

## Relativistic superdense star models of pseudo spheroidal space-time <sup>1</sup>

Ramesh Tikekar<sup>1</sup> and Kanti Jotania<sup>2</sup>  
Inter-University Centre for Astronomy and Astrophysics,  
Post Bag 4, Pune 411 007, INDIA

Permanent address:

<sup>1</sup>*Department of Mathematics, Sardar Patel University,  
Vallabh Vidyanagar-388 120, INDIA.  
E-mail: tikekar@iucaa.ernet.in*

<sup>2</sup>*Department of Physics, M. N. College  
Visnagar - 384 315, Gujarat, INDIA.  
E-mail: kanti@iucaa.ernet.in*

### Abstract

The physically viable models of compact stars like SAX (J1808.4-3658) can be obtained using Vaidya-Tikekar ansatz prescribing spheroidal geometry for their interior space-time. We discuss here the suitability of an alternative ansatz in this context. The models of superdense star are proposed using a general three parameter family of solutions of relativistic field equations obtained adopting the alternative ansatz. The set up is shown to admit physically viable models of superdense stars and strange matter stars such as Her. X-1.

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# 1 Introduction

Recent studies indicate that some of the observed physical features of stellar objects such as X-ray pulsar, Her. X-1 [1, 2], X-ray buster 4U 1820-30[3], SAX [4] can be understood if their physical content is prescribed to be strange matter-quark matter ( $u, d, s$ ). These objects seem to be the promising strange star (SS) candidates. The possible existence of SS is a consequence of the belief that strange matter could be the absolute ground state of strongly interacting matter. Such strange stars are predicted to be formed (i) through conversion of prototype neutron stars in the presence of strange matter seeds in their interior, (ii) during the collapse of supernova cores. The problem of developing theoretical models of such compact objects in relativistic set up is a highly formidable task. It is expected to provide deeper insight into their interior structure. A relativistic model of SS (Her. X-1) has been discussed by Sharma and Mukherjee [5] following the method proposed by Vaidya-Tikekar [6, 7].

The conventional approach of obtaining relativistic models of superdense stars in equilibrium consists of prescribing the equation of state (EOS) for the fluid forming the interior of the star and solving the appropriate system of Einstein's equations connecting the interior geometry with the dynamical variables of their physical content. The non-linearity of the relativistic equations of the highly complex hydrodynamical system can be tackled by adopting numerical procedures for obtaining their plausible solutions. In the case of superdense objects such as neutron stars the precise nature of the behaviour of the matter in the central core regions is not known with certainty and so the reliable information about the EOS is not available. Vaidya and Tikekar [6, 7] have shown that in such situations the alternative approach of prescribing suitable geometry for the 3-space associated with the interior space-time of such configurations, making Einstein's equations tractable, leads to physically viable models of superdense stars in equilibrium. Vaidya-Tikekar [6] and Tikekar [7] in this context proposed specific ansatz characterized by two geometrical parameters  $K$  and  $R$  prescribing specific 3-spheroidal geometries for the 3-space of the interior space-time of the star and discussed the various features and suitability of the two parameter class of models obtained. Knutsen [8] had shown that the models of the former class are stable against radial pulsation modes. Various general features of the superdense star models based on Vaidya-Tikekar ansatz have

been extensively studied by Maharaj and Leach [9] and Mukherjee, Paul and Dadhich [10], Gupta and Jassim [11] on the basis of the general solutions of the Einstein's equations admissible in this context.

The interior 3-space of the compact star of the model discussed by Sharma and Mukherjee [5] has the geometry of a 3-spheroid in accordance with the Vaidya-Tikekar scheme. The model is found to have a scaling property [12]. Sharma *et al* [13] have shown that the *realistic strange star equation of state* (shortened as ReSS) approximated to linear form is derivable from models based on Vaidya-Tikekar ansatz.

The interest in promoting understanding of compact SS object warrants the search for alternative approaches suitable for their description in relativistic set up. With this motivation, we have examined the suitability of the ansatz prescribing 3-pseudo spheroidal geometry for the 3-space of the interior space-time of a compact spherical star adopted by Tikekar and Thomas [14]. The geometrical parameter  $R$  in this ansatz does not impose any limitation on the region of validity of the models. Physically viable models of superdense star based on the two parameter class of solutions, for the specific choice of the parameter  $K = 2$  were obtained in [14].

