

# One Parameter Solution of Spherically Symmetric Accretion in Various Pseudo-Schwarzschild Potentials

**Aveek Sarkar**

*National Centre For Radio Astrophysics, TIFR, Post Bag 3 Ganeshkhind,  
Pune 411 007, India*

*Email: sarkar@ncra.tifr.res.in*

**Tapas K. Das**

*Inter University Centre For Astronomy And Astrophysics, Post Bag 4 Ganeshkhind,  
Pune 411 007, India*

*Email: tapas@iucaa.ernet.in*

## Abstract <sup>1</sup>

In this paper we have solved the hydrodynamic equations governing the spherically symmetric isothermal accretion (wind) onto (away from) compact objects using various pseudo-Schwarzschild potentials. These solutions are essentially one parameter solutions in a sense that all relevant dynamical as well as thermodynamic quantities for such a flow could be obtained (with the assumption of a one-temperature fluid) if *only one* flow parameter (temperature of the flow  $T$ ) is given. Also we have investigated the transonic behaviour of such a flow and showed that for a given  $T$ , transitions from subsonic to the supersonic branch of accretion (wind) takes place at different locations depending on the potentials used to study the flow and we have identified these transition zones for flows in various such potentials.

---

**Published in the International Journal of Modern Physics D, 2002, Volume 11, Issue 03, pp. 427-432.**

---

## 1 Introduction

A number of modified Newtonian potentials of various forms are available in literature which accurately approximate some general relativistic effects important to study accretion discs around a Schwarzschild black hole. Such potentials may be called 'pseudo-Schwarzschild' potentials because they nicely mimic the space-time around non-rotating/slowly rotating compact astrophysical bodies. Recently we have shown that<sup>1</sup> (Das & Sarkar, 2001, hereafter PI) though the potentials discussed in this paper were originally proposed to mimic the relativistic effects manifested in disc accretion, it is quite reason-

---

<sup>1</sup>Send any correspondence/ offprint requests to Tapas K. Das ([tapas@iucaa.ernet.in](mailto:tapas@iucaa.ernet.in))

able to use most of these potentials in studying various dynamical as well as thermodynamic quantities also for spherical Bondi-type<sup>2</sup> accretion of adiabatic fluid onto Schwarzschild black holes. In PI, the space variation of various dynamical and thermodynamic quantities had been studied for a two parameter adiabatic accretion and wind system i.e., flow parametrized by the specific energy and the polytropic constant of the flow. In this paper, we would like to continue our investigation of spherical accretion for the same set of pseudo potentials used in PI but for isothermal accretion (wind) which may be characterized by a single parameter, i.e., the flow temperature which can be approximated by the proton temperature for an one-temperature fluid system.

Owing to the fact that, close to a spherically accreting Schwarzschild black hole, the ratio of electron number density to the photon number density is proportional to  $\sqrt{r}$  (where  $r$  is the radial distance measured from the central accretor in units of Schwarzschild radius  $r_g = \frac{2GM_{BH}}{C^2}$ ), a lesser number of electrons would be available per photon with decrease of  $r$ , and close to the accretor, momentum transfer by photons on the accreting fluid (or wind) might be an efficient process which may keep the flow temperature roughly constant (at least upto the sonic point) and hence isothermality may not be an entirely unjustified assumption<sup>3</sup>. However, this efficiency for momentum transfer falls off for large  $r$  and far away from the black hole isothermality assumption may brake down.

## 2 Governing equations

We take the following pseudo potentials:

$$\begin{aligned} \Phi_1 &= -\frac{1}{2(r-1)}, \quad \Phi_2 = -\frac{1}{2r} \left[ 1 - \frac{3}{2r} + 12 \left( \frac{1}{2r} \right)^2 \right] \\ \Phi_3 &= -1 + \left( 1 - \frac{1}{r} \right)^{\frac{1}{2}}, \quad \Phi_4 = \frac{1}{2} \ln \left( 1 - \frac{1}{r} \right) \end{aligned} \quad (1)$$

to study the flow. Whereas  $\Phi_1$  and  $\Phi_2$  had been proposed by Paczyński and Wiita<sup>4</sup> (1980) and Nowak and Wagoner<sup>5</sup> (1991) respectively,  $\Phi_3$  and  $\Phi_4$  are suggested by Artemova et. al.<sup>6</sup> (1996) (see [6] and PI for detail discussion about these potentials). Hereafter, we will denote any  $i$ th potential as  $\Phi_i$  where  $\{i = 1, 2, 3, 4\}$  corresponds to  $\{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}$  respectively.

