantum mechanical* description. Specifically, the three terms are (i) a factor (1/2) that is typical of the ground state energy of a quantum oscillator, (ii) a Planckian distribution $N(\Omega)$ and, most importantly, (iii) a term proportional to $\sqrt{N(N+1)}$, which is the root mean square fluctuations about the Planckian distribution. The implications of this result are discussed. [S0556-2821(97)02722-7]

PACS number(s): 04.62.+v, 04.70.Dy

It is a well-known result in quantum field theory that the Minkowski vacuum appears as a thermal state in a system confined to the right (or the left) Rindler wedge [1]. It is also known that the response of a uniformly accelerating detector in the Minkowski vacuum is a thermal spectrum [2,3]. In both these situations, one obtains the thermal spectrum in the strict sense of the word: Not only that the mean occupation number in any mode is Planckian, but the fluctuations about the mean value is also characterized by the standard thermal noise. These results suggest that the quantum fluctuations in the Minkowski vacuum appear as thermal fluctuations in the uniformly accelerated frame.

In contrast with quantum theory, classical field theory does not admit any intrinsic fluctuations. The absence of concepts such as ''vacua'' and ''fluctuations'' in classical field theory may lead us to believe that nontrivial phenomena such as the one mentioned above will not have any classical analogue.

We report here an interesting and curious effect that arises purely in the context of *classical field theory* and has a formal similarity with well-known quantum-mechanical results mentioned in the first paragraph. We find that when a real, monochromatic, plane wave mode of a scalar field is Fourier analyzed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum consists of three terms, none of which has a simple physical interpretation in terms of purely *classical* concepts. However, they closely resemble terms which have interesting *quantum-mechanical* interpretation. We shall now present the details of the analysis.

Consider a massless, scalar field Φ which satisfies the Klein-Gordon equation. (Our results can be trivially generalized to other higher spin boson fields.) In flat spacetime, the basis solutions to the Klein-Gordon equation in Minkowski coordinates $(t,\mathbf{x})$ can be taken to be plane waves labeled by the wave vector $\mathbf{k}$:

$$\Phi(t,\mathbf{x})=\cos(\omega t-\mathbf{k}\cdot\mathbf{x}), \tag{1}$$

where $\omega=|\mathbf{k}|$. Now, consider an observer who is moving on an arbitrary trajectory $(t(\tau),\mathbf{x}(\tau))$, parametrized by the proper time τ. How will this observer view the above Minkowski plane wave mode?

The moving observer will see the scalar field varying with respect to his (her) proper time in a manner determined by the function $\Phi(t(\tau),\mathbf{x}(\tau))$. To determine the exact decomposition of the wave, we should Fourier analyze the Minkowski mode in the frame of the observer. The Fourier transform of the Minkowski plane wave with respect to the proper time τ of the observer in motion is described by the integral

$$\tilde{\Phi}(\Omega)=\int_{-\infty}^{\infty}d\tau\,\cos[\omega t(\tau)-\mathbf{k}\cdot\mathbf{x}(\tau)]e^{-i\Omega\tau}. \tag{2}$$

This expression gives the amplitude of a component with frequency Ω (as defined by the moving observer) present in the original monochromatic wave.

We shall now specialize to the case of an observer who is accelerating uniformly along the x axis with a proper acceleration g. The world line of such an observer in Minkowski coordinates $(t,\mathbf{x})$ is given by the relations

$$t=t_0+g^{-1}\sinh(g\tau),\quad y=y,$$
$$x=x_0+g^{-1}\cosh(g\tau),\quad z=z, \tag{3}$$

where t_0 and x_0 are constants and τ is the proper time as measured by the clock in the frame of the uniformly accelerated observer.

