

Role of Modified Chaplygin Gas in Accelerated Universe

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In this paper we have considered a model of modified Chaplygin gas and its role in accelerating phase of the universe. We have assumed that the equation of state of this modified model is valid from the radiation era to Λ CDM model. We have used recently developed statefinder parameters in characterizing different phase of the universe diagrammatically.

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I. INTRODUCTION

Recent observations of the luminosity of type Ia supernovae indicate [1, 2] an accelerated expansion of the universe and lead to the search for a new type of matter which violate the strong energy condition $\rho + 3p < 0$. The matter content responsible for such a condition to be satisfied at a certain stage of evolution of the universe is referred to as *dark energy*. There are different candidates to play the role of the dark energy. The type of dark energy represented by a scalar field is often called *quintessence*. The transition from a universe filled with matter to an exponentially expanding universe does not necessarily require the presence of the scalar field as the only alternative. In particular one can try another alternative by using an exotic type of fluid - the so-called Chaplygin gas which obeys an equation of state like $p = -B/\rho$ [3] where p and ρ are respectively the pressure and energy density and B is a positive constant. Subsequently the above equation was modified to the form $p = -B/\rho^\alpha$ with $0 \leq \alpha \leq 1$. This model gives the cosmological evolution from an initial dust like matter to an asymptotic cosmological constant with an epoch that can be seen as a mixture of a cosmological constant and a fluid obeying an equation of state $p = \alpha\rho$. This generalized model has been studied previously [4, 5].

In the paper [4], the flat Friedmann model filled with Chaplygin fluid has been analyzed in terms of the recently proposed “*statefinder*” parameters [6]. In fact trajectories in the $\{s, r\}$ plane corresponding to different cosmological models demonstrate qualitatively different behaviour. The statefinder diagnostic along with future SNAP observations may perhaps be used to discriminate between different dark energy models. The above statefinder diagnostic pair are constructed from the scale factor $a(t)$ and its derivatives upto the third order as follows:

$$r = \frac{\ddot{a}}{aH^3} \quad \text{and} \quad s = \frac{r-1}{3(q-\frac{1}{2})} \quad (1)$$

where H and q ($= -\frac{a\ddot{a}}{a^2\dot{a}^2}$) are the Hubble parameter and the deceleration parameter respectively. These parameters are dimensionless and allow us to characterize the properties of dark energy in a model independent manner. The parameter r forms the next step in the hierarchy of geometrical cosmological parameters after H and q .

In section II, all the discussions are valid in general for $k = 0, \pm 1$, but in section III we have specifically considered the simple case of a spatially flat universe ($k = 0$), which naturally

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follows from the simplest version of the inflationary scenario and is confirmed by recent CMB experiments [7]. In our present work we consider a more general modified Chaplygin gas obeying an equation of state [8]

$$p = A\rho - \frac{B}{\rho^\alpha} \quad \text{with} \quad 0 \leq \alpha \leq 1 \quad (2)$$

This equation of state shows radiation era (when $A = 1/3$) at one extreme (when the scale factor $a(t)$ is vanishingly small) while a Λ CDM model at the other extreme (when the scale factor $a(t)$ is infinitely large). At all stages it shows a mixture. Also in between there is also one stage when the pressure vanishes and the matter content is equivalent to a pure dust. We have further described this particular cosmological model from the field theoretical point of view by introducing a scalar field ϕ and a self-interacting potential $V(\phi)$ with the effective Lagrangian

$$\mathcal{L}_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad (3)$$

In the paper of Gorini et al [4], it has been shown that for the simple flat Friedmann model with Chaplygin gas can equivalently described in terms of a homogeneous minimally coupled scalar field ϕ . In this case FRW equations for Chaplygin gas fits in Barrow's scheme [9]. Following Barrow [10], Kamenshchik et al [3, 11] have obtained homogeneous scalar field $\phi(t)$ and a potential $V(\phi)$ to describe Chaplygin cosmology. In the section II, we have obtained the corresponding expressions for a modified Chaplygin gas as generalization of the previous work.

It has been possible to express $V(\phi)$ in terms of the scale factor $a(t)$ for arbitrary values of the constant α and have explicitly shown how $V(\phi)$ varies as the scale factor interpolates between radiation and Λ CDM stages for different values of the constant A . In the next stage evolution of the model universe has been studied in the $\{r, s\}$ plane for the entire physically realistic history of the universe and compared to that for either pure Chaplygin gas model or a generalized Chaplygin gas model as given in [5].