For isothermal equation of state,

$$P = \frac{R\rho T}{\mu} = C_s^2 \rho \quad (2)$$

where  $P, \rho, T$  and  $C_s$  are the pressure, density, constant temperature and the constant isothermal sound speed of the flow respectively;  $R$  and  $\mu$  being universal gas constant and the mean molecular weight.  $C_s$  and  $T$  can be related as,

$$C_s^2 = \Theta T \quad (3)$$

Where  $\Theta = \frac{\kappa}{\mu m_H}$ , a constant,  $m_H \sim m_p$  being the mass of the Hydrogen atom. Integrating the radial momentum conservation equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} + \Phi_i' = 0$$

and the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho u r^2) = 0$$

using eqn. (2) and (3), we obtain two conservation equations for our flow as (in system of units used in PI):

$$\frac{u_i^2}{2} + \Theta T \ln \rho_i + \Phi_i = \mathbf{C} \quad (4a)$$

$$\dot{M} = 4\pi \rho_i u_i r^2 \quad (4b)$$

Where  $\mathbf{C}$  is a constant,  $u$  and  $\dot{M}$  being the dynamical flow velocity and the mass accretion rate respectively. Subscript  $i$  indicates that respective quantities are measured for a particular  $\Phi_i$ . The space rate of change of dynamical velocity can be written as:

$$\frac{du_i}{dr} = \frac{\frac{2\Theta T}{r} - \Phi_i'}{u_i - \frac{\Theta T}{u_i}} \quad (4c)$$

where  $\Phi_i' = \frac{d\Phi_i}{dr}$ . From eqn. (4b), one can easily obtain the sonic point conditions as,

$$u_i = \sqrt{\frac{r_c^i \Phi_i'}{2}} = C_s = \Theta T \quad (4d)$$

where  $\Phi_i' \Big|_c$  is the value of  $\frac{d\Phi_i}{dr}$  at the corresponding sonic point  $r_c^i$ .  $\frac{du_i}{dr}$  at the sonic point may be obtained by solving the following equation

$$\left(\frac{du}{dr}\right)_{c,i}^2 + 0.25 \left(\Phi_i' \Big|_c\right)^2 \Theta^{-1} T^{-1} + 0.5 \Phi_i'' \Big|_c = 0 \quad (4e)$$

where  $\Phi_i'' \Big|_c$  is the value of  $\frac{d^2\Phi_i}{dr^2}$  at sonic point. Mach number of the flow  $M_i$  for a given  $T$  and  $\Phi_i$  can be expressed as,

$$M_i = u_i \Theta^{-0.5} T^{-0.5} = \sqrt{T \dot{M} \rho_i^{-2} r^{-2} + 2\Phi_i \Theta^{-1} T^{-1} + (2T^2 - 1) \ln \rho_i} \quad (5)$$

where  $\dot{M}$  is the mass accretion rate defined in eqn. (4b).

### 3 Results

For a given flow temperature  $T$ , one can solve eqn. (4d) to find out the sonic point  $r_c^i$  for any particular  $i$ th potential  $\Phi_i$ . In figure 1 we show the variation of sonic points (for  $\Phi_{i=1-4}$ ) with constant flow temperature. We observe that for all  $\Phi_{i=1-4}$ 's, sonic point anticorrelates with  $T$  non-linearly and monotonically. This is obvious because as sound speed  $C_s$  is constant throughout isothermal flow, it is only the dynamical velocity  $u$  whose change would control the change of Mach number, and as  $C_s$  is proportional to  $\sqrt{T}$ , the more will be the flow temperature, the more will be  $C_s$  and the less will be the Mach number  $M$  ( $M = \frac{u}{C_s}$ ) for a particular radial distance and the less will be the location of  $r_c$  where  $u$  becomes equal to  $C_s$ , i.e.,  $M$  becomes unity. It is evident from the figure that for a given flow temperature (fixed  $T$ ):

