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Response of distance measures to the equation of state

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ABSTRACT

We show that the distance measures (such as the luminosity and angular diameter distances) are *linear* function*als* of the equation of state function $w(z)$ of the dark energy to a fair degree of accuracy in the regimes of interest. That is, the distance measures can be expressed as a sum of (i) a constant and (ii) an integral of a weighting function multiplied by the equation of state parameter $w(z)$. The existence of such an accurate linear response approximation has several important implications. (1) Fitting a constant-w model to the data drawn from an evolving model has a simple interpretation as a weighted average of $w(z)$. (2) Any polynomial (or other expansion coefficients) can also be expressed as weighted sums of the true w. (3) A replacement for the commonly used heuristic equation for the effective w, as determined by the cosmic microwave background, can be *derived* and the result is found to be quite close to the heuristic expression commonly used. (4) The reconstruction of $w(z)$ by Huterer & Starkman can be expressed as a matrix inversion. In each case the limitations of the linear response approximation are explored and found to be surprisingly small.

Key words: methods: statistical – cosmological parameters – cosmology: theory.

1 INTRODUCTION

Current cosmological data suggests that the expansion of the Universe is accelerating (e.g. Riess et al. 1998; Perlmutter et al. 1999; Efstathiou et al. 2002; Lewis & Bridle 2002; Melchiorri et al. 2002). A number of models have been proposed to explain this fact, the simplest of which is the cosmological constant. While a cosmological constant is enough to explain the current data, its constancy leads to a fine-tuning problem (Sahni & Starobinsky 2000; Peebles & Ratra 2002; Padmanabhan 2002b). A simple phenomenological generalization of the cosmological constant is to model the dark energy component that drives the acceleration as an ideal fluid with an equation of state given by $P = w\rho$ in which the equation of state parameter w is allowed to vary with time. In this parametrization the cosmological constant corresponds to $w = -1$, while for other viable models $-0.6 \lesssim w \lesssim -1$ at the present epoch. Such a parametrization indeed arises naturally in several models, such as quintessence (Ratra & Peebles 1988; Wetterich 1988), K-essence (Armendariz-Picon, Damour & Mukhanov 1999) and a tachyonic scalar field (Gibbons 2002; Padmanabhan 2002a; Padmanabhan & Roy 2002; Bagla, Jassal & Padmanabhan 2003). However, the precise form of $w(z)$ is model-dependent.

In view of this there has been considerable interest in attempts to summarize the current (and future) data in terms of a few numbers. There are several ways of doing this, such as using a polynomial approximation for $w(z)$ (e.g. Weller & Albrecht 2002) or in terms of derivatives of the expansion factor (called 'statefinders' by Sahni et al. 2003). To be able to fit data using a polynomial form we have to choose a low-order polynomial since realistic data contain only a finite amount of information. On the other hand, we need to use enough polynomial orders so that the data are adequately described. Saini, Weller & Bridle (2003) investigate in some detail the issue of how to go about this, and show that the current data and near-future data seem to require at most a low-order polynomial. (This, of course, does not imply that the true equation of state is also of a low-order polynomial form.)

These methods attempt to capture the effect of an unknown function $w(z)$, which is equivalent to an infinite number of parameters, in terms of a finite (and often small) number of parameters. In the limited redshift range probed by supernova data, the true variations in $w(z)$ might be such that a low-order polynomial fit is a reasonable approximation. Alternatively one can expand the function $w(z)$ in terms of some set of basis functions which are complete in the given redshift interval. If the basis functions are chosen judiciously, only a small set of expansion coefficients will be required to describe the function $w(z)$ within the limits of the experimental accuracy. Owing to the random noise in cosmological data, any of these choices could fit the data adequately in a maximum likelihood sense. Since the distance measures are integrals over non-linear expressions involving $w(z)$, any parameter determined by such a maximum likelihood analysis represents some non-trivial average of the true $w(z)$. Our main aim in this paper is to show that the actual *functional* relation of the cosmological distance measures to $w(z)$ is not far from being

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linear. This allows us to deduce an approximate relation between the fitted coefficients and the true $w(z)$. This allows one to obtain a quantitative description of the averaging involved in reducing a function $w(z)$ to a finite set of numbers.