II. MODIFIED CHAPLYGIN GAS IN FRW MODEL

The metric of a homogeneous and isotropic universe in FRW model is

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (4)$$

where $a(t)$ is the scale factor and k ($= 0, \pm 1$) is the curvature scalar.

The Einstein field equations are

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{1}{3}\rho \quad (5)$$

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) \quad (6)$$

where ρ and p are energy density and isotropic pressure respectively (choosing $8\pi G = c = 1$).

The energy conservation equation is

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \quad (7)$$

Using equation (2) we have the solution of ρ as

$$\rho = \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+\alpha)}} \right]^{\frac{1}{1+\alpha}} \quad (8)$$

where C is an arbitrary integration constant.

Due to complicated form of the equation of state (2), the scale factor $a(t)$ can not be solved from the Einstein field equations (5) and (6) for arbitrary k . However for $k = 0$, the solution of scale factor $a(t)$ has the form

$$a^{\frac{3(1+A)}{2}} {}_2F_1[x, x, 1+x, -\frac{B}{C(1+A)}a^{\frac{3(1+A)}{2x}}] = \frac{\sqrt{3}}{2}(1+A)C^x t \quad (9)$$

where $x = \frac{1}{2(1+\alpha)}$ and ${}_2F_1$ is the hypergeometric function.

Now for small value of scale factor $a(t)$, we have

$$\rho \simeq \frac{C^{\frac{1}{1+\alpha}}}{a^{3(1+A)}}, \quad (10)$$

which is very large and corresponds to the universe dominated by an equation of state $p = A\rho$.

Also for large value of scale factor $a(t)$,

$$\rho \simeq \left(\frac{B}{1+A} \right)^{\frac{1}{1+\alpha}} \quad \text{and} \quad p \simeq - \left(\frac{B}{1+A} \right)^{\frac{1}{1+\alpha}} = -\rho \quad (11)$$

which correspond to an otherwise empty universe with a cosmological constant $\left(\frac{B}{1+A} \right)^{\frac{1}{1+\alpha}}$.

For accelerating universe q must be negative i.e., $\ddot{a} > 0$ i.e.,

$$a^{3(1+\alpha)(1+A)} > \frac{C(1+A)(1+3A)}{2B} \quad (12)$$

This expression shows that for small value of scale factor we have decelerating universe while for large values of scale factor we have accelerating universe and the transition occurs when the scale factor has the expression $a = \left[\frac{C(1+A)(1+3A)}{2B} \right]^{\frac{1}{3(1+\alpha)(1+A)}}$.

Considering now the subleading terms in equation (8) at large values of a , we can obtain the following expressions for the energy density and pressure:

$$\rho \simeq \left(\frac{B}{1+A} \right)^{\frac{1}{1+\alpha}} + \frac{C}{1+\alpha} \left(\frac{1+A}{B} \right)^{\frac{1}{1+\alpha}} a^{-3(1+\alpha)(1+A)} \quad (13)$$

and

$$p \simeq - \left(\frac{B}{1+A} \right)^{\frac{1}{1+\alpha}} + \frac{C}{1+\alpha} \left(\frac{1+A}{B} \right)^{\frac{1}{1+\alpha}} \{ \alpha + (1+\alpha)A \} a^{-3(1+\alpha)(1+A)} \quad (14)$$

Equations (13) and (14) describe the mixture of a cosmological constant equal to $\left(\frac{B}{1+A}\right)^{\frac{1}{1+\alpha}}$ with a matter whose equation of state is given by

$$p = \{\alpha + (1 + \alpha)A\}\rho, \quad (15)$$

which for a pure Chaplygin gas reduces to $p = \alpha\rho$.

Now we consider this energy density and pressure corresponding to a scalar field ϕ having a self-interacting potential $V(\phi)$. The Lagrangian of the scalar field has been given in equation (3). The analogous energy density ρ_ϕ and pressure p_ϕ for the scalar field are the following:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \rho = \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+\alpha)}} \right]^{\frac{1}{1+\alpha}} \quad (16)$$

and

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) = A\rho - \frac{B}{\rho^\alpha} = A \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+\alpha)}} \right]^{\frac{1}{1+\alpha}} - B \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+\alpha)}} \right]^{-\frac{\alpha}{1+\alpha}} \quad (17)$$

Hence for flat universe (i.e., $k = 0$) we have

$$\dot{\phi}^2 = (1+A) \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+\alpha)}} \right]^{\frac{1}{1+\alpha}} - B \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+\alpha)}} \right]^{-\frac{\alpha}{1+\alpha}} \quad (18)$$

and

$$V(\phi) = \frac{1}{2}(1-A) \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+\alpha)}} \right]^{\frac{1}{1+\alpha}} + \frac{1}{2}B \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+\alpha)}} \right]^{-\frac{\alpha}{1+\alpha}} \quad (19)$$

