

Field Theories from the Relativistic Law of Motion

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Abstract

From the relativistic law of motion we attempt to deduce the field theories corresponding to the force law being linear and quadratic in 4-velocity of the particle. The linear law leads to the vector gauge theory which could be the abelian Maxwell electrodynamics or the non-abelian Yang-Mills theory. On the other hand the quadratic law demands spacetime metric as its potential which is equivalent to demanding the Principle of Equivalence. It leads to the tensor theory of gravitational field - General Relativity. It is remarkable that a purely dynamical property of the force law leads uniquely to the corresponding field theories.

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1 Introduction

Until the advent of the general relativity (GR), the equations of motion for particle and field were two separate and independent statements, and no relation between them was sought. The field does however determine motion of particle by appearing on the right hand side of Newton's second law of motion. Whereas the equation of motion for the field is prescribed independently by the field theory without any reference to particle motion. GR brought forth the first instance of a relation between the two. In here the particle equation follows from Einstein's equation for gravitational field¹. Intuitively, it could be understood as follows: Gravitation is described by the curvature of spacetime and this fact is stated by Einstein's equation. Solution of this equation determines the geometry of spacetime which now incorporates gravitational field. Motion under gravity would then be free motion relative to spacetime geometry determined by the solution of the field equation. This is how the particle equation is determined by the field equation.

In this context, the two questions naturally arise are the following:

- (a) Like gravitation, there exists another classical field of electromagnetism, why can't such a relation between the equations of motion of particle and field be sought for it?
- (b) For gravitation, the particle motion is derived from the field equation in GR, how about the other way round; i.e. field equation from particle equation.

For the electromagnetic field, the question was first posed by Feynman

long back and he obtained the half of the Maxwell equations (the source free homogeneous, the Bianchi set) by considering commutation relation between position and velocity. Since Feynman thought that this would lead to something profound and fundamental but soon realized that that was not the case, and he left it at that. It has been brought to light in 1990 by Dyson's paper on Feynman's derivation of the Maxwell equations². This gave rise to some activity in terms of discussions, rederivation and application to other situations³⁻⁶. All of them with a sole exception⁶ concerned with the Bianchi set. Recently, we have derived the complete set of Maxwell equations for the non relativistic case⁷. We begin by demanding the differential operator in the Newton's second law be self-adjoint which immediately leads to the Bianchi set. Note that in the familiar terms self-adjointness of the differential operator is equivalent to demanding the force being linear in velocity and derivable from a potential. Of course, it is taken that field is produced by certain charge, and a priori there is no reason why pseudo scalar charge cannot exist. Allowing for pseudo scalar charge would lead to another set of two (Bianchi) equations. Then the solvability of the system of equations in terms of the vector fields leads uniquely to the entire set of the Maxwell equations and a universal chiral relation between the scalar and pseudo scalar charges. This was all done in the non relativistic framework. In here we shall start with the relativistic equation of motion with 4-force being linear in 4-velocity and then the same arguments would readily lead to a vector gauge theory in an elegant and cogent manner. It could be the abelian Maxwell or the non-abelian Yang-Mills theory. This consideration is independent of the background spacetime which could be flat or curved.

Note that it was the linear force law which lead to the vector gauge theory. Naturally the question arises, what field theory would the quadratic law lead to? It turns out that the quadratic law would demand spacetime metric to be its potential, which would imply the Principle of Equivalence. For force to be globally non-removable, the spacetime must be curved and the field would then be described by the curvature of spacetime. This is how precisely gravitation is described in GR. Thus, the quadratic law would lead to GR with the proper prescription of the energy momentum tensor.

It is remarkable that purely a dynamical consideration on the force law determines uniquely the corresponding field theories. This is in the same spirit as the Bertrand's theorem for the central force in classical mechanics which picks out only the inverse square and linear law on the demand of closed orbit. This was however restricted only to central force. In our case, the demand is on the velocity dependence of the force law, otherwise it is all general.