The plan of this paper is as follows. In Section 2 we derive the linear response approximation relating the coordinate distance and the equation of state of the dark energy, and show that within a reasonable range of parameters for the dark energy it works to better than ∼2 per cent. In Section 3 we use this approximation to relate the fitted coefficients of a polynomial form to the true equation of state for the case of fitting to the luminosity distance, and show that they are related through a weighted integral of the true equation of state. We then generalize this result to the case of an arbitrary functional form which is linear in the parameters. In Section 4 we derive a similar relationship for the cosmic microwave background (CMB) where, in the simplest case, only one measurement of the angular distance to the last scattering surface is available. In Section 5 we show that the linear approximation also enables a non-parametric estimation of $w(z)$. Our conclusions are presented in Section 6.

2 RESPONSE OF GEOMETRY TO THE EQUATION OF STATE

The luminosity and angular diameter distances depend on the dark energy through the coordinate distance $r[w, z]$ according to the equations $D_{\mathrm{L}}(z) = (1+z)\, r[w, z]$ and $D_{\mathrm{A}}(z) = r[w, z]/(1+z)$. The square brackets in this notation explicitly show that at any redshift z the coordinate distance requires full knowledge of the function $w(z)$. In a flat universe the coordinate distance $r[w, z]$ is given by

$$r[w, z] = (1+g)^{1/2} \int_1^{1+z} \mathrm{d}x\, x^{-3/2} [g + Q[w, x]]^{-1/2}, \tag{1}$$

where $x = 1 + z$, $g = \Omega_{\mathrm{m}}/(1 - \Omega_{\mathrm{m}})$,

$$Q[w, z] = \exp\left[3 \int_1^{1+z} \mathrm{d}x\, w(x)/x\right], \tag{2}$$

and we have set c and H_0 equal to unity. To explore the behaviour of $r[w, z]$ when different equation of state functions $w(z)$ are used, we need to understand the sensitivity of r to $w(z)$. This can be characterized by the *functional derivative* of r with respect to $w(z)$. Since this is not a routine weapon in the arsenal of the astronomer, we shall briefly introduce the concept before applying it.

In the case of a real function $f(x)$, the sensitivity of the function to the independent variable x can be characterized by the derivative $\mathrm{d}f/\mathrm{d}x = f'(x)$. Broadly speaking, a large value for $f'(a)$ indicates that f is relatively more sensitive to the independent variable around the point $x = a$; and a small value for $f'(a)$ will indicate relative insensitivity of f to the independent variable around $x = a$. This follows directly from the definition of the derivative of a function

$$f'(a) = \lim_{\epsilon \to 0} \frac{f(a+\epsilon) - f(a)}{\epsilon}. \tag{3}$$

In the case of f depending not on a single variable but on a *function* $p(x)$, we need to study how f changes if the function $p(x)$ is changed slightly around a point $x = b$. This is best done by changing the function $p(x)$ to a new function $p_1(x) \equiv p(x) + \epsilon\delta_{\mathrm{D}}(x - b)$ which adds a 'spike' at $x = b$ with a strength proportional to ϵ. We can now evaluate the value of f for both $p(x)$ and $p_1(x)$. The difference in the numerical values of f in the limit of $\epsilon \to 0$ is a good measure of the sensitivity of f to the functional form of $p(x)$ around $x = b$. More formally, this functional derivative is defined by

$$\frac{\delta f[p, x]}{\delta p(b)} = \lim_{\epsilon \to 0} \frac{f[p + \epsilon\delta_{\mathrm{D}}(x - b), x] - f[p, x]}{\epsilon}. \tag{4}$$

Just as the ordinary derivative of a function depends on the location at which it is evaluated, the functional derivative depends on the form of $p(x)$ around which the sensitivity is measured, $x = a$, as well as the point $x = b$ at which the input function is perturbed.

For the purposes of this paper we need the response of the coordinate distance $r[w, z]$ at a redshift z to a change in the equation of state at a different redshift, z'. This can be computed around a given fiducial $w(z) = w^{\mathrm{fid}}(z)$ from the functional derivative

$$\frac{\delta r[w^{\mathrm{fid}}, z]}{\delta w(z')} = \lim_{\epsilon \to 0} \frac{r\left[w^{\mathrm{fid}} + \epsilon\delta_{\mathrm{D}}(z - z'), z\right] - r[w^{\mathrm{fid}}, z]}{\epsilon} \tag{5}$$

defined exactly as in equation (4) [also see Huterer & Turner (2001) for a similar application of this idea].