In this paper we have examined various aspects of geometrical and physical relevance of the general three-parameter family of solutions of the Einstein's equations on the background of pseudo spheroidal space-time and discussed their suitability to describe relativistic models of superdense stars such as Her. X-1, SAX and similar ultra compact objects. In Section-2 we have obtained (i) the metric of the space-time whose  $t = \text{constant}$  sections have the geometry of a 3-pseudo spheroid and (ii) the relativistic field equations for fluid distributions in equilibrium on it. The general solution of the field equations is discussed in Section-3. The law governing the density variation in the configurations described by the models of this set up, their physical viability and other relevant aspects are discussed in subsequent sections. The numerical procedures have been used to investigate the behaviour of the dynamical variables of superdense fluid content of the stars and their profiles have been presented for certain selected prototype compact star models.

## 2 Metric of pseudo spheroidal space-time and field equations

A 3-pseudo spheroid immersed in a four dimensional Euclidean space with metric

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2, \quad (1)$$

will have the cartesian equation

$$\frac{w^2}{b^2} - \frac{(x^2 + y^2 + z^2)}{R^2} = 1, \quad (2)$$

where  $b$  and  $R$  are constants. The sections  $w = \text{constant}$  are spheres or pseudo-spheres while the sections  $x = \text{constant}$ ,  $y = \text{constant}$ ,  $z = \text{constant}$  are respectively hyperboloids of two sheets. On introducing the parametrization

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta, \quad w = b \sqrt{1 + \left(\frac{r}{R}\right)^2}, \quad (3)$$

the metric on the 3-pseudo spheroid assumes the form

$$ds^2 = \frac{1 + K \left(\frac{r}{R}\right)^2}{1 + \left(\frac{r}{R}\right)^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4)$$

where

$$K = 1 + \left(\frac{b}{R}\right)^2. \quad (5)$$

The pseudo spheroidal 3-space of (4) is spherically symmetric and regular everywhere. It is flat when  $K = 1$  and is open hyperboloidal when  $K = 0$ . We shall prescribe that the interior space-time of the superdense star in equilibrium is described by the metric

$$ds^2 = e^{\nu(r)} dt^2 - \frac{1 + K \left(\frac{r}{R}\right)^2}{1 + \left(\frac{r}{R}\right)^2} dr^2 - r^2 [d\theta^2 + \sin^2 \theta d\phi^2]. \quad (6)$$

The 3-space of the space-time of (6) has the pseudo spheroidal geometry which is distinct from the spheroidal geometry adopted in [6]–[10]. Accordingly, the superdense star models of this class are expected to have distinct

features which we will be discussed in this paper. The matter content of the star will be assumed to be a perfect fluid in equilibrium with energy-momentum tensor

$$T_{ij} = \left( \rho + \frac{p}{c^2} \right) u_i u_j - \frac{p}{c^2} g_{ij}, \quad (7)$$

with  $\rho$  and  $p$  respectively denoting matter density and fluid pressure. For a static star the unit four-velocity field  $u^i$  of the fluid has the expression

$$u^i = \left( 0, 0, 0, e^{-\frac{\nu}{2}} \right). \quad (8)$$

The Einstein's field equations relating the metric parameters of the interior space-time of the star with its dynamic variables

$$R_{ij} - \frac{1}{2} R g_{ij} = -\frac{8\pi G}{c^2} T_{ij}, \quad (9)$$

lead to the following system of three equations :

$$\frac{8\pi G}{c^2} \rho = \frac{3(K-1)}{R^2} \left( 1 + \frac{K r^2}{3 R^2} \right) \left( 1 + \frac{K r^2}{R^2} \right)^{-2}, \quad (10)$$

$$\frac{8\pi G}{c^4} p = \left( 1 + \frac{r^2}{R^2} \right) \left( 1 + \frac{K r^2}{R^2} \right)^{-1} \left[ \frac{\nu'}{r} + \frac{1}{r^2} \right] - \frac{1}{r^2}, \quad (11)$$

$$\begin{aligned} \left( 1 + \frac{r^2}{R^2} \right) \left( 1 + K \frac{r^2}{R^2} \right) \left( \nu'' + \frac{\nu'^2}{2} - \frac{\nu'}{r} \right) - \frac{2(K-1)}{R^2} \left( \frac{\nu'}{2} + \frac{1}{r} \right) \\ + \frac{2(K-1)}{R^2} \left( 1 + K \frac{r^2}{R^2} \right) = 0. \end{aligned} \quad (12)$$

It is apparent from the equation (10) that the ansatz prescribing pseudo spheroidal geometry for the interior space-time of the star determines the law of variation of density of matter content of the configuration. The non-negativity of  $\rho$  constrains the geometrical parameter so that  $K > 1$ . The matter density is observed to be decreasing radially outward.