From equation (18) we have the relation between scale factor $a(t)$ and scalar field ϕ as

$$\phi = \frac{2}{\sqrt{3}(1+\alpha)(1+A)} \text{ Sinh}^{-1} \left\{ \sqrt{\frac{C(1+A)}{B} \frac{1}{a^{\frac{3}{2}(1+\alpha)(1+A)}}}} \right\} \quad (20)$$

From equation (19) we get

$$\begin{aligned} V(\phi) = & \frac{1}{2}(1-A) \left(\frac{B}{1+A} \right)^{\frac{1}{1+\alpha}} \text{ Cosh}^{\frac{2}{1+\alpha}} \left\{ \frac{\sqrt{3}\sqrt{1+A}(1+\alpha)}{2} \phi \right\} \\ & + \frac{1}{2}B \left(\frac{B}{1+A} \right)^{-\frac{\alpha}{1+\alpha}} \text{ Cosh}^{-\frac{2\alpha}{1+\alpha}} \left\{ \frac{\sqrt{3}\sqrt{1+A}(1+\alpha)}{2} \phi \right\} \end{aligned} \quad (21)$$

Since $2V(\phi) = \rho - p = (1-A)\rho + \frac{B}{\rho^\alpha}$. So when $\rho \rightarrow \infty$ i.e., when $a \rightarrow 0$ the situations are completely different for $A \neq 1$ and for $A = 1$. In the second case $V(\phi) \rightarrow 0$ as $\rho \rightarrow \infty$, but in the first case $V(\phi) \rightarrow \infty$ as $\rho \rightarrow \infty$. This is why we get qualitatively two different pictures in two cases - as reflected in the figures 2 and 5. In the other limit $a \rightarrow \infty$, $V(\phi) \rightarrow \left(\frac{B}{1+A}\right)^{\frac{1}{1+\alpha}}$ for the above two cases.

For example, $A = \frac{1}{3}$, $B = C = \alpha = 1$, $V(\phi) \rightarrow \frac{\sqrt{3}}{2}$ as $a \rightarrow \infty$

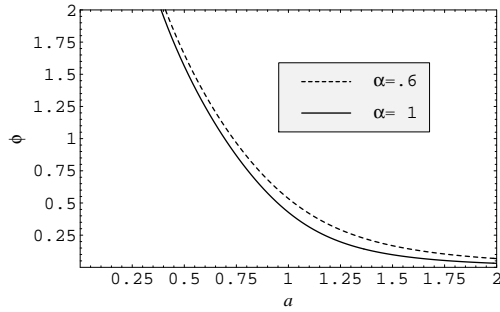


Fig.1

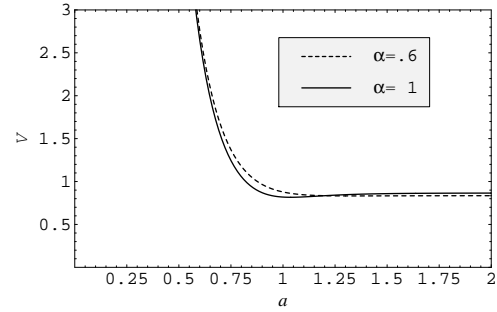


Fig.2

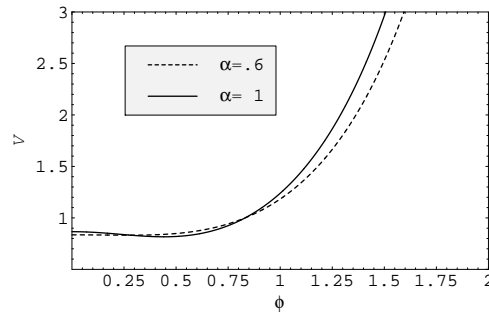


Fig.3

Figs. 1 - 3 shows variation of ϕ and V against a and ϕ for $A = 1/3$ and $\alpha = 0.6, 1$.