For the rest of this section we switch from discussing the coordinate distance to the luminosity distance and change the independent variable from redshift z to $x = 1 + z$. Multiplying by $(1 + z)$ we obtain the response function for the luminosity distance $\delta D_{\mathrm{L}}[w^{\mathrm{fid}}, x]/\delta w(x') \equiv K_w(x, x')$. The subscript w on the kernel denotes that it depends on w^{fid}, the fiducial equation of state around which the approximation holds. Evaluating the expression in equation (5), we obtain

$$K_w(x, x') = \begin{cases} -\dfrac{3x(1+g)^{1/2}}{2x'} \displaystyle\int_{x'}^{x} \frac{\mathrm{d}y}{y^{3/2}} \frac{Q[w^{\mathrm{fid}}, y]}{\left(g + Q[w^{\mathrm{fid}}, y]\right)^{3/2}} \\ 0 \qquad (\text{for } x < x'). \end{cases} \tag{6}$$

The kernel K_w is a function of two parameters, the redshift at which $w(z)$ is perturbed and the redshift at which we consider the change in the luminosity distance. A surface plot of this function is shown in Fig. 1. Since the effect of varying $w(z)$ at a redshift z' is felt only at $z > z'$, the kernel is zero in half the plane. For small δw we can use this result to approximate the calculation of the luminosity distance as

$$\begin{aligned} D_{\mathrm{L}}[w^{\mathrm{fid}} + \delta w, z] &\simeq D_{\mathrm{L}}[w^{\mathrm{fid}}, z] + \delta D_{\mathrm{L}} \\ &\equiv D_{\mathrm{L}}[w^{\mathrm{fid}}, z] + \int_0^z K_w(z, z')\delta w(z')\,\mathrm{d}z'. \end{aligned} \tag{7}$$

To be useful this linear response approximation should hold to a good accuracy for a reasonable range of $w(z)$. The *SuperNova Acceleration Probe* (*SNAP*) survey is expected to observe about 2000 Type Ia supernovae (SNe), up to a redshift $z \sim 1.7$, each year (Aldering et al. 2002). A single supernova will measure the luminosity distance with a relative error of ∼7 per cent. If we bin the supernovae in redshift intervals of 0.02, this will give a relative error in the luminosity distance of about ∼1 per cent. Saini et al. (2003) show that, given this level of precision and given the present uncertainty in the value of Ω_{m}, the data seem to require at the most a linear polynomial order in $w(z)$. To show how accurately equation (7) gives the luminosity distance in this restricted case, in Fig. 2 we plot in the w_0–w_1 plane the percentage difference between the true luminosity distance and the one computed through equation (7) at $z = 1.5$. The kernel K was computed for $w^{\mathrm{fid}}(z) = -0.5$ in this calculation. Within the region $-1 < w(z) < 0$ the approximation is better than 2 per cent. Therefore the accuracy of the linear approximation is quite acceptable with respect to the projected uncertainties in the luminosity distance obtained from a *SNAP* class experiment. The approximation works even better at smaller redshifts but less so at higher redshifts, although the measurement errors are expected

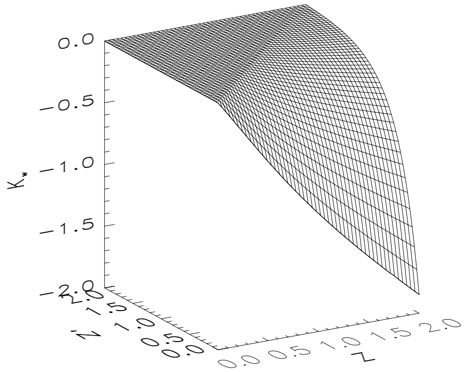

Figure 1. A surface plot of the kernel K_w for supernovae distributed uniformly up to $z = 2.0$.