### 3 General three parameter family of solutions

The choice of new variables

$$\psi^2 = e^\nu, \quad (13)$$

$$u^2 = \frac{K}{(K-1)} \left( 1 + \left( \frac{r}{R} \right)^2 \right) \quad (14)$$

reduces the linear differential equation (12) to the following standard form

$$(1-u^2) \frac{d^2\psi}{du^2} + u \frac{d\psi}{du} + (1-K)\psi = 0. \quad (15)$$

We introduce new variable defined as

$$u = \cosh \xi. \quad (16)$$

and integrate the equation (15) considering the two cases :

- (i)  $1 < K < 2$ ,
- (ii)  $2 < K$ .

The family of superdense star models in [14] is based on the analytic closed form solution of equation (15) for  $K = 2$ . The expressions for  $\psi$  in the respective cases are given below in the suitable forms :

**Case (i) :  $1 < K < 2$**

$$\begin{aligned} \psi = & A \left[ n\sqrt{u^2-1} \cosh(n\xi) - u \sinh(n\xi) \right] \\ & + B \left[ n\sqrt{u^2-1} \sinh(n\xi) - u \cosh(n\xi) \right]. \end{aligned} \quad (17)$$

**Case(ii) :  $2 < K$**

$$\begin{aligned} \psi = & A \left[ n\sqrt{u^2-1} \cos(n\xi) - u \sin(n\xi) \right] \\ & + B \left[ n\sqrt{u^2-1} \sin(n\xi) + u \cos(n\xi) \right]. \end{aligned} \quad (18)$$

In above  $A$  and  $B$  are constants of integration and

$$n^2 = |K - 2|.$$

The solutions (17) and (18) in the limit  $K \rightarrow 2$ , degenerate to the specific solution for  $K = 2$  of [14]

$$\begin{aligned}
 ds^2 = & -\frac{1 + K \left(\frac{r}{R}\right)^2}{1 + \left(\frac{r}{R}\right)^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + \left\{ A \sqrt{1 + \left(\frac{r}{R}\right)^2} \right. \\
 & + B \left[ \sqrt{1 + \left(\frac{r}{R}\right)^2} \ln \left( \sqrt{2} \sqrt{1 + \left(\frac{r}{R}\right)^2} + \sqrt{1 + 2 \left(\frac{r}{R}\right)^2} \right) \right. \\
 & \left. \left. - \frac{1}{\sqrt{2}} \sqrt{1 + 2 \left(\frac{r}{R}\right)^2} \right] \right\}^2 dt^2. \tag{19}
 \end{aligned}$$

#### 4 Density variation in the configuration

The field equation (10) shows that the ansatz prescribing pseudo spheroidal geometry of the interior space-time of star in equilibrium determines the law of variation of density of its matter content. Further it implies that at the centre  $r = 0$  the matter density  $\rho_0$  will be

$$\frac{8\pi G}{c^2} \rho_0 = \frac{3(K-1)}{R^2}. \tag{20}$$

In view of equation (5),  $K > 1$ , accordingly  $\rho_0$  is positive. The density decreases radially outward from this maximum value at the centre. On the boundary  $r = a$ , of the star it attains the value

$$\frac{8\pi G}{c^2} \rho_a = \frac{3(K-1)}{R^2} \left( 1 + \frac{K a^2}{3 R^2} \right) \left( 1 + K \frac{a^2}{R^2} \right)^{-2}. \tag{21}$$

The ansatz further determines the ratio  $\alpha = \rho_a/\rho_0$  which will have the explicit expression

$$\alpha = \frac{\rho_a}{\rho_0} = \left( 1 + \frac{K a^2}{3 R^2} \right) \left( 1 + K \frac{a^2}{R^2} \right)^{-2}. \tag{22}$$

The space-time in the exterior region  $r > a$  is described by the Schwarzschild exterior metric

$$ds^2 = \left( 1 - \frac{2m}{r} \right) dt^2 - \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 - r^2 [d\theta^2 + \sin^2 \theta d\phi^2] \tag{23}$$

and the metric (6) should be continuous with the metric (23) at the boundary ( $r = a$ ). This is achieved by stipulating the continuity of the metric coefficients ( $g_{00}$  &  $g_{11}$ ). Further the fluid pressure is expected to vanish at the boundary.