Let us first consider the case of a plane wave mode that is traveling along the x axis; i.e., the wave vector is given by $\mathbf{k}=(k,0,0)$. To see how this plane wave will be viewed by the observer accelerating uniformly along the x axis, we substitute the coordinate transformations (3) in the Fourier integral (2), and obtain that [4]

$$\tilde{\Phi}(\Omega)=\left(\frac{e^{-i\phi}}{2g}\right)\Gamma(i\Omega g^{-1})(e^{(\Omega/4\Omega_0)}e^{i\beta}+e^{-(\Omega/4\Omega_0)}e^{-i\beta}), \tag{4}$$

where

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$$\phi=\Omega g^{-1}\ln(\omega g^{-1}), \quad \Omega_0=(g/2\pi), \quad \beta=\omega(t_0-x_0). \tag{5}$$

In evaluating the above integral we have assumed that the plane wave is traveling to the right, i.e., $k=\omega$. The resulting power spectrum per logarithmic interval in frequency is given by $\mathcal{P}(\Omega)\equiv\Omega|\tilde{\Phi}(\Omega)|^2$ and can be written in a remarkable form:

$$\mathcal{P}(\Omega)=\left(\frac{\pi}{g}\right)\left\{\frac{1}{2}+N(\Omega)+\sqrt{N(N+1)}\,\cos(2\beta)\right\}, \tag{6}$$

where

$$N(\Omega)=\left(\frac{1}{\exp(\Omega/\Omega_0)-1}\right). \tag{7}$$

We shall now consider various features of this result.

To begin with we note that this result is purely classical and hence $\hbar$ does not appear anywhere. In ordinary units, $\Omega_0=(g/2\pi c)$ has the correct dimensions (viz., per second), for a frequency. The quantity N is a Planckian in terms of *frequencies* and is again independent of $\hbar$. Usually, one tries to express the Planckian distribution in terms of energies of the ''quanta'' labeled by frequency Ω and in such a case we need to write frequencies as, say, $\Omega=(E/\hbar)$, thereby *artificially* introducing $\hbar$, but the result, stated as a power spectrum in frequency space, makes perfect conceptual sense as it stands. For example, radio astronomers often measure the power spectrum of a source in frequency space (or the temporal correlation function, which is the Fourier transform of the power spectrum) and do not think in terms of photons. Of course, to obtain a quantity with the dimension of temperature we again need to introduce a $\hbar$ into the quantity Ω_0. But, since there is no real concept of temperature in the situation we are considering, we will not do so.

The analysis done above could have been carried out even before the days of quantum theory—it uses only classical relativity. Had it been done, there would have been no simple way of understanding the terms which arise in Eq. (6). But with our knowledge of quantum theory, one can offer a suggestive interpretation of the three terms in the power spectrum (6). The first term—viz., the factor 1/2—is typical of the ground state energy of a quantum oscillator. The second term N is a Planckian distribution in Ω, as already mentioned. *Note that these two terms are totally independent of the original frequency ω of the plane wave.*

The third term is still more remarkable. When we vary the constants t_0 and x_0 this term varies between $-\sqrt{N(N+1)}$ and $+\sqrt{N(N+1)}$. The magnitude of this fluctuation is exactly what one would have obtained for a strictly thermal distribution of massless bosonic quanta in quantum field theory. Thus, a classical plane wave, viewed in the accelerated frame, has a power spectrum reminiscent of the Planck spectrum with associated thermal fluctuations.

To avoid possible misunderstanding, we stress the following fact: The system we are considering has no fluctuations or temperature in the sense of statistical physics. Being a purely classical system, it does not have any quantum fluctuations either. But the terms which we get in the accelerated frame have the most natural interpretation in terms of notions like thermal spectrum and its fluctuations. [The quantity β is related to t_0 and x_0 by Eq. (5). If the original plane wave had an extra phase δ, then the argument of the cosine term will pick up 2δ additionally. For a specific choice of the constants δ, t_0, and x_0, it is possible to kill the fluctuations in the power spectrum. It is also easy to verify that one *cannot* choose the constants to cancel the first two terms as well. But—in general—all the three terms are present in the power spectrum.] We believe this result is unlikely to be a mere curiosity and deserves attention. We shall now comment on several related aspects of this result.