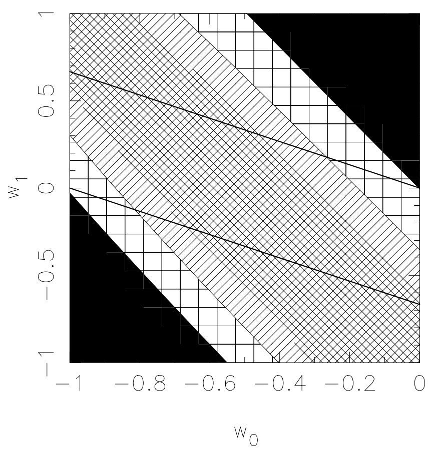

Figure 2. This figure illustrates the accuracy of the linear response approximation by plotting the relative difference in the luminosity distance computed using the exact expression compared with that from the approximate expression in equation (7). This is calculated for the special class of models in which the equation of state is a linear function of redshift, $w(z) = w_0 + w_1 z$. The shaded regions give the relative error $\delta D_{\rm L}/D_{\rm L} <$ 0.005, 0.01, 0.02, 0.05 from the inner hatched region diagonally outwards. The two straight lines mark the regions in which $-1 < w(z) < 0$ in the range $0 < z < 1.5$.

to be larger here. Therefore the conclusions drawn from this approximation are expected to remain valid until an unprecedented accuracy is achieved in the measurement of luminosity distance.

3 INTERPRETATION OF THE FITTING PARAMETERS

Since we have no fundamental understanding of the nature of dark energy, it is necessary to assume some suitable, versatile, functional form for $w(z)$ to fit the cosmological data. Since the luminosity distance depends on the equation of state through an integral relation, and the cosmological data are often fitted through the maximum likelihood method, we can always add a small quickly varying term to any well-fitting $w(x)$, while still retaining a good fit.[1] Owing to this integral dependence, any flexible parametrization eventually recovers only some integrated property of the underlying equation of state. The simple polynomial forms for $w(z)$ usually assumed for the purposes of fitting the cosmological data are often viewed as the first few terms in a series expansion of $w(z)$. Since the behaviour of $w(z)$ is not necessarily polynomial-like, the recovered coefficients of the polynomial are not related in a simple manner to the true coefficients of the series expansion. In this section we use the linear response approximation described above to compute the expectation value for the coefficients of the polynomial approximating the $w(z)$, and relate them to the 'true' input $w(z)$. This will show that such fitting functions serve a useful purpose of measuring some integrated properties of the true equation of state of the dark energy.

3.1 Interpreting the fit with a constant w

The simplest fit to the cosmological data is the *constant-*w model, $w^{\rm fit}(z) = w_0^{\rm fit}$. We are interested in relating this to the true $w(z) = w^{\rm true}(z)$. For an arbitrary equation of state this relation is non-trivial and non-linear but, given the approximations considered in Section 2, we can construct the relationship analytically. Let us first consider fitting the luminosity distance to redshift. This requires a knowledge of the present-day matter density $\Omega_{\rm m}$ as well and, for simplicity, we shall assume that this is known. [When $\Omega_{\rm m}$ is unknown, both $\Omega_{\rm m}$ and w become biased: see Maor, Brustein & Steinhardt (2001).] Our approach could be extended to the case in which $\Omega_{\rm m}$ is not known; however, this is beyond the scope of the present paper.

In the maximum likelihood reconstruction the quantity

$$\chi^2 = \int_0^{z_{\max}} {\rm d}z \left\{ \frac{D_{\rm L}^{\rm fit}(z) - \left[D_{\rm L}^{\rm true}(z) + n(z)\right]}{\sigma(z)} \right\}^2 \tag{8}$$

is minimized, where $n(z)$ is the noise in the measurement at redshift z and $\sigma(z)$ is the variance. (We have replaced the conventional summation with an integral.) We now approximate the fitting function and the luminosity distance by

$$D_{\rm L}^{\rm fit}(z) \approxeq D_{\rm L}[w^{\rm fid}, z] + \Delta w \int_0^{z_{\max}} K_w(z, z')\,{\rm d}z', \tag{9}$$

$$D_{\rm L}(z) \approxeq D_{\rm L}[w^{\rm fid}, z] + \int_0^{z_{\max}} K_w(z, z'')\delta w(z'')\,{\rm d}z'', \tag{10}$$

where we have assumed a constant $w^{\rm fid}$ and defined $\Delta w = w_0^{\rm fit} - w^{\rm fid}$ and $\delta w(z) = w^{\rm true}(z) - w^{\rm fid}$. Minimizing χ^2 and taking the expectation value, we obtain

