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# Quantum Fluctuations and Nonavoidance of the Singularity in Bianchi Type I Cosmology

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### *Abstract*

An effective metric is defined and used for analyzing the quantum fluctuations in a classical geometry. Earlier work showing that quantum (conformal) fluctuations avoid the classical singularity in the case of spherically symmetric collapse is briefly reviewed. It is shown that this result does *not* extend to anisotropic Bianchi type I cosmology. Here the dispersion in the fluctuations increases too slowly to quench the classical singularity. The singularity persists in the space-time described by the effective metric.

## §(1): *Introduction*

The method of path integrals was used recently to discuss the behavior of quantum fluctuations near classical space-time singularities [1-3]. In all the cases considered the fluctuations diverge as the classical singularity is approached. But it was also noted in [2] that the degree of divergence is different in different cases. It is of interest to see whether this has any significance in the context of singularities. In other words, we ask the question as to how the divergence affects the classical singularity and especially whether the singularity is always removed because of this divergence in the fluctuation.

In order to analyze the situation, we introduce the concept of an effective (or average) metric which has been previously used [3] to discuss a collapsing dust ball solution. This is presented briefly in the next section. We find that the effective metric here is completely nonsingular and that the quantum fluctuations stop the collapse. We present in the next section a similar effective metric analysis for conformal fluctuations in an anisotropic Bianchi type I metric. Here, however, we find that the fluctuations do *not* diverge fast enough to avoid the classical singularity. The singularity persists in the effective metric.

Misner [4] has considered the problem of singularities in a quantum cosmology using a Hamiltonian approach, and arrived at the conclusion that the singularity is not avoided in a mixmaster cosmology. Our result suggests that (at least in this particular case) the two methods do yield the same conclusion.

## §(2): *The Effective Metric Concept*

Consider the time-dependent conformal fluctuations $\Omega(t)$ about a classical solution $\bar{g}_{ik}(x)$. We use the path integral formalism to calculate the quantum probability amplitude for a transition from a value $\Omega_1$ at $t_1$ to $\Omega_2$ at $t_2$ of the conformal factor. This is formally given by

$$K(\Omega_2, t_2; \Omega_1, t_1) = \int \exp \frac{iS}{\hbar}\, \mathfrak{D}\Omega(t). \tag{1}$$

This kernel can be calculated in closed form for some standard classical solutions $\bar{g}_{ik}(x)$ (for more details see [1], [2], and [5].) We shall assume that, at $t = t_1$, the "wave function(al)" for $\Omega(t)$ is given by a Gaussian distribution:

$$\psi(\Omega_1) = \left(\frac{1}{2\pi\Delta_1^2}\right)^{1/4} \exp\left(-\frac{(\Omega_1-1)^2}{2\Delta_1^2}\right) \tag{2}$$

Using the kernal (1), we can find the wave function at any later time as

$$\psi(\Omega, t) = \int K[\Omega, t; \Omega_1, t_1]\ \psi(\Omega_1)\, d\Omega_1 \tag{3}$$

We define the effective metric simply as the average metric obtained using $\psi(\Omega, t)$ as the wave function. That is,

$$g_{ik}^{\text{eff}} \equiv \langle g_{ik}\rangle = \int \psi^*(\Omega, t)\, g_{ik}(\Omega, x^i)\, \psi(\Omega, t)\, d\Omega \tag{4}$$

Since the metric is of the form $g_{ik}(\Omega, x^i) = \Omega^2(t)\,\bar{g}_{ik}(x^i)$ one only has to compute,

$$\langle \Omega^2(t)\rangle = \int \psi^*(\Omega, t)\, \Omega^2(t)\, \psi(\Omega, t)\, d\Omega \tag{5}$$

In most of the cases of interest the kernel is of quadratic type and hence a Gaussian wave packet is propagated as a Gaussian packet with a spreading dispersion and the same mean. Then one has, simply,

$$\langle \Omega^2(t)\rangle = 1 + \Delta^2(t) \tag{6}$$

In this case the effective metric is, just,

$$\langle g_{ik}\rangle = [1 + \Delta^2(t)]\ \bar{g}_{ik}(x) \tag{7}$$

If one uses this method, in the case of collapsing dust ball given by the metric,

$$ds^2 = dt^2 - Q^2(t)\left(\frac{dr^2}{1-\alpha r^2} + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right) \tag{8}$$

one finds that the $Q(t)$ is replaced by another function (near the singularity)

$$\overline{Q}(t) = Q(t)\left\{1 + \left(\frac{\tau_1}{\tau}\right)^4\left[\frac{A}{\Delta^2}\left(1-\frac{\tau}{\tau_1}\right)^2 + \Delta^2\left(1-\frac{2\tau}{\tau_1}\right)^2\right]\right\}^{1/2} \tag{9}$$

where $A, \Delta, \tau_1$ are constants, and $\tau = (-t)^{1/3}$. Since $Q$ goes as $\tau^2$ near $\tau = 0$, one sees that $\overline{Q}(0)$ is finite, thus avoiding the singularity. An explicit calculation of the scalar curvature confirms this result.