$$r_c^1 > r_c^4 > r_c^3 > r_c^2 \quad (6)$$

For a fixed  $T$ , it is now quite straightforward to simultaneously solve eqn. (4a-4e) and eqn. 5 to get the integral curves of motion for all  $\Phi_{i=1-4}$ 's to study the transonicity of the flow. In figure 2, for  $T = 5 \times 10^{10}$  oK, we plot all the integral curves of motion for accretion as well as for all  $\Phi_{i=1-4}$ 's described in §1. For a given flow temperature and for a particular distance  $r = r_o$ , if we define  $M_o^i$  to be the Mach number attained at  $r_o$  for a specific  $\Phi_i$ , it is clear from the figure that

$$M_o^1 > M_o^4 > M_o^3 > M_o^2 \quad (7)$$

while the sequence is just reversed for the wind. As velocity profile for the isothermal flow is identical in nature of its Mach number profile with a scale shift of  $(\Theta T)^{-0.5}$ , one can conclude that  $\Phi_1$  produces the steepest velocity gradient for accretion and flattest velocity gradient for wind. Combining eqn. (6) with (7), one can interpret that for isothermal accretion, Paczyński and Wiita<sup>4</sup> (1980) potential  $\Phi_1$  would produce the maximum spatial rate of change of kinetic energy of the *accretion* for a given  $T$  while the same is produced by potential proposed by Nowak and Wagoner<sup>5</sup> (1991), i.e.,  $\Phi_2$ , for *wind*. Similarly, if for any  $\Phi_i$ , one defines the spatial gradient of Mach number to be  $\frac{dM^i}{dr}$ , which might be regarded as the measure of the 'transonicity' of the flow (because more will be the value of  $\frac{dM^i}{dr}$ , the faster the flow will become supersonic, i.e., the further will be the sonic point away from the black hole), we find that for accretion branch,  $\left| \frac{dM^i}{dr} \right|$ , e.g., the absolute value of  $\frac{dM^i}{dr}$ , increases non-linearly and monotonically as the flow moves towards the black hole and sequence for  $\left| \frac{dM^i}{dr} \right|$  follows the same trend shown in eqn. (7). However, for wind branch,  $\left| \frac{dM^i}{dr} \right|$  does *not* monotonically increase as the flow moves away from the accretor rather  $\left| \frac{dM^i}{dr} \right|$  attains a maximum value in the *subsonic part* of the wind and then starts falling non-linearly, see figure 3. The location of the maxima can be obtained by solving the following non-trivial equation for  $r_p$  (the location of the peak):

$$\left( r_p^i \right)^2 \Psi_{\mathbf{p}}^i(r_p, u_p) + r_p^i \Omega_{\mathbf{p}}^i(r_p, u_p) - 2\Theta T = 0 \quad (8)$$

Where  $\Psi^i(r, u)$  and  $\Omega^i(r, u)$  are two complicated functions of  $r$  and  $u$  defined as:

$$\Psi^i(r, u) = \frac{1}{u_i^2} \frac{du_i}{dr} \left( \frac{u_i^2 + \Theta T}{u_i^2 - \Theta T} \right) \Phi_i - \Phi_i''$$

$$\Omega^i(r, u) = -\frac{2}{u^2} \frac{du_i}{dr} \left( \frac{u_i^2 + \Theta T}{u_i^2 - \Theta T} \right) \Theta T$$

The superscript  $i$  indicates the value obtained for a particular  $i$ th potential. Also we obtain that the location of the peak maintains the following sequence:

$$r_p^1 < r_p^4 < r_p^3 < r_p^2 \quad (9)$$

Thus we successfully analyze various aspects of transonic behaviour of the spherically symmetric, isothermal accretion characterized by its flow temperature  $T$ .