The continuity of  $g_{11}$  implies that

$$\frac{m}{a} = \frac{(K-1)a^2}{2R^2} \left(1 + K \frac{a^2}{R^2}\right)^{-1}. \quad (24)$$

Equation (22) determines  $a^2/R^2$  in terms of  $K$  and  $\alpha$  as

$$\frac{a^2}{R^2} = \frac{[1 + \sqrt{1 + 24\alpha} - 6\alpha]}{6K\alpha}. \quad (25)$$

Subsequently equation (24) determines  $m/a$  also in terms of  $K$  and  $\alpha$ . The remaining boundary conditions determine the constants  $A$  and  $B$ . We write equations (24) and (25) alternatively in the form convenient for computations

$$\frac{a^2}{R^2} = \frac{1 + \sqrt{1 + 24\alpha} - 6\alpha - \left(\frac{2m}{a}\right) (1 + \sqrt{1 + 24\alpha})}{6\alpha}, \quad (26)$$

$$K = \frac{1 + \sqrt{1 + 24\alpha} - 6\alpha}{1 + \sqrt{1 + 24\alpha} - 6\alpha - \left(\frac{2m}{a}\right) (1 + \sqrt{1 + 24\alpha})}, \quad (27)$$

which determine  $a^2/R^2$  and  $K$  in terms of  $m/a$  and  $\alpha$ .

The ratio  $m/a$  provides a measure of the compactification of the stellar configurations. We shall refer to it as *compactification factor* of the object. This factor classifies the stellar objects in the various categories as normal stars ( $\sim 10^{-5}$ ), white dwarfs ( $\sim 10^{-3}$ ), neutron stars ( $\sim 10^{-1}$  to  $1/4$ ), ultra compact stars ( $\sim 1/4$  to  $< 1/2$ ) and black holes ( $\sim 1/2$ ). The parameter  $\alpha$  is a measure of the density contrast of the star. These two parameters provide information of astrophysical relevance. Hence, observational information about their values for a star could be of help in determining the structure of its space-time in this set up as follows :

The star model is characterised by three parameters  $\rho_a$ - the surface density,  $\alpha$ - the *density contrast* and  $(m/a)$ - the *compactification factor*. The parameters describing the geometry of the space-time  $K$ ,  $R$ , and boundary radius  $a$  are

determined in terms of the above physical parameters by equations (27), (20) and (26). The physical viability of the star model is eventually examined by studying the implications of the physical conditions like

$$\rho > 0, \quad (\rho - 3p/c^2) > 0$$

expected to be fulfilled throughout the configuration.

## 5 Discussion

The two parameter solution of equation (19) has been known to admit physically viable models of superdense stars stable against radial pulsation modes [14]. We examine here the suitability of the ansatz using the general three parameter solution obtained earlier. Following Rhodes and Ruffini [15], we adopt the value  $\rho_a = 2 \times 10^{14} \text{ gm-cm}^{-3}$  for the matter density on the boundary  $r = a$  of the star. Prescribing a value for  $K > 1$  and for the density contrast  $\alpha < 1$ , we compute the central density  $\rho_0$ . Equation (20) then determines  $R^2$ , equation (25) determines boundary radius  $a$  of the configuration and subsequently, equation (24) fixes up the star mass  $m$ . The appropriate solution of the form (17) or (18) will be used depending upon the value of  $K$  to determine the remaining parameters.

We have examined the behaviours of  $\rho(r)$ ,  $p(r)$  and the EOS  $p = p(\rho)$  for star models of this class with different values of the parameters ( $K$ ,  $\alpha$ ) using numerical procedures. The set up is found to provide physically viable models of superdense stellar configurations. The estimates of the star size and star masses of the models obtained for some assumed values of  $K$  and  $\alpha$  are displayed, below in a tabular form :

**Table : I**

| $K$ | $\alpha$ | $R$ kms | $a/R$  | $a$ kms | $m/a$  | $M/M_\odot$ |
|-----|----------|---------|--------|---------|--------|-------------|
| 1.6 | 0.3      | 12.03   | 0.8464 | 10.18   | 0.1001 | 1.02        |
| 2.4 | 0.5      | 23.7    | 0.4720 | 11.20   | 0.102  | 1.14        |
| 6.0 | 0.1      | 20.05   | 0.7895 | 15.83   | 0.3288 | 3.54        |

