

Naked singularities in low energy, effective string theory

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Solutions to the equations of motion of the low energy, effective field theory emerging out of heterotic string theory are constructed by making use of well-known duality symmetries. Beginning with four-dimensional solutions of the Einstein-massless scalar field theory in the canonical frame we first construct the corresponding solutions in the string frame. Thereafter, using the T and S duality symmetries of the low energy string effective action we arrive at the corresponding uncharged, electrically charged and magnetically charged solutions. We also verify the status of the energy conditions for the solutions in the string frame. Several of the metrics found here are shown to possess naked singularities although the energy conditions are obeyed. Dual solutions exhibit a duality in the conservation/violation of the NEC/ANEC. Additionally, those backgrounds which conserve the NEC/ANEC seem to serve as counterexamples to cosmic censorship in the context of string theory.

I. INTRODUCTION

Solutions to the background field equations of string theory representing black holes, cosmologies etc. are abundant. Extensive investigations about various features of these solutions have been carried out over the last decade or so. It is therefore probably surprising that there are still newer solutions with distinguishing characteristics. The fact that the solution-set has not been completely exhausted is largely due to the nonlinearity of the equations (in the same way as in GR) as well as the plethora of fields that arise in compactified, low-energy effective string theory. Infact there still does exist a lot of scope so far as solution-construction is concerned.

In this paper, we add to the list of solutions by first looking at the well-known solutions of the Einstein-scalar field system which represent naked singularities. Among these solutions are the ones constructed by Janis, Newman and Winicour [3] which have been recently shown to be equivalent to the Wyman solutions [4] in [5]. There are further generalisations of these solutions for the non-static case discussed by Roberts [6] and more recently in [7] in the context of scalar field collapse. Higher dimensional generalisations, of the Wyman solutions have been constructed in [8] with essentially the same features (singular event horizons). All these metrics when written in the string frame are solutions of the background field equations in that frame—in the Einstein frame they are also solutions to those equations when written in the Einstein canonical frame. Given these solutions, we construct newer solutions by exploiting the symmetries of the low energy effective string theory. In particular, we first use T-duality to construct charged solutions (electric). Then, we use S-duality to write down the magnetic counterpart of the electric solution in the string frame. We also use other kinds of transformations—basically subsets of the full T-duality group, such as the Buscher formulae to find solutions in a different class.

It should be mentioned that in a recent paper [13] Kiem and Park have obtained general solutions of the Einstein-Maxwell-dilaton theory in D dimensions. These authors exploit the fact the D dimensional theory can be reduced to an effective two dimensional dilaton gravity model by a spherisymmetric ansatz for the D dimensional metric. Some of the solutions in their paper do already exist in the literature (notably the purely dilatonic solutions in diverse dimensions). Our aim here is to demonstrate how, using duality properties one can indeed obtain many other solutions which actually reduce to the known solutions under a specific choice of parameters. The *new* solutions in this paper are : (a) the magnetically charged solution in the string frame and its various limiting cases (b) the T dual pure scalar field solution in the string frame obtained using the Buscher formulae (b) the electrically charged solution in the string frame obtained by using an $O(2, 1)$ transformation of the string frame JNW solution.

After arriving at the various solutions we then move on towards constructing their physical properties and also check out matters related to the nature of the singularities, energy conditions etc. Most of the metrics we obtain actually have naked singularities even though they seem to obey the energy conditions. This prompts us to comment that within the context of string theory we have examples of violations of cosmic censorship. This fact was noted for the Einstein-scalar system by Roberts and there do exist recent verifications of this in the context of scalar field

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collapse through the numerical work of Choptuik and others [14] which indicate the existence of point-mass black holes (essentially naked singularities!).

It may be argued that there is no need to look into the properties of matter in the string frame because the Einstein frame solutions do not have any problem with the energy conditions. However, it is an acceptable fact that no consensus has been reached as yet about which frame is *more* important in string theory. This is largely due to the fact that certain symmetries appear to be valid in the string frame whereas others hold good in the Einstein frame. Moreover, since the string frame metric is the σ -model metric it is this metric that a 'string sees'.