To begin with, the existence of the three terms is a direct consequence of our choosing a *real* plane wave which—in classical field theory—*is* mandatory. If the same analysis is repeated for a complex mode for the scalar field, say, $\Phi(t,x)=\exp[-i(\omega t-kx)]$, then the resultant power spectrum per logarithmic frequency interval will be $\mathcal{P}(\Omega)=(2\pi/g)N(\Omega)$, where $N(\Omega)$ is given by Eq. (7). We do not get the zero-point term or the fluctuations. Of course, in classical field theory, one must use *real* modes, which is exactly what we have done here.

A more intriguing aspect is the limit of our expression when we consider $\omega\to 0$. In this limit, the field in the inertial frame reduces to an unimportant constant—which could be thought of as closest to the concept of a ''vacuum'' in the classical theory. The Fourier integral as well as the phase ϕ in Eq. (5) diverges when $\omega\to 0$, but the power spectrum—which is the squared modulus of the amplitude—is well defined:

$$\mathcal{P}(\Omega)|_{\omega\to 0}=\left(\frac{\pi}{g}\right)\left\{\frac{1}{2}+N(\Omega)+\sqrt{N(N+1)}\right\}. \tag{8}$$

Treating ω as a ''regulator'' and setting it to zero at the end of the calculation, one can say that the accelerated observer will see these terms even in the limit of $\omega\to 0$. This is very reminiscent of the inertial vacuum appearing as a Planckian spectrum to the accelerated observer (in quantum field theory) in a manner which is completely independent of the original wave mode. Mathematically, this result arises because our limiting procedure does not commute with that of Fourier transforming the mode. If we take the $\omega\to 0$ limit first and then take the Fourier transform, we will—of course—get the square of the Dirac δ function as the power spectrum. But when we compute the power spectrum first and *then* take the limit of $\omega\to 0$ we get a different—and finite—result. Once again, the situation is reminiscent of regularization procedures (such as the ''$i\epsilon$ prescription'') in quantum theory in which the order of operations matter.

In a way, this limiting value turns out to be a more generic feature. In the above discussion we have assumed that the wave and the observer are moving along same direction, viz., the x axis. But the result for the $\omega\to 0$ should hold irrespective of this condition. Direct analysis shows that this is indeed the case [5].

A somewhat similar analysis, viz., Fourier analyzing the Minkowski modes in the frame of a uniformly accelerated observer, was carried out earlier by Gerlach [6]. He had constructed a linear superposition of Minkowski modes such that the modulus square of the amplitude of these modes (which represents the total classical energy of these modes) is equivalent to that of the ground state energy of a quantum oscillator. Fourier analyzing such a field configuration with respect to the proper time of a uniformly accelerating observer, Gerlach had obtained a power spectrum similar in form to Eq. (6). He had presented his result as a ''heuristic derivation of the thermal spectrum'' that arises in quantum field theory due to the inequivalent quantization in Minkowski and Rindler coordinates. *Our results and emphasis are different in several ways.* To begin with, the effect we are reporting here is a feature of classical field theory and no quantum processes are involved. It is physically motivated in a clear and simple manner and we do not have to resort to any superposition of modes. Secondly, we obtain a Planckian spectrum for a real, monochromatic plane wave, when the wave is moving in the same direction as that of the uniformly accelerated observer. On the other hand, for such a case, Gerlach, in order to obtain the Planckian spectrum, needed to resort to a suitable averaging scheme because of the particular superposition of modes he had chosen. Thirdly, we would like to draw attention to the zero-frequency limit of the wave, when it takes a life of its own in the accelerated frame. This result, as far as we know, has not been noted in the literature before. Finally, Gerlach had offered no explanation for the appearance of the factor $\cos(2\beta)$ as the coefficient of the fluctuation term. Our analysis clearly shows that it arises due to the shift in the origin of the Minkowski coordinates.