### Acknowledgements

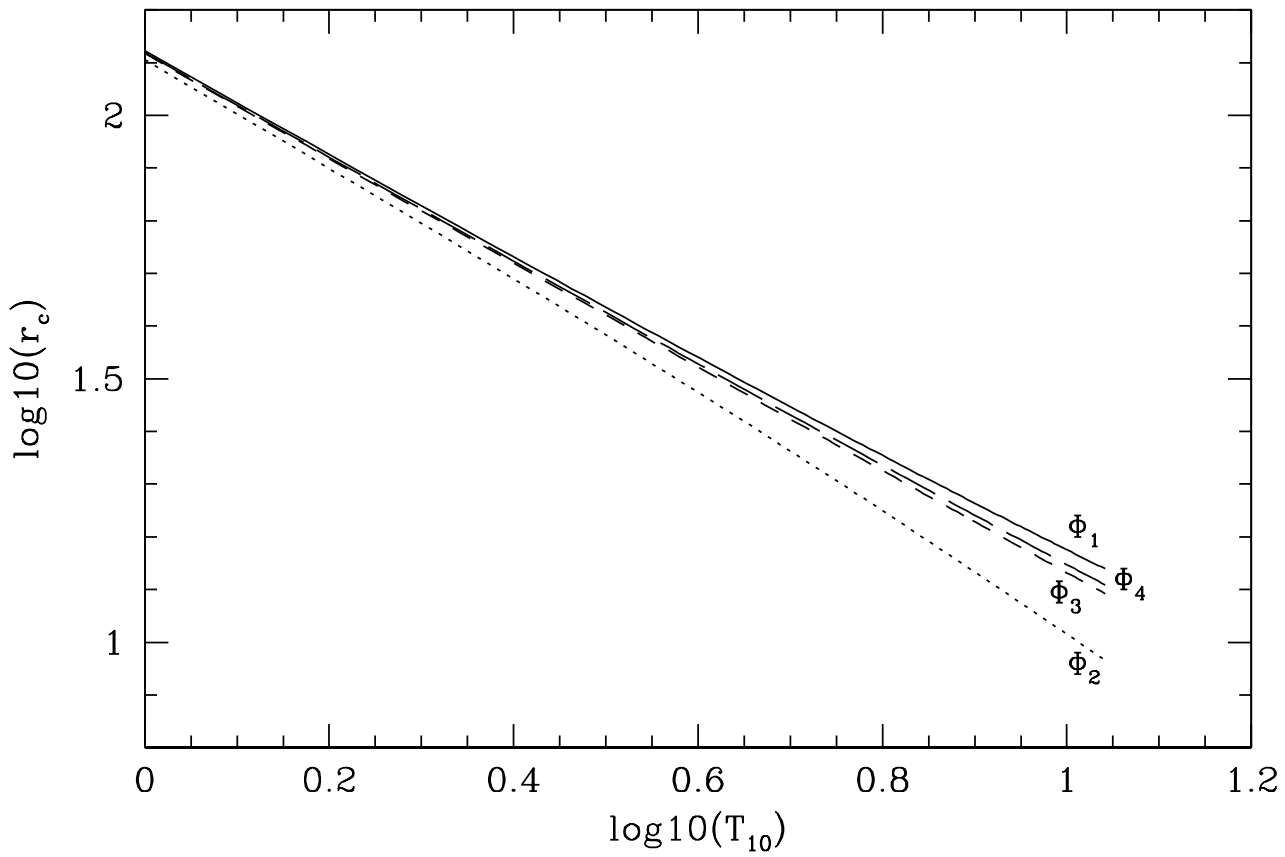
We are thankful to Prof. I. D. Novikov and Prof. P. J. Wiita for useful discussions.