$$\Delta w = \int_0^{z_{\max}} \Phi_w(z')\delta w(z')\,{\rm d}z', \tag{11}$$

where

$$\Phi_w(z) = \frac{\iint K_w(x, x')K_w(x, 1+z)/\sigma^2(x)\,{\rm d}x\,{\rm d}x'}{\iiint K_w(x, x')K_w(x, x'')/\sigma^2(x)\,{\rm d}x\,{\rm d}x'\,{\rm d}x''}, \tag{12}$$

where all integrals are in the range $(0, z_{\max})$. The noise term has dropped out since its expectation value is zero. Adding $w_0^{\rm fit}$ to both sides of equation (11), we obtain

$$w_0^{\rm fit} = \int_0^{z_{\max}} \Phi_w(z')w^{\rm true}(z')\,{\rm d}z'. \tag{13}$$

The fitted constant, $w_0^{\rm fit}$, is just a weighted average of the true equation of state, with weighting function Φ_w. This weighting function is shown in Fig. 3 for the case of supernovae distributed evenly from $z = 0$ to 2.0. It decreases steadily with redshift, as might be expected

[1] For example, $w(x)$ and $w(x) + \theta\,\sin(x/\theta)$, where $\theta \to 0$, would both fit the data equally well though none of the derivatives agrees. Although this is a contrived example, this point has been well illustrated more realistically by Maor et al. (2001).

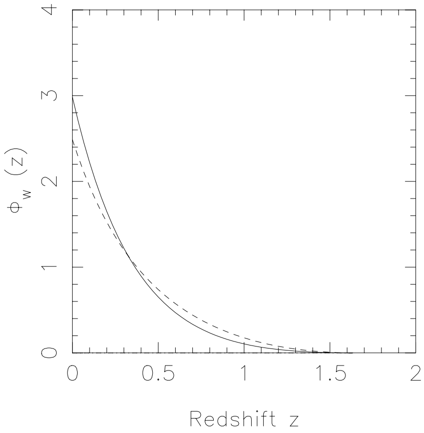

Figure 3. The weighting function for the constant-w estimator for supernovae distributed uniformly up to $z = 2.0$. The solid line shows the weighting function computed with $w^{\rm fid} = -1$ and the dashed one that for $w^{\rm fid} = 0$. The small difference between these plots shows that the linear approximation holds to a good accuracy.

from the fact that all the supernovae are affected by the equation of state at $z = 0$, whereas the equation of state at $z > 2.0$ affects no supernovae. This result shows that the value of $w_0^{\rm fit}$ obtained by fitting to supernova data remains invariant if the $w^{\rm true}(z')$ is changed in such a way that the weighted integral is unaffected.

If the linear response approximation did not hold to a high accuracy then equation (13) would lead to different results depending on the value for $w^{\rm fid}$ used to compute the weighting function Φ_w. Therefore a further test for the linear response approximation is provided by the variation of the weighting function evaluated for a different $w^{\rm fid}$. In Fig. 3 we show the weighting function computed for two different values of $w^{\rm fid}$, namely $w^{\rm fid} = -1.0$ and 0. The figure shows that the variation in the weighting function Φ_w is small.

To quantify this difference further, we consider the change in $w_0^{\rm fit}$ with the change in the weighting function

$$\Delta w_0^{\rm fit} = \int_0^{z_{\rm max}} \Delta\Phi_w(z') w^{\rm true}(z')\, {\rm d}z'. \tag{14}$$

Squaring both sides and using the Schwartz inequality we obtain the bound:

$$\left(\Delta w_0^{\rm fit}\right)^2 \leqslant \int_0^{z_{\rm max}} (\Delta\Phi_w)^2\, {\rm d}z \int_0^{z_{\rm max}} (w^{\rm true})^2\, {\rm d}z. \tag{15}$$

Since we have assumed that $|w| < 1$ in the relevant range, the second integral on the right-hand side has a maximum value $z_{\rm max}$. (We have computed the weighting function for $z_{\rm max} = 2.0$.) With these numbers we can now compute $(\Delta w^{\rm fit})^2$ by considering the difference of Φ_w evaluated at $w_0 = -1$ and 0. We obtain $|\Delta w^{\rm fit}| \sim 0.1$. This shows that the expected error in the determination of w through the linear response approximation is about 10 per cent if we linearize the luminosity distance function around an *incorrect* initial guess. In practice, of course, one could do much better by first fitting the cosmological data with a constant w and then expanding around that value so that the approximation works better than when it is expanded about an arbitrary point.