The paper is organised as follows. We first write down the relevant actions and the equations of motion in II and also the known solutions of the Einstein-massless-scalar system. Then, using the duality transformations we enumerate the various charged as well as uncharged solutions. In Section III the properties of the various solutions are discussed—the singularities as well as the energy conditions. Section IV contains a summary and concluding remarks. The Appendix to the paper contains relevant expressions for the Riemann, Ricci, Einstein tensors and the Ricci scalar for the class of metrics under consideration.

We follow the sign conventions of MTW [9].

II. ACTION, EQUATIONS AND SOLUTIONS

Our starting point is the four dimensional, low energy effective action of heterotic theory compactified on a six-torus. The field content is—the graviton ($g_{\mu\nu}$), antisymmetric tensor field ($B_{\mu\nu}$), Maxwell field, (A_μ), dilaton (ϕ). We have retained only one of the 16 U(1) gauge fields present in the action written in, say [10] and also set the various moduli fields (M) to zero.

The action is given as :

$$S_{eff} = \int d^4x \sqrt{-g} e^{-2\phi} \left[R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - F_{\mu\nu} F^{\mu\nu} \right] \quad (1)$$

The corresponding equations of motion obtained by performing variations with respect to the various fields $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ , and A_μ are given as follows :

$$R_{\mu\nu} = -2\nabla_\mu \nabla_\nu \phi + 2F_{\mu\lambda} F_\nu^\lambda + \frac{1}{4} H_{\mu\lambda\sigma} H_\nu^{\lambda\sigma} \quad (2)$$

$$\nabla^\nu (e^{-2\phi} F_{\mu\nu}) + \frac{1}{12} e^{-2\phi} H_{\mu\nu\rho} F^{\nu\rho} = 0 \quad (3)$$

$$\nabla^\mu (e^{-2\phi} H_{\mu\nu\rho}) = 0 \quad (4)$$

$$4\nabla^2 \phi - 4(\nabla\phi)^2 + R - F^2 - \frac{1}{12} H^2 = 0 \quad (5)$$

The above set of equations are written for the metric in the string (Brans-Dicke) frame.

Assuming the Maxwell and antisymmetric tensor fields to be zero we get the following spherically symmetric, static solution :

$$ds_{str.}^2 = - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} dt^2 + \left(1 - \frac{2\eta}{r}\right)^{\frac{\sigma-m}{\eta}} dr^2 + \left(1 - \frac{2\eta}{r}\right)^{1+\frac{\sigma-m}{\eta}} r^2 d\Omega^2 \quad (6)$$

$$\phi = \frac{\sigma}{2\eta} \ln \left(1 - \frac{2\eta}{r}\right) \quad (7)$$

where m is the mass, σ is the scalar charge and η is given by $\eta^2 = m^2 + \sigma^2$. For $\sigma = 0$, this solution reduces to the Schwarzschild solution. Reality of the metric coefficients indicates that we confine ourselves to the domain $r \geq \eta$. Even for $m = p\eta, \sigma = q\eta$, with p, q as integers we end up with $p^2 + q^2 = 1$ which contradicts the assumption that p, q be integers. We shall also see later on that the metric has a naked singularity at $r = 2\eta$.

We can rewrite the above metric in the Einstein canonical frame (as written in the papers of Wyman [4] and Roberts [6] by employing the standard relations between the two metrics $g_{\mu\nu}^{str.} = e^{2\phi} g_{\mu\nu}^E$. Therefore, just as all vacuum solutions of GR are also solutions of low energy string theory, these scalar field solutions are also solutions of the string effective action. This simple fact has however not been noticed in the literature till now.

Before we get into constructing the charged solutions using the duality properties of the action and the equations of motion let us look at the various limiting values of the parameters m, σ and η in the solution stated above.