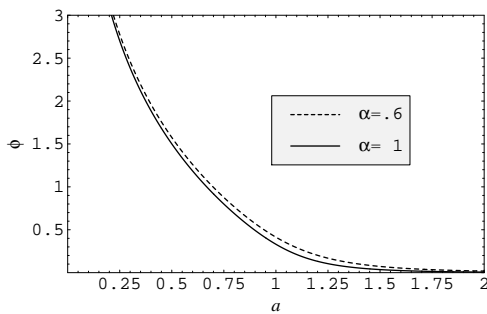


Fig.4

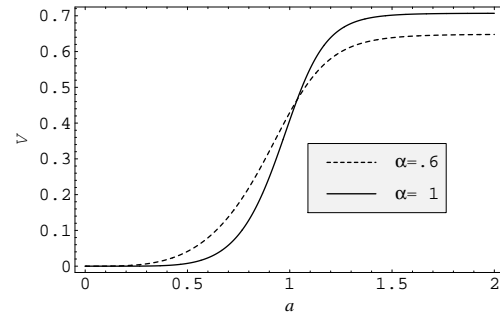


Fig.5

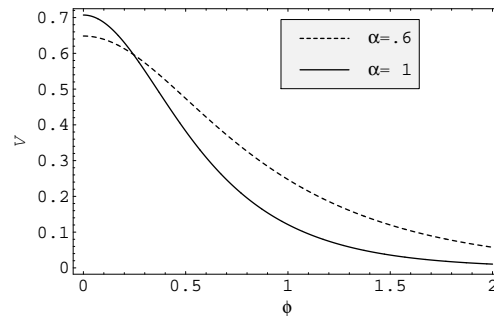


Fig.6

Figs. 4 - 6 shows variation of ϕ and V against a and ϕ for $A = 1$ and $\alpha = 0.6, 1$.

and $A = 1, B = C = \alpha = 1, V(\phi) \rightarrow \frac{1}{\sqrt{2}}$ as $a \rightarrow \infty$.

The graphical representation of ϕ against a and $V(\phi)$ against a and ϕ respectively have been shown in figures 1 - 3 for $A = 1/3$ and figures 4 - 6 for $A = 1$. From figures 1, 4 we have seen that scalar field ϕ sharply falls when scale factor $a(t)$ increases, both for $A = 1/3$ and $A = 1$. In figure 2, we see that potential function $V(\phi)$ sharply decreases from extremely large value and thereafter it increases to a fixed value for $A = 1/3$. On the other hand for $A = 1$, the nature of $V(\phi)$ against a is as stated in the following. For small a , $V(\phi)$ varies insignificantly from zero value and then increases sharply to a constant value. In figure 3, $V(\phi)$ decreases slightly and then increases to infinitely large value for $A = 1/3$ while for $A = 1$, in figure 6, $V(\phi)$ decreases to zero asymptotically starting from a extremely large value. So $V(\phi)$ demonstrates completely different behaviour for two different values of $A = 1$ and $A = 1/3$. Further the figures show how $V(\phi)$ varies with ϕ with different values chosen for α .

III. THE ROLE OF STATEFINDER PARAMETERS IN FRW UNIVERSE

The statefinder parameters have been defined in equation (1). Trajectories in the $\{r, s\}$ plane corresponding to different cosmological models, for example Λ CDM model diagrams correspond to the fixed point $s = 0, r = 1$.

For Friedmann model with flat universe (i.e., $k = 0$),

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{1}{3}\rho \quad (22)$$

and

$$q = -\frac{\ddot{a}}{aH^2} = \frac{1}{2} + \frac{3}{2}\frac{p}{\rho} \quad (23)$$

So from equation (1) we get

$$r = 1 + \frac{9}{2} \left(1 + \frac{p}{\rho}\right) \frac{\partial p}{\partial \rho}, \quad s = \left(1 + \frac{\rho}{p}\right) \frac{\partial p}{\partial \rho} \quad (24)$$

Thus we get the ratio between p and ρ :

$$\frac{p}{\rho} = \frac{2(r-1)}{9s} \quad (25)$$

For modified Chaplygin gas, velocity of sound can be written as

$$v_s^2 = \frac{\partial p}{\partial \rho} = A(1 + \alpha) - \frac{\alpha p}{\rho} \quad (26)$$

From equations (24) and (25) we get the relation between r and s :

$$18(r-1)s^2 + 18\alpha s(r-1) + 4\alpha(r-1)^2 = 9sA(1+\alpha)(2r+9s-2) \quad (27)$$

In the $\{r, s\}$ plane the above equation has only one asymptote parallel to s -axis namely, $r = 1 + \frac{9}{2}A(1 + \alpha)$ and the asymptote intersects the curve in only one point $\left(1 + \frac{9}{2}A(1 + \alpha), \frac{\alpha(1+\alpha)A}{(1+\alpha)A-\alpha}\right)$.