All the above results follow for the Friedmann metric also. Here, however, there is an additional curiosity, if we take the open model with $V \to \infty$ (see [2]). Then the constant $A$ goes to zero and we get

$$\overline{Q}(t) = Q(t)\left[1 + \left(\frac{\tau_1}{\tau}\right)^4\left(1-\frac{2\tau}{\tau_1}\right)^2\Delta^2\right]^{1/2}. \tag{10}$$

Thus the effective metric is purely classical and is determined by just two parameters, $\Delta$ and $\tau_1$. The physical significance (if any!) of this result is still obscure.

## §(3): *Fluctuations in the Anisotropic Metrics*

Consider the anisotropic Bianchi type I solution given by the metric,

$$ds^2 = dt^2 - A_1^2(t)\,(dx^1)^2 - A_2^2(t)\,(dx^2)^2 - A_3^2(t)\,(dx^3)^2 \tag{11}$$

where,

$$A_\mu(t) = \overline{Q}\left(\frac{t^{2/3}}{Q}\right)^{2\sin\alpha_\mu}, \quad \mu = 1, 2, 3 \tag{12}$$

$$\alpha_\mu = \alpha + \frac{2\pi}{3}(\mu - 1), \qquad \alpha = \text{const} \tag{13}$$

and

$$\overline{Q}^3 = \tfrac{9}{2}\, Mt(t+\Sigma), \qquad M, \Sigma > 0 \tag{14}$$

This describes a dust-filled universe with a density given by

$$\rho = \frac{3M}{4\pi \bar{Q}^3} \tag{15}$$

The scalar curvature for the metric is

$$\bar{R} = \frac{4}{3t(t+\Sigma)} \tag{16}$$

Thus we have a singularity as $t \to 0$.

One now considers conformal fluctuations in this metric. It was shown in [2] that near the singularity, the fluctuations have a dispersion which goes as

$$\Delta_2 = -\frac{4\pi}{27M\Sigma V\Delta_1} \ln \frac{t}{t_1} \qquad (\text{as } t \longrightarrow 0) \tag{17}$$

$$\equiv -c \ln \frac{t}{t_1} \tag{18}$$

So, from our general discussion the effective metric near the singularity is

$$ds_{\text{eff}}^2 = [1 + \Delta_2^2(t)]\; ds^2 \tag{19}$$

We want to compute the effective scalar curvature. The scalar curvatures of two conformally related metrics are related by

$$R_{\text{eff}} = \frac{\bar{R}}{\Omega^2} - \frac{6}{\Omega^3}\, \Omega^{;i}_{;i} \tag{20}$$

in our case, $\Omega = [1 + \Delta^2(t)]^{1/2}$ and $\bar{R}$ is given by equation (16). A direct though somewhat tedious computation gives (near the singularity, i.e., $t \to 0$),

$$R_{\text{eff}} \cong \frac{2}{[3ct \ln (t/t_1)]^2} \tag{21}$$

Thus $R_{\text{eff}}$ *does* diverge as $t \to 0$. This fact is due to the original result that the dispersion increases only logarithmically.

Thus we find that conformal fluctuation alone is not sufficient to remove the singular behavior from an anisotropic cosmology.

## §(4): *Discussion*

Our result may be physically interpreted as follows. As claimed in [2], it is the case that quantum conformal fluctuations of the Bianchi type I models do lead to some space-times which are nonsingular. However, when we consider the *totality* of all final states, the average, as computed by the effective metric, describes a singular final state.

In classical relativity it is well known that the introduction of shear helps in arriving at a final singularity. The fact that there is no singularity in the effective metric in the shear-free Friedmann case and that there is one in the shearing model discussed here suggests the validity of this result in the quantum relativity also. It would therefore be of interest to examine how the introduction of rotation in the quantum case affects the outcome vis-a-vis singularity. We hope to consider this problem in a later paper.

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