The above table indicates that the configurations have mass and size typical of neutron stars and strange stars. The table also indicates that in this set

up the models of ultra compact objects with larger compaction factor will be admissible. Such models can be obtained on choosing appropriately high density contrast and larger values of  $K$ . The behaviours of the dynamical variables  $\rho(r)$ ,  $p(r)$  and the EOS in the respective cases have been graphically displayed in **Figures- 1, 2 and 3** . Here and what follows, appropriate conversion factor of  $G$ ,  $c$  and  $\rho$  has been used in graphs. It is evident from these figures that all the physical viability requirements such as non-negativity of  $\rho(r)$ ,  $p(r)$  and  $\rho - (p/c^2) > 0$  are fulfilled throughout the interior in all above models. However, the strong energy conditions  $\rho - 3(p/c^2) > 0$  are not met with throughout the interior of the models with  $K = 2.4$  and  $6$ . The graphical representations of the EOS of these models indicate that it may be possible to approximate them to the linear form  $p = \kappa(\rho - \beta)$  for appropriately chosen constants  $\kappa$  and  $\beta$ .

Recently, Sharma *et al* [13] have shown that EOS for SAX (ReSS) approximated to above linear form is derivable from models based on Vaidya-Tikekar ansatz. The two model, which they have discussed in this context, have

$$\text{(EOS SS1): mass } M = 1.435M_{\odot}, a = 7.07 \text{ kms, } \rho_0 = 4.68 \times 10^{15} \text{ gm-cm}^{-3}$$

$$\text{(EOS SS2): mass } M = 1.323M_{\odot}, a = 6.55 \text{ kms, } \rho_0 = 5.5 \times 10^{15} \text{ gm-cm}^{-3}$$

Here mass  $M = (c^2/G) m$

We find that the alternative ansatz under consideration is suitable to describe of such strange matter stars as well. To explore these possibilities we assigned the values for  $((m/a), \alpha, \rho_a)$  appropriate for the above models. The equation (27) determines the parameter  $K$ . The equation (26) fixes the value of  $(a/R)$ . This coupled with the value of  $R$  determined from equation(20) on prescribing the central matter density  $\rho_0$  leads to estimates of  $a$  and  $M$  for the configuration. The details of some these representative models are indicated in the following table :

**Table : II**

| $m/a$ | $\alpha$ | $\rho_0 \text{ gm} \cdot \text{cm}^{-3}$ | $K$   | $a/R$  | $R \text{ kms}$ | $a \text{ kms}$ | $M/M_{\odot}$ |
|-------|----------|------------------------------------------|-------|--------|-----------------|-----------------|---------------|
| 0.3   | 0.2      | $4.68 \times 10^{15}$                    | 13.75 | 0.3658 | 20.9            | 7.65            | 1.56          |
| 0.3   | 0.2      | $5.5 \times 10^{15}$                     | 13.75 | 0.3658 | 19.30           | 7.06            | 1.44          |
| 0.25  | 0.2      | $4.68 \times 10^{15}$                    | 4.4   | 0.4657 | 10.78           | 6.97            | 1.18          |

The requirement  $K > 1$  constrains the permissible values of density contrast and compactification parameters. The models of this class have the same  $(m/a)$  ratio and the mass, size comparable with those of the models of Sharma *et al.* However, they accommodate more density variation from center to the boundary. We examined the behaviour of  $\rho(r), p(r)$  and the EOS  $p = p(\rho)$  for the above models using numerical procedures. It is observed that all the above models are physically viable and the behaviour of their fluid content is similar to the models of **Figure 2 and 3**. The **Figure 4** indicates the behaviour of dynamical variables and the EOS of the fluid content of the last model of Table II as a representative case. It is apparent that the EOS gets approximated to a linear form as mentioned earlier. In the absence of the definite EOS for fluid content of ultra compact objects this approach certainly provides insight into the plausible structure of their interior space-time. The causal behaviour, adiabatic features and stability against radial pulsations of the above models are being critically examined.