2 Maxwell's equations

The relativistic law of motion would read as

$$m du^i/ds = f^i \tag{1}$$

defining the equation of motion for a test particle, where s is the proper time. Since 4-velocity is by definition orthogonal to 4-acceleration, $f^i u_i = 0$ always.

The requirement of linearity would mean $f_a = F_{ab} u^b$. Now the orthogo-

nality $f_a u^a = 0$ would imply $F_{ab} = -F_{ba}$. Further for it to be derivable from a potential would require the potential to be $A_a u^a$, which would lead to

$$F_{ab} = \nabla_{[a} A_{b]} \quad (2)$$

where $A_a = (\phi, \mathbf{A})$ is the gauge potential. This would satisfy the Bianchi identity,

$$\nabla^b {}^*F_{ab} = 0 \quad (3)$$

where ${}^*F_{ab} = \frac{1}{2}\eta_{abcd}F^{cd}$ is the dual of F_{ab} and η_{abcd} is the 4-volume form. Defining

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (4)$$

$$\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t \quad (5)$$

then in the familiar terms, the Bianchi equation reads as

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t \quad (6)$$

where \mathbf{E} is a polar vector and \mathbf{B} is an axial vector. Note that $F_{ab}(\mathbf{E}, \mathbf{B})$ is composed of a polar vector \mathbf{E} and an axial vector \mathbf{B} . Under the duality transformation $\mathbf{F} \rightarrow {}^*\mathbf{F}$, $\mathbf{E} \rightarrow \mathbf{B}$, $\mathbf{B} \rightarrow -\mathbf{E}$. The source for the polar field is scalar charge while the axial field is produced by motion of charge.

On the other hand we could as well have begun with $P_{ab}(\mathbf{H}, \mathbf{D})$ where the axial field \mathbf{H} is produced by a pseudoscalar charge and polar field \mathbf{D} by motion of this charge. That is with a field of the type ${}^*\mathbf{F}$ instead of \mathbf{F} . A priori there is no reason to have only F_{ab} with scalar charge and not P_{ab} with pseudoscalar charge. Then we will again have the Bianchi identity for P_{ab} ,

$$\nabla^b {}^*P_{ab} = 0 \quad (7)$$

leading to

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{H} = \partial \mathbf{D} / \partial t. \quad (8)$$

Note that P_{ab} has the same structure as that of $*F_{ab}$. We now have four equations in four vector fields which cannot be solved because to specify a vector field both its divergence and curl must be given. That is for each vector, there should be two equations. We have only four equations for four vectors while there should be twice as many. The only way this system could be solved is to reduce the vectors from four to two by prescribing a linear relation between the two sets. That is assuming a linear relation between $*F_{ab}$ and P_{ab} because they are of the similar type. Thus we write

$$P_{ab} = \alpha *F_{ab} \quad (9)$$

where α is a proportionality constant. Then eq.(7) would become

$$\nabla^b F_{ab} = 0 \quad (10)$$

because $** = -1$. If we identify the polar field (\mathbf{E}) and the axial field (\mathbf{B}) as the electric and magnetic fields produced by some charge distribution (not in our consideration) then this is the other set of the Maxwell equations. We have thus deduced the complete set of the field equations for vacuum. Of course the above linear relation would also imply that the two kinds of charges are no longer independent and they must also be related by a chiral relation of the type,

$$q_p = q_s \tan \theta \quad (11)$$

where θ is a universal constant for a given family of particles. Thus there could exist only one kind of charge, calling it electric or magnetic is simply

a matter of giving name⁷. In a different context, the above relation was also considered by Schwinger⁸.

Thus the linear relativistic law of motion leads uniquely to the classical electrodynamics. The above deduction would run through similarly with the appropriate derivative operator for the non-abelian gauge potential as well leading to the Yang-Mills equations. The linear law thus leads to the vector gauge theories. Further, in our consideration the background metric did not enter specifically and hence it would remain valid for both flat as well as curved spacetime.