3.2 Generalization

We have derived above an expression for the expectation value of $w^{\rm fit}$ for the case of a *constant* w fit. The corresponding expressions for the case of more complicated fitting functions are, however, quite cumbersome. We can simplify the notation a little by considering discretized expressions. Suppose the luminosity distance is known at a large number of uniformly distributed redshifts z_i where $i = 1, N$. We consider a model $w(z)$ which is given at redshifts z'_k where $k = 1, M$.

After having chosen a $w^{\rm fid}$, which for simplicity is taken to be a constant, we can define the vectors $\boldsymbol{d} \equiv \{\delta D_{\rm L}(z_i)\}$ and $\boldsymbol{w} \equiv \{\delta w(z'_i)\}$ and matrix $\mathbf{K} \equiv \{K(z_i, z'_j)\, \delta z\}$, where $\mathbf{K}$ is a $N \times M$ matrix and δz denotes the redshift interval between the bins. With these definitions equation (7) becomes

$$\boldsymbol{d} = \mathbf{K}\boldsymbol{w}, \tag{16}$$

where both $\boldsymbol{d}$ and $\boldsymbol{w}$ are small quantities. Next we give the equivalent discrete version of the weighting function obtained in the previous subsection. We have $w^{\rm fit}(z) = w^{\rm fid} + \Delta w$, where Δw is a constant; therefore, using the notation defined above we have $\boldsymbol{w}^{\rm fit} = \boldsymbol{u}\, \Delta w$, where $\boldsymbol{u} = (1, 1, \cdots, 1)$, and similarly $w^{\rm true} = w^{\rm fid} + \delta w(z)$, $\boldsymbol{w}^{\rm true} = \{\delta w(z_i)\}$. With these definitions equations (11) and (12) translate to

$$\Delta w = \frac{\boldsymbol{u}^{\rm T}\mathbf{M}}{\boldsymbol{u}^{\rm T}\mathbf{M}\boldsymbol{u}} \boldsymbol{w}^{\rm true}; \quad \mathbf{M} = \mathbf{K}_w^{\rm T}\mathbf{K}_w. \tag{17}$$

If instead we fit the data with a linear model $w^{\rm fit} = w^{\rm fid} + \Delta w_0 + \Delta w_1 z$ then, following the same procedure that led to equation (11), we find

$$\Delta w_0 = \frac{[(\boldsymbol{z}^{\rm T}\mathbf{M}\boldsymbol{z})\boldsymbol{u}^{\rm T}\mathbf{M} - (\boldsymbol{u}^{\rm T}\mathbf{M}\boldsymbol{z})\boldsymbol{z}^{\rm T}\mathbf{M}]}{(\boldsymbol{u}^{\rm T}\mathbf{M}\boldsymbol{u})(\boldsymbol{z}^{\rm T}\mathbf{M}\boldsymbol{z}) - (\boldsymbol{u}^{\rm T}\mathbf{M}\boldsymbol{z})^2} \boldsymbol{w}^{\rm true}, \tag{18}$$

$$\Delta w_1 = \frac{[(\boldsymbol{u}^{\rm T}\mathbf{M}\boldsymbol{u})\boldsymbol{z}^{\rm T}\mathbf{M} - (\boldsymbol{u}^{\rm T}\mathbf{M}\boldsymbol{z})\boldsymbol{u}^{\rm T}\mathbf{M}]}{(\boldsymbol{u}^{\rm T}\mathbf{M}\boldsymbol{u})(\boldsymbol{z}^{\rm T}\mathbf{M}\boldsymbol{z}) - (\boldsymbol{u}^{\rm T}\mathbf{M}\boldsymbol{z})^2} \boldsymbol{w}^{\rm true}, \tag{19}$$

where we have also defined the vector $\boldsymbol{z} = \{z_i\}$. This generalizes the concept of the weighting function to the linear case. It should be noted that the weighting function for Δw_0 is different from that in equation (17), since we are also fitting for w_1. However, as expected, these results agree for the special case when $\boldsymbol{w}^{\rm true} = \alpha\boldsymbol{u}$.