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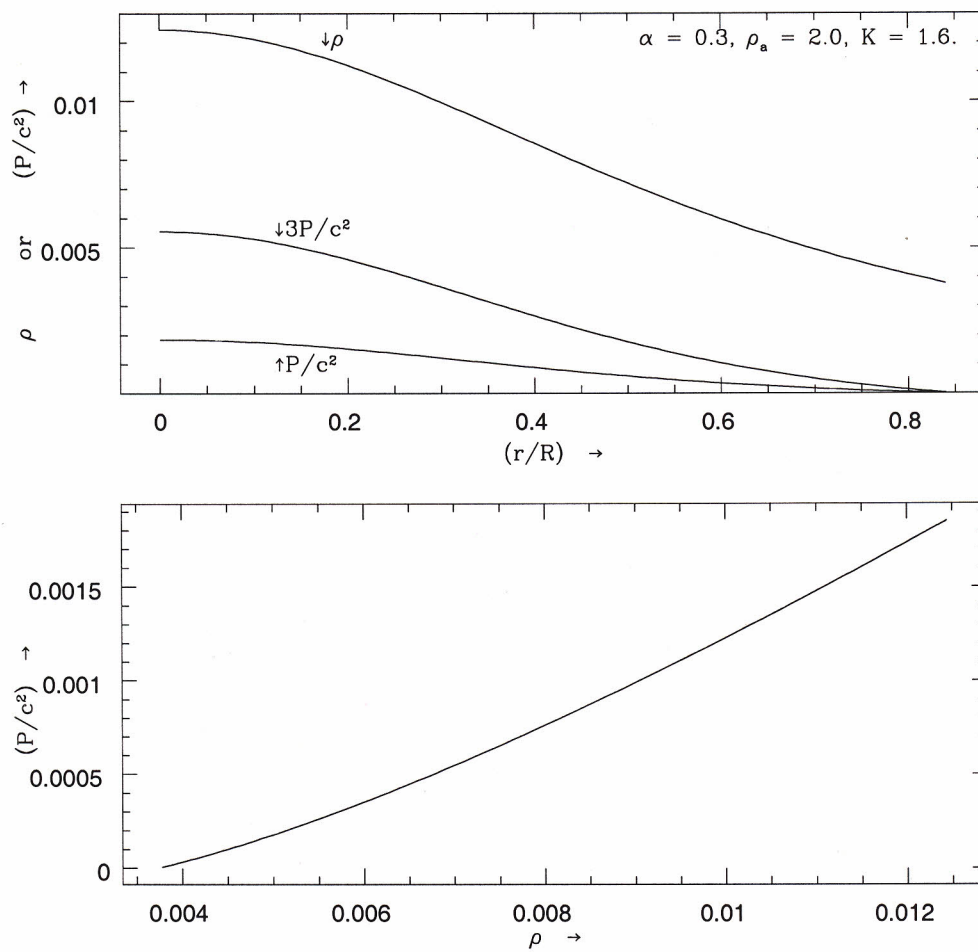


Figure 1: (Top): The plot of  $\rho$ ,  $p$ ,  $3p \rightarrow \frac{r}{R}$  with suitable parameters indicated in figure. (Bottom): The plot of EOS for the model of Top part of the figure