3 Einstein's equations

For the quadratic law, we must have

$$f^a = -T_{bc}^a u^b u^c. \quad (12)$$

For this equation to be derivable from an Action principle, the Lagrangian must have a quadratic term in velocity which should be a scalar and it could be written as

$$L = \frac{1}{2} p_{ab} u^a u^b. \quad (13)$$

This Lagrangian on variation would give the equation of motion,

$$p_{ab} \frac{du^b}{ds} = -P_{bc,a} u^b u^c, \quad (14)$$

where

$$P_{bc,a} = (1/2)(p_{ba,c} + p_{ac,b} - p_{bc,a}). \quad (15)$$

Here something remarkable has happened. That the equation of motion for the quadratic law is independent of mass of the particle. This is the property which is known as the Principle of Equivalence (PE). It is important to note that PE is the characterizing feature of the quadratic law and not necessarily of gravity alone. Thus all forces that are quadratic in velocity would obey PE.

The 4-force should be orthogonal to the 4-velocity, which would imply,

$$u^a p_{ab} \frac{du^b}{ds} = -\frac{1}{2} p_{ba,c} u^b u^c u^a = 0 \quad (16)$$

because of symmetry in all the three indices. This could be true only if p_{ab} is antisymmetric, which is impossible. The only way the equation can be made to have some sensible meaning is that $p_{ab} = g_{ab}$ define the spacetime metric. Then the equation would become the geodesic equation, $\dot{u}^a = u^b \nabla_b u^a = 0$ for the metric g_{ab} with $P_{ab,c}$ defining the Christoffel symbols. Then the 4-acceleration would be by definition orthogonal to 4-velocity.

We have thus reached the fundamental and profound conclusion that the quadratic force requires the spacetime metric as its potential. It is profound for the reason that it is the first instance of a force law making a demand on the spacetime geometry which marks an important break from the classical paradigm of given inert spacetime background. So far we have made no reference to any particular field. A number of inferences readily follow: (a) It now ceases to be an external force and instead becomes a property of the spacetime geometry, which would be felt by timelike as well as null particles alike. (b) It must therefore link to all particles and energy forms alike including photons. That is, it has the universal linkage. (c) Since the force would now

arise through the Christoffel symbols, which would be globally non-zero only when spacetime is curved having non-zero Riemann curvature. If force is to be globally non-removable, then spacetime must be a curved Riemannian manifold. (d) Another way to state PE is that the equation of motion must be free of mass of particle. When that happens, force could only be derived from spacetime geometry.

In the relativistic law of motion, null particles must also be included. The characteristic feature of null particles is that they only respond to the spacetime geometry and not to any external force, and their propagation vector is null $u^a u_a = 0$. Any field which is to be shared by both ordinary and null particles has to be the property of spacetime geometry. We turn the question around to ask when could equation of motion be independent of mass of particle? The answer would be only when motion is purely driven by the spacetime metric.

All this is true for any force law which is quadratic. This is another matter that gravitation and inertial forces are the only known examples of such a force law. The former is globally non-removable while the latter is by a suitable coordinate transformation. This is why inertial forces were called fictitious because they could be removed by a suitable choice of spacetime geometry. What removal here means is incorporation into geometry of spacetime, which would now act as their potential. The distinguishing criterion between inertial forces and gravitational field is the Riemann curvature of spacetime. The metric that incorporates inertial forces would have vanishing curvature while that for gravitation would have non-zero curvature.

Since the spacetime is a Riemannian manifold, its curvature would satisfy

the differential Bianchi identity

$$R_{a(bcd;e)} = 0 \tag{17}$$

which on contraction would as is well-known give

$$\nabla_b G^{ab} = 0 \tag{18}$$

where G^{ab} is the Einstein tensor. It integrates to give

$$G^{ab} = \kappa T^{ab} + \Lambda g^{ab} \tag{19}$$

with T^{ab} being divergence free. Now if we identify T^{ab} with the energy momentum tensor of matter/energy distribution, Λ with the cosmological constant and κ involving gravitational constant, then this is the familiar Einstein equation with the proper Newtonian limit. Note that the two terms on the right are the sources for the curvature of spacetime which incorporates in its geometry the quadratic force law of the gravitational field. We have thus deduced the Einstein field equation for gravitation with sources. The vacuum is defined by $T_{ab} = 0 = \Lambda$.