### References

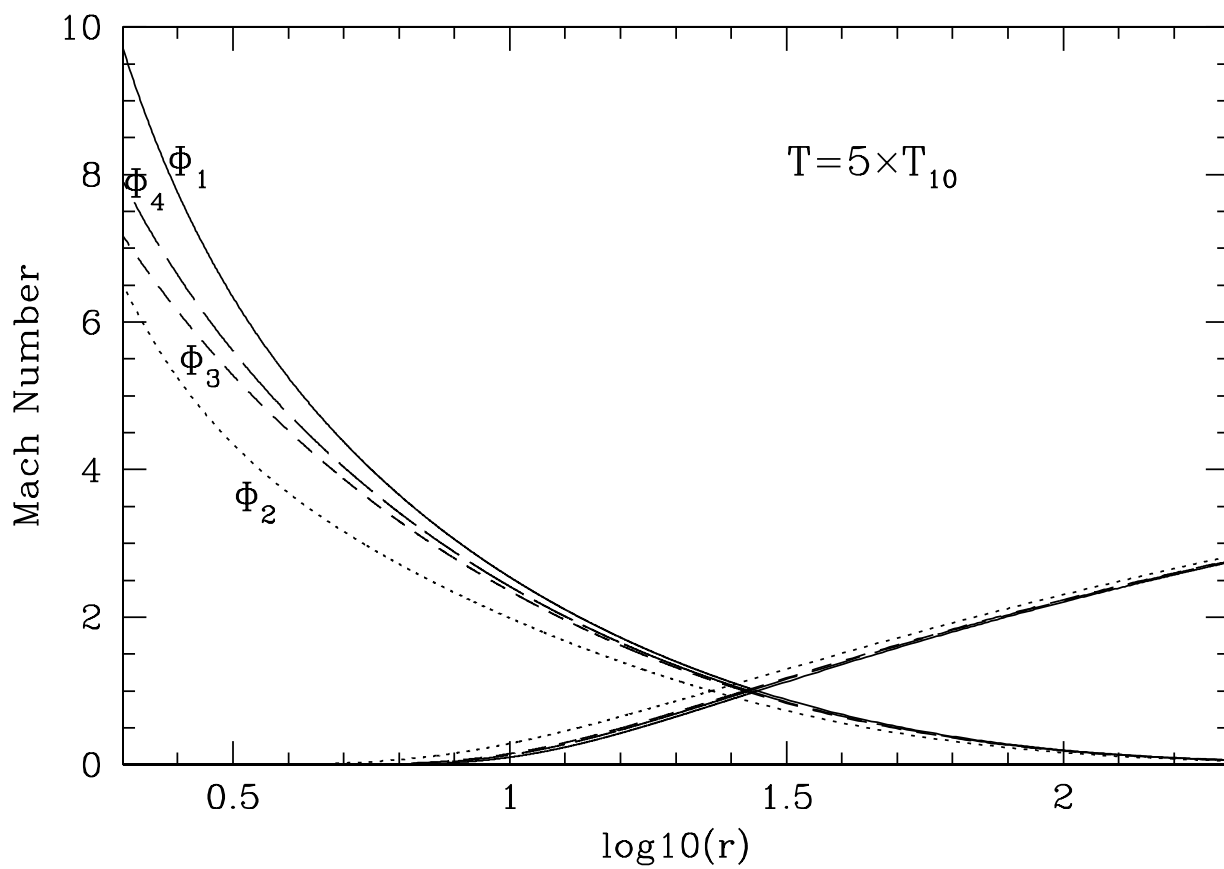
- [1] T. K. Das and A. Sarkar , *Astron. Astrophys.* 374, 1150 (2001) (PI)
- [2] H. Bondi, *Monthly Notices Roy. Astron. Soc.* 112, 195 (1952)
- [3] T. K. Das, *Monthly Notices Roy. Astron. Soc.* 318, 294 (2000)
- [4] B. Paczyński and P. J. Wiita, *Astron. Astrophys.* 88, 23 (1982)
- [5] A. M. Nowak and R. V. Wagoner, *Astrophys. J.* 378, 656 (1991)
- [6] I. V. Artemova, G. Bjornsson and I. D. Novikov *Astrophys. J.* 461, 565 (1996)