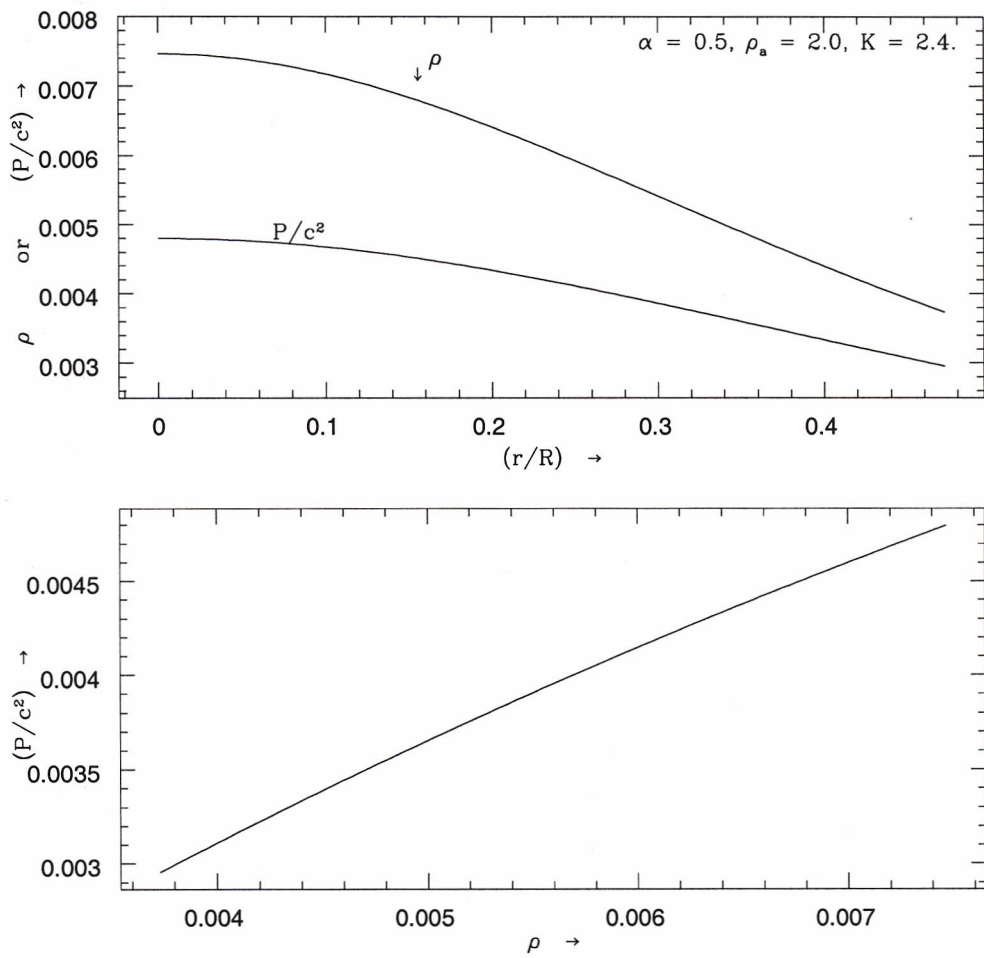


Figure 2: (Top): The plot of  $\rho, p \rightarrow \frac{r}{R}$  with suitable parameters indicated in figure. (Bottom): The plot of EOS for the model of Top part of the figure