It is also worthwhile to consider the following random superposition of real modes for the scalar field:

$$\Phi(t,x)=\int_{-\infty}^{\infty} d\omega A(\omega)\cos[\omega(t-x)+\theta(\omega)], \tag{9}$$

where $A(\omega)$ is a stochastic variable satisfying the relation

$$\langle A(\omega)A(\omega')\rangle = P(\omega)\,\delta(\omega-\omega'). \tag{10}$$

Further, we shall assume that $\theta(\omega)$ is a random phase factor distributed uniformly in the range $(0,2\pi)$. Also, we shall consider $P(\omega)$ to be an arbitrary function of ω such that $C=\int_{-\infty}^{\infty} d\omega P(\omega)$ is a finite constant. We can now set $t_0=x_0=0$ in Eq. (3) without any loss of generality. Computing the power spectrum as before and averaging the power spectrum per unit logarithmic interval over the stochastic variables $A(\omega)$ and $\theta(\omega)$, we obtain that

$$\langle \mathcal{P}(\Omega)\rangle = \left(\frac{\pi C}{g}\right)\left\{\frac{1}{2}+N(\Omega)\right\}. \tag{11}$$

In this case, the random phases have averaged out the fluctuation term, viz., the factor $\sqrt{N(N+1)}$ that had appeared in the power spectrum (6). A similar result has been obtained earlier by Boyer [7]. Modeling the zero-point fluctuations of a quantum field as a random superposition of Minkowski plane wave modes, he had shown that the correlation function of a uniformly accelerating observer ''in a random classical scalar zero-point radiation'' exactly matches the correlation function of an inertial observer in a thermal background. Our analysis here shows that the effect reported by Boyer arises when a random superposition of Minkowski real modes is Fourier analyzed in the frame of a uniformly accelerating observer [cf. Eq. (11)]. *But notice that such an approach has killed a very interesting* $\sqrt{N(N+1)}$ *term which was originally present.*

In conclusion, we would like to stress those aspects of our results which are unexpected and contrast them with those which could have been anticipated with some hindsight.

To begin with, the following fact is well known: In quantum field theory, the amplitude for transition of an Unruh-DeWitt detector, up to first order in perturbation theory, is described by the Fourier transform of the normal mode of the quantum field with respect to the proper time of the detector in motion [2,3]. When the scalar field is decomposed in terms of the Minkowski modes, the transition probability, per unit proper time, of a uniformly accelerating Unruh-DeWitt detector turns out to be a thermal spectrum (see, for instance, Ref. [8]). It might, therefore, seem that when a traveling wave is Fourier transformed with respect to the proper time of a uniformly accelerated observer, the resulting power spectrum will have a Planckian nature.

However, there are some subtleties involved. To begin with, the modes of the quantum field are complex while here we are dealing with real plane wave modes. This makes the vital difference. As we have mentioned before, while a complex mode like $\exp-i(\omega t-kx)$ will give a Planckian distribution, it will *not* yield the two other terms we have obtained in our analysis. In this sense, the real wave is quite different from the complex one. We stress the fact that, when a real Minkowski mode is Fourier transformed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum not only contains a Planckian distribution, but also contains a term proportional to the root mean square fluctuations about the Planckian. *We know of no simple way to guess at this answer.*

Secondly, the effect survives in the power spectrum even in the limit of $\omega\to 0$ if the procedure is interpreted as a regularization scheme. This is the closest to what one can call a ''classical vacuum''—and our result shows that such a mode, with infinitesimal frequency, leads to a thermal ambience in the accelerated frame which is *totally independent of the properties of the original wave*. This result suggests that there is a deep connection between plane waves, accelerated frames, and thermal ambience even at the classical level. This connection could be worth exploring.

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