The dynamical characterization of electromagnetic and gravitational fields is thus by the linear and quadratic character of the force law. In this sense gravity is the generalization of the classical electrodynamics or rather the Yang-Mills field. From the field theoretic viewpoint it is the zero mass spin 1 photon characterizes the electromagnetic field. This is essentially brought about by studying the effects of rotation on the transverse components of the vector potential⁹. The existence of a 4-vector potential for the field determines that the mediating particle must be of spin 1 and since the interaction

is long range, it should be massless. In our consideration, it is the linear character of the force law that demands a 4-vector potential. This is how both classical and quantum characterizations meet at the level of gauge potential.

In the case of gravitation, the quadratic law requires the spacetime to be curved and the metric itself plays the role of potential. We can construct an action for fields and by taking the weak field limit and write the plane wave solutions as $h_{ab} = \epsilon_{ab} \exp(ik \cdot x)$, where ϵ_{ab} is the polarization tensor¹⁰. Then following the similar procedure it can be shown that only the transverse components of ϵ_{ab} are physically relevant which would correspond to the spin 2 massless graviton.

In essence our path is reverse from the one followed in field theories¹¹. There in order to have a field theory of massless particles with spin, we write the field in terms of creation and annihilation operators. Then the demand of the Lorentz and gauge invariance restricts the force field for spin 1 field to an antisymmetric tensor $F_{ab} = \partial_a A_b - \partial_b A_a$ with a coupling current J^a satisfying the conservation equation $\partial_a J^a = 0$. This then leads to the Maxwell's equations in vacuum. Similar demand for the field of a spin 2 massless particle would lead to a tensor R_{abcd} having the same symmetry properties of the Riemann tensor. To identify this field with that of in the linearized gravity we require the introduction of a symmetric tensor h_{ab} and a coupling current T^{ab} satisfying the conservation equation $\partial_a T^{ab} = 0$. Note that although the above construction was in the linear regime, one can recover full GR by lifting the restriction of linearity. The important point is that no sooner we strike the Riemann curvature tensor, we have landed in the Einstein's theory of gravitation.

4 Discussion

Let us now reflect on the route from field to particle motion. Given the field equation, how do we deduce the particle equation ? For the vector gauge theory, we are given a 4-vector potential. Its interaction with the particle should be given by a scalar which could only be formed by the scalar product of the 4-potential with the 4-velocity of particle (which is the only 4-vector particle could have of its own). This would lead to an interaction term of the form $(A_i u^i)^n$ and for a gauge invariant theory n can not be different from 1. This would then lead to the familiar particle equation of motion. Alternatively we could have demanded that the field which is a 2-form and appears linearly in the particle equation. In the case of gravity, the field equation yields a curved spacetime with its metric serving as potential. The particle equation would then follow from the invariant constructed of the metric and 4-particle velocity. Since it is quadratic in velocity, the equation would be free of particle's mass and would automatically obey PE. It would be given by the geodesic equation. Though the Newtonian equation of motion for particle is also independent of mass but it does not obey in the strict sense PE because the equation does not refer to null particles. For the relativistic equation, the requirement of mass independence is equivalent to PE as well as to spacetime metric serving as potential for motion.

In summary, we have illustrated a simple and elegant method to deduce the corresponding field theories to linear and quadratic force law. The linear law yields the vector gauge theories; Maxwell theory for the abelian and the Yang-Mills theory for the non-abelian case. This deduction is independent of

the spacetime background which could be flat or curved. The quadratic law marks a fundamental break by requiring spacetime metric to be its potential and it leads to the tensor theory. This is the most important statement. Since gravitation also shares this property, the Einstein equation for gravitation could be deduced. It is always insightful and illuminating to see interrelations between various physical concepts and statements. This exercise is an attempt to understand the relation between motion of field and particle.

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