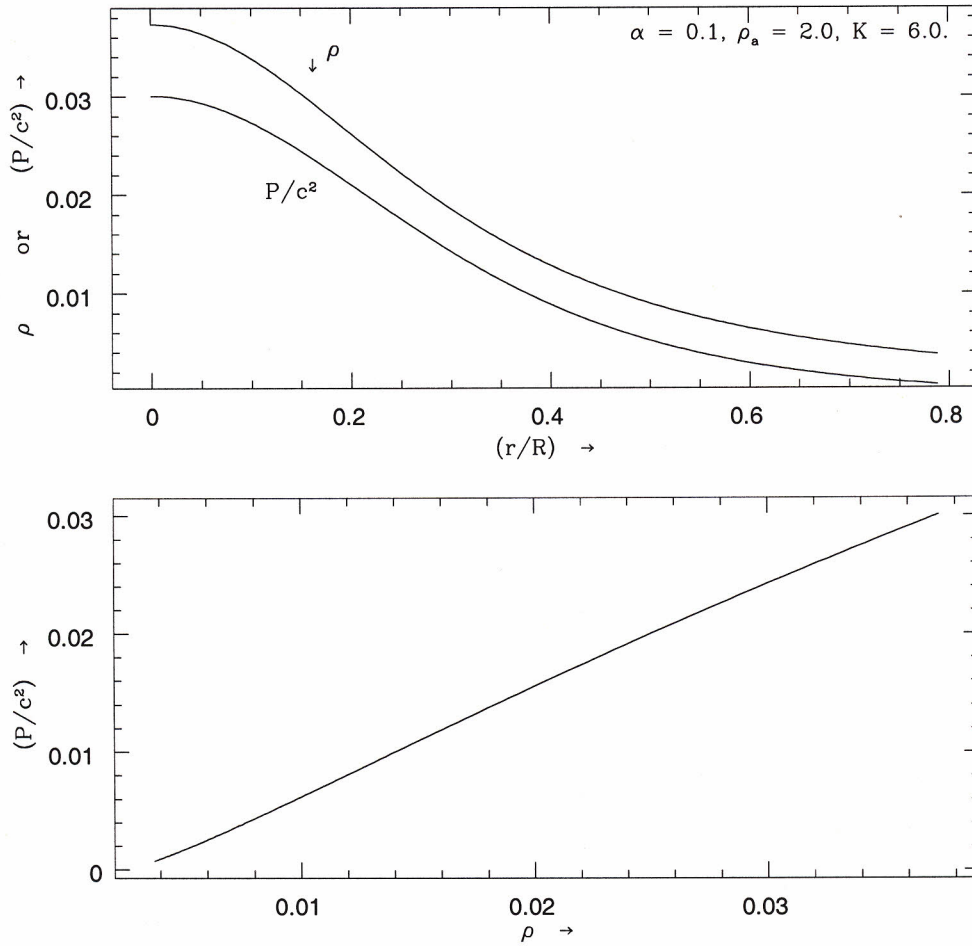


Figure 3: (Top): The plot of  $\rho, p \longrightarrow \frac{r}{R}$  with suitable parameters indicated in figure. (Bottom): The plot of EOS for the model of Top part of the figure

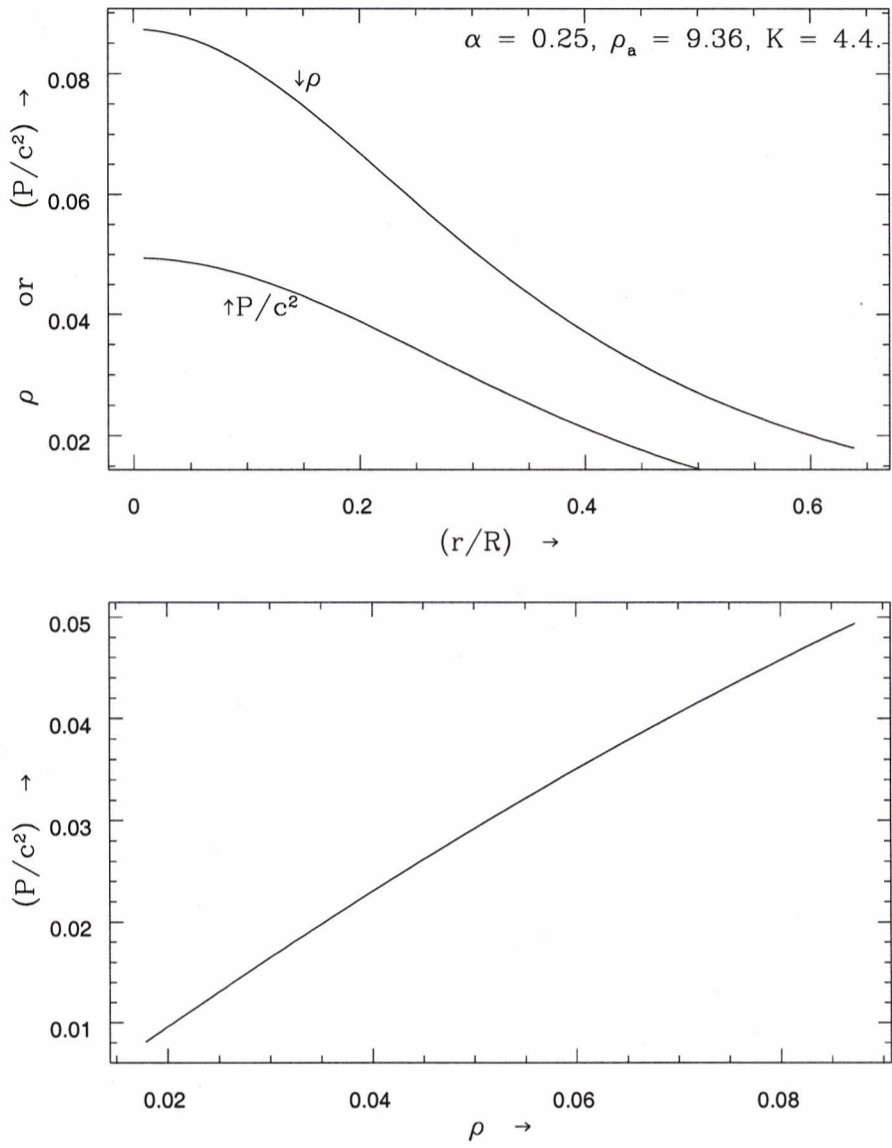


Figure 4: (Top): The plot of  $\rho, p \longrightarrow \frac{r}{R}$  with suitable parameters indicated in figure. (Bottom): The plot of EOS for the model of Top part of the figure