In the most general case we could approximate $w(z)$ as a linear combination of an arbitrary set of functions F_i (e.g. Gerke & Efstathiou 2002) as

$$w^{\rm fit}(z) = w^{\rm fid} + \sum_{i=1}^{N} c_i F_i(z). \tag{20}$$

In this case the coefficients c_is are related to the true equation of state as follows. Defining the N vectors $\boldsymbol{F}_i = \{F_i(z_k)\}$ we obtain $\boldsymbol{w} = \sum_{i=1}^{N} c_i \boldsymbol{F}_i$. On performing a maximum likelihood analysis we find an expression for the coefficients c_i as follows:

$$c_i = \sum_{i=1}^{N} a_{ij}^{-1} y_j, \tag{21}$$

where we have defined the two matrices

$$a_{ij} = \boldsymbol{F}_i^{\rm T}\mathbf{M}\boldsymbol{F}_j, \tag{22}$$

$$y_i = \boldsymbol{F}_i^{\rm T}\mathbf{M}\boldsymbol{w}^{\rm true}. \tag{23}$$

4 EFFECTIVE *W* SEEN BY THE CMB

Current and near-future supernova surveys will probe redshifts only up to around $z = 2$. We can also compute the effective equation of state probed by the CMB at $z_{\rm cmb} \sim 1000$. In the simplest case the CMB gives us the angular distance to the last scattering surface and

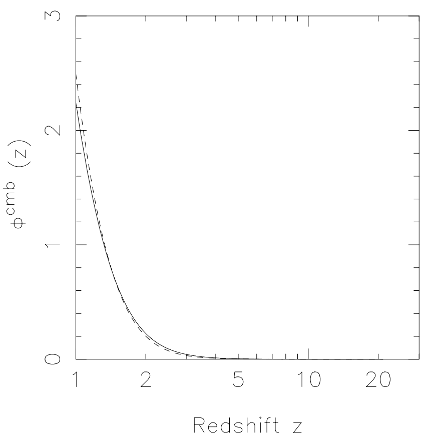

Figure 4. The weighting function that relates the effective equation of state w_{eff} and the true equation of state $w(z)$ as described in the text. The solid line shows the weighting function computed from the linear response approximation using $w^{\mathrm{fid}} = -1$, while the dashed one shows the weighting function inferred from Ω_Q. The two functions agree to a good accuracy.

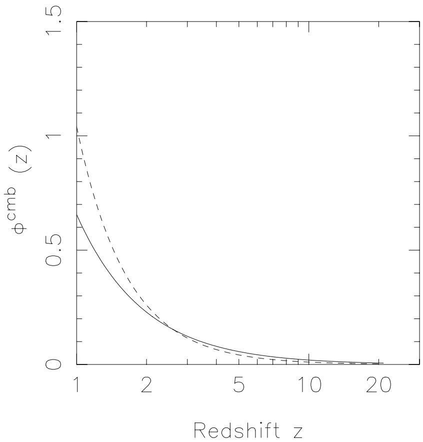

Figure 5. The weighting function for the CMB evaluated using $w^{\mathrm{fid}} = 0$ (solid line). The dashed line again shows the weighting function inferred from Ω_Q. The two functions do not agree to a good accuracy, as discussed in the text.

therefore it does not give detailed information on the behaviour of $w(z)$. Huey et al. (1999) show that a w_{eff} that keeps the CMB temperature and the matter power spectrum unaffected to an accuracy of 5–10 per cent is given by

$$w_{\mathrm{eff}} = \frac{\int \Omega_Q(a) w(a)\,\mathrm{d}a}{\int \Omega_Q(a)\,\mathrm{d}a}, \tag{24}$$

where the integral extends up to the CMB redshift. This is in the same spirit as the weighting function corresponding to fitting a constant equation of state described earlier. Following a similar procedure to that in the previous section, we obtain

$$w_{\mathrm{eff}} = \int \Phi^{\mathrm{cmb}}(z) w(z)\,\mathrm{d}z, \tag{25}$$

$$\Phi^{\mathrm{cmb}}(z) = \frac{K'(1+z_{\mathrm{cmb}}, 1+z)}{\int_0^{z_{\mathrm{cmb}}} K'(1+z_{\mathrm{cmb}}, 1+z)\,\mathrm{d}z}, \tag{26}$$

where $K' = K/(1+z)^2$ owing to the difference between D_{L} and D_{A}.