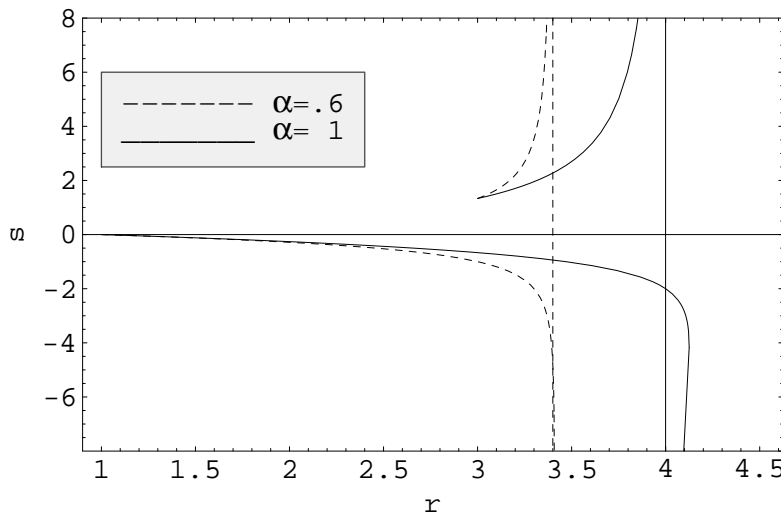


Fig.7

Fig. 7 shows the variation of s against r for different values of α ($= 0.6, 1$) and for $A = 1/3$.

Figure 7 shows the variation of s with the variation of r for $A = 1/3$ and for $\alpha = 0.6, 1$. The portion of the curve on the positive side of s which is physically admissible is only the values of r greater than $\{1 + \frac{9}{2}A(1 + A)\}$. The part of the curve between $r = 1$ and $r = 1 + \frac{9}{2}A(1 + A)$ with positive value of s is not admissible (we have not shown that part in the figure 7) because for the Chaplygin gas under consideration we face a situation, where the magnitude of the constant B becomes negative. It can be shown in the following way.

Since we consider the range of A as $0 < A < 1/2$, we immediately get the inequality relation

$$r - 1 - 9A < 0 \quad (28)$$

for all points lying in the part of the curve between $r = 1$ and $r = 1 + \frac{9}{2}A(1 + A)$ as stated above. In this range positive values of the parameter s is explicitly given by

$$s = \frac{1}{2} \frac{(r-1)(1-2A)}{9A+(1-r)} \left[1 + \sqrt{1 + \frac{8(9A+1-r)}{9(1-2A)^2}} \right] \quad (29)$$

The above expression for s is obtained by solving equation (27). One must note that the other root containing negative sign before the square root symbol is to be left out since we consider only the positive magnitude of s .

Again from (25) and (2), we get

$$\frac{p}{\rho} = A - \frac{B}{\rho^{\alpha+1}} = \frac{2(r-1)}{9s}$$

which in view of (29) leads to the relation

$$\frac{B}{\rho^{\alpha+1}} = \frac{1}{2} - \frac{(1-2A)}{2} \sqrt{1 + \frac{8(9A+1-r)}{9(1-2A)^2}} \quad (30)$$

One can show by a simple algebraic exercise that the right hand side of (30) is clearly negative for all points corresponding to $r < 1 + \frac{9}{2}A(1 + A)$ and hence B becomes negative, which is unacceptable in view of the Chaplygin gas model we are considering. This upper limit for r corresponds to $r = 3$ for $A = 1/3$ that is when we choose our universe to be a mixture of radiation and Chaplygin gas.

Thus the curve in the positive side of s starts from radiation era and goes asymptotically to the dust model. But the portion in the negative side of s represents the evolution from dust state ($s = -\infty$) to the Λ CDM model ($s = 0$). Thus the total curve represents the evolution of the universe starting from the radiation era to the Λ CDM model.

IV. DISCUSSIONS

In this work, we have presented a model for modified Chaplygin gas. In this model we are able to describe the universe from the radiation era ($A = 1/3$ and ρ is very large) to Λ CDM model (ρ is small constant). So compare to Chaplygin gas model, the present model describe universe to a large extent. Also if we put $A = 0$ with $\alpha = 1$, then we can recover the results of the Chaplygin gas model. In figure 7, for $\{r, s\}$ diagram the portion of the curve for $s > 0$ between $r = 1$ to $r = 1 + \frac{9}{2}A(1 + A)$ is not describable by the modified Chaplygin gas under consideration. For example, if we choose $r = 1.03$, $A = 1/3$ then from the curve $s = 0.01$ which corresponds to $q = 3/2$ and hence we have from the equation of state, $B < 0$ which is not valid for the specific Chaplygin gas model considered here. At the large value of the scale factor we must have some stage where the pressure becomes negative and hence B has to be chosen positive. It follows therefore that a portion of the curve as mentioned above should not remain valid.

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