## Figure Captions

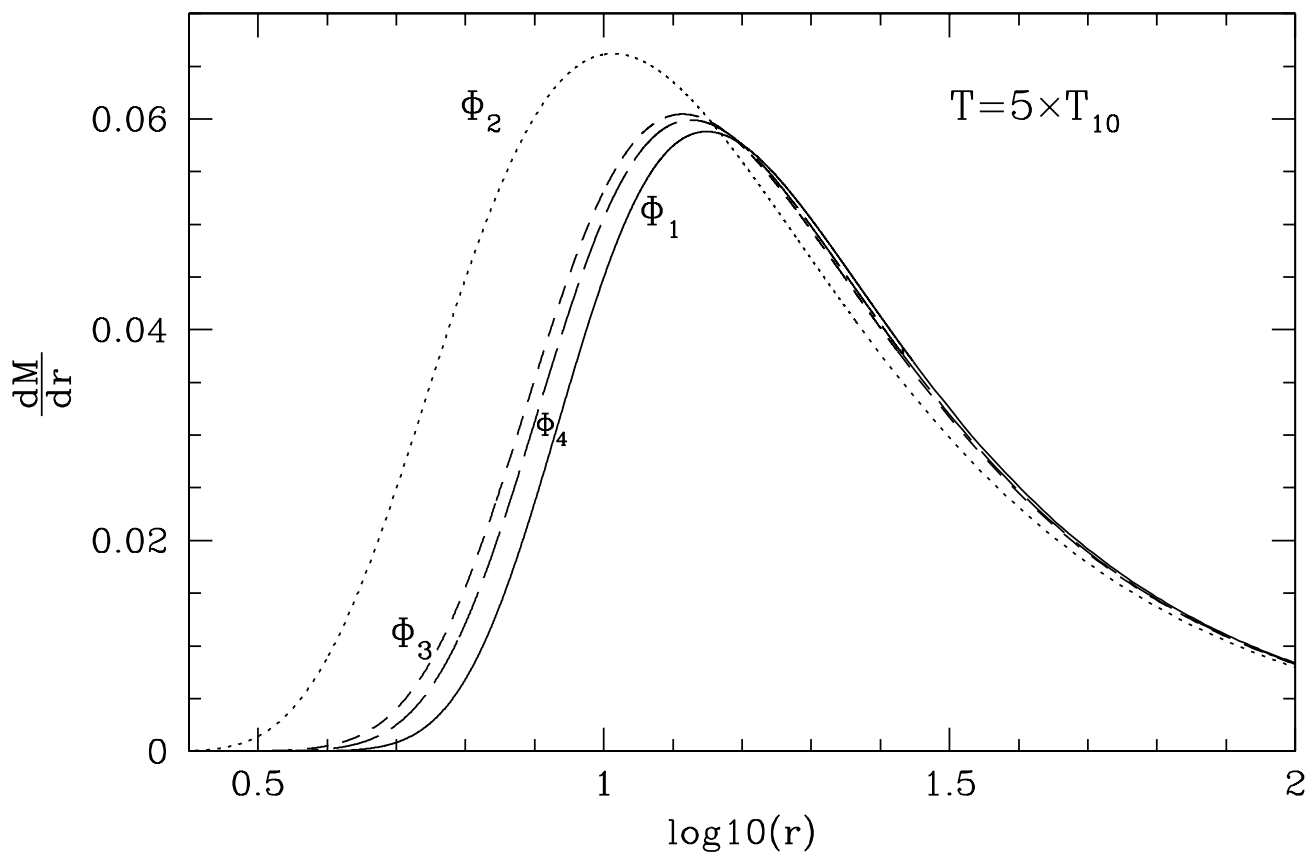
1. **Fig. 1:** Variation of sonic point with flow temperature for various  $\Phi_{i=1-4}$ 's. While flow temperature  $T$  (in units of  $T_{10} = 10^{10} \text{ } ^\circ K$ ) is plotted (in logarithmic scale) along x axis, sonic point  $r_c^i$  (in logarithmic scale) is plotted along y axis.
2. **Fig. 2:** The integral curves of motion drawn for various  $\Phi_{i=1-4}$ . While the radial distance from the black hole (in units of  $r_g$ ) is plotted along x axis in logarithmic scale, Mach number  $M$  of the flow is plotted along y axis. The sonic points for various  $\Phi_{i=1-4}$ 's are obtained as  $\{r_c^1, r_c^2, r_c^3, r_c^4\} = \{28.05, 23.14, 26.6, 27.09\} r_g$  respectively.
3. **Fig. 3:** Variation of gradient of Mach number for a given  $T$  with respect to spatial distance. While the radial distance from the accretor (in units of  $r_g$ ) is plotted along x axis in logarithmic scale, the spatial rate of change of Mach number  $\left(\frac{dM}{dr}\right)$  is plotted along y axis. It is clear that the curves obtained for all  $\Phi_{i=1-4}$  produces a 'peak' location of which is designated as  $r_p$  in eqn. (8).  $r_p$  is different for different  $\Phi_i$ 's for a particular given flow temperature  $T$ .



**Fig. 1:**  
**Aveek Sarkar and Tapas K. Das**  
**One Parameter Solution of Spherically Symmetric Accretion in Various**  
**Pseudo-Schwarzschild Potentials**



**Fig. 2:**  
 Aavek Sarkar and Tapas K. Das  
 One Parameter Solution of Spherically Symmetric Accretion in Various  
 Pseudo-Schwarzschild Potentials



**Fig. 3:**  
**Aveek Sarkar and Tapas K. Das**  
**One Parameter Solution of Spherically Symmetric Accretion in Various**  
**Pseudo-Schwarzschild Potentials**