In Fig. 4 (solid line) we plot this weighting function as calculated using the linear approximation by expanding around $w^{\mathrm{fid}} = -1$. It has the same rough shape as the weighting function for supernovae but this time extends to higher redshift. We see that, despite the high redshift of the CMB, the weighting function falls off close to zero by redshift $z \sim 5$. This is due to the fact that the dark energy term in the Friedman equations is increasingly unimportant as we go to high redshifts. For example, it is well known that the cosmological constant is dynamically unimportant at CMB redshifts.

Overlaid in Fig. 4 (dashed line) is the weighting function from equation (24), converting the integral from the scale factor a to redshift z. The good agreement shows that weighting with Ω_Q works well for models close to the cosmological constant. In Fig. 5 we show the same plot but this time expanding around $w^{\mathrm{fid}} = 0$. The two results do not agree to a good accuracy. This shows that the weighting with Ω_Q should not work so well for models that have an average w closer to zero. These figures also show that the linear response approximation works less well for the CMB since the weighting function changes by a large amount when we expand around $w^{\mathrm{fid}} = -1$ and 0. This means that the weighting function would only apply for those models for which $|w - w_{\mathrm{eff}}|$ is small.

5 NON-PARAMETRIC RECONSTRUCTION OF THE EQUATION OF STATE PARAMETER

Although the fitting functions that we have considered so far provide useful information about the true equation of state of the dark energy, the linear approximation discussed above in principle allows a non-parametric reconstruction of the equation of state.

Consider a given SN data set. To a first approximation we can fit it with a constant w, even though this may not be a good fit. We set the fiducial equation of state w^{fid} to this best-fitting constant value in all that follows. We next compute the difference between the SN distances and the D_{L} computed for w^{fid} to obtain the residuals δD_{L}. We use the notation described in the first paragraph of Section 3.2 so these are given by $\boldsymbol{d}^{\mathrm{s}}$, where the superscript s signifies that these are obtained from measured supernova distances. These are related to the $w(z)$ in bins through equation (16), where $\boldsymbol{w} = \{w(z_k) - w_0\}$ with w_0 given by the constant-w fit. Since these are noisy estimates of distances we need to minimize the χ^2 function with respect to $\boldsymbol{w}$ to obtain the maximum likelihood estimator for $\boldsymbol{w}$: that is, we minimize $\chi^2 = (\boldsymbol{d}^{\mathrm{s}} - \mathbf{K}\boldsymbol{w})^{\mathrm{T}}(\boldsymbol{d}^{\mathrm{s}} - \mathbf{K}\boldsymbol{w})$ and obtain

$$\boldsymbol{w} = (\mathbf{K}^{\mathrm{T}}\mathbf{K})^{-1}\mathbf{K}^{\mathrm{T}}\boldsymbol{d}^{\mathrm{s}}, \tag{27}$$

where we have assumed the noise on $\boldsymbol{d}^{\mathrm{s}}$ is constant with redshift, although the result could be extended for the general case.

This equation also allows the trivial calculation of the Fisher matrix $\mathbf{F} = \mathbf{K}^{\mathrm{T}}\mathbf{K}$ corresponding to the uncertainties on the reconstructed equation of state. It was noted by Huterer & Starkman (2002) that the Fisher matrix was a surprisingly weak function of the model parameters, which matches with the result obtained above and shows that it reflects on the validity of the linear response approximation.

Although equation (27) gives a formal solution of the problem, this estimation is, in general, very noisy. Huterer & Starkman (2003) show that even for SNAP-like data there are only a few principal components that are well determined by this method. It is clear that this lack of resolution is largely due to the fact that no constraints are imposed on the behaviour of $\boldsymbol{w}$. We discuss some of the ways of rectifying this in a separate paper.

6 CONCLUSIONS

We have shown that, in the relevant range of parameters expected for the dark energy, the luminosity distance is a linear functional to the equation of state to a surprisingly high level of accuracy. This approximation allows us to find the relationship between the usual polynomial models for $w(z)$ and the true underlying equation of state of the dark energy. Although the usual interpretation of these polynomial approximations is in terms of a series expansion of $w(z)$, we show that the coefficients of the polynomial approximation are related to the true equation of state through an integral relation. Only in the exact polynomial-like equation of state would these approximations measure the real $w(z)$. We show that the fitting of cosmological data with such forms is still useful since the parameters of such models measure certain, well-defined, integrated properties of the underlying equation of state. Finally, the approximation allows a formal, non-parametric way to measure the equation of state.

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