For $m = 0$ we note that $\eta = \pm\sigma$. If we assume $\eta = -\sigma$ then we have :

$$ds^2 = -\frac{1}{1 - \frac{2\eta}{r}} (-dt^2 + dr^2) + r^2 d\Omega^2 \quad (8)$$

On the other hand, with $\eta = +\sigma$ the metric turns out to be :

$$ds^2 = \left(1 - \frac{2\eta}{r}\right)^2 \left[\frac{1}{1 - \frac{2\eta}{r}} (-dt^2 + dr^2) + r^2 d\Omega^2 \right] \quad (9)$$

A. Charged Solutions

Given the above metrics, we can now move on towards constructing the corresponding electrically charged solutions by using the standard boosting procedure. This is an application of the target space duality symmetry of the string effective action (more precisely the set of transformations which gives the electrically charged solution form a subgroup of $O(2,1)$).

The only new part in the charged metric is in its g_{00} component. Additionally there does appear changes in the dilaton field and a non-zero vector potential appears which actually generates the electric charge of the solution.

We have :

$$\tilde{g}_{00} = \frac{g_{00}}{[1 + (1 + g_{00}) \sinh^2 \alpha]^2} = -\frac{\left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}}}{\left[1 + \left(1 - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}}\right) \sinh^2 \alpha\right]^2} \quad (10)$$

$$\tilde{A}_0 = -\frac{(1 + g_{00}) \sinh 2\alpha}{2\sqrt{2} [1 + (1 + g_{00}) \sinh^2 \alpha]} = -\frac{\left(1 - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}}\right) \sinh 2\alpha}{2\sqrt{2} \left[1 + \left(1 - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}}\right) \sinh^2 \alpha\right]} \quad (11)$$

$$e^{-2\tilde{\phi}} = e^{-2\phi} [1 + (1 + g_{00}) \sinh^2 \alpha] = \left(1 - \frac{2\eta}{r}\right)^{-\frac{\sigma}{\eta}} \left[1 + \left(1 - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}}\right) \sinh^2 \alpha\right] \quad (12)$$

The only nonzero component of the field strength of the Maxwell field can be shown to be equal to :

$$F_{rt} = \frac{(m + \sigma) \sinh 2\alpha}{r^2} \frac{\left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta} - 1}}{\sqrt{2} \left[1 + \left(1 - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}}\right) \sinh^2 \alpha\right]^2} \quad (13)$$

Therefore, as $r \rightarrow \infty$, $F_{rt} \rightarrow \frac{(m+\sigma) \sinh 2\alpha}{\sqrt{2} r^2}$ from which we can read off the expression for the electric charge. Also, the mass M can be obtained by comparing the Einstein frame metric with the Schwarzschild solution. We therefore have,

$$Q_E = \frac{(m + \sigma) \sinh 2\alpha}{\sqrt{2}} ; \quad M = m + (m + \sigma) \sinh^2 \alpha \quad (14)$$

Note that for $\sigma = 0$ the scalar field reduces to zero and we get back the uncharged, Schwarzschild black hole for $\alpha = 0$ and the electrically charged stringy black hole for $\alpha \neq 0$. The presence of a third parameter σ is responsible for a different relation between M and Q_E which can be written as :

$$Q_E^2 = 2(M - m)(M + \sigma) \quad (15)$$

Note, that this relation has a symmetry under the interchange $m \rightarrow -\sigma; \sigma \rightarrow -m$.

For $\sigma = 0$ it, ofcourse, reduces to the standard relation between the charges and masses of the charged black hole. .
If one assumes $m = 0$ straightaway we get :

$$Q_E^2 = 2M(M + \sigma)$$

It is easy to see that, , holding $\sigma \sinh^2 \alpha$ fixed and setting $\sigma \rightarrow 0, \alpha \rightarrow \infty$ we obtain :

$$Q_E^2 = 2M^2 \quad (16)$$

which may be interpreted as the minimum (maximum) electric charge for a given positive (negative) scalar charge. This limit is quite similar to the usual *extremal* limit one talks about in the electric/magnetic solutions. The metric for the electric solution in this limit takes the standard form :

$$ds^2 = - \left(1 + \frac{2M}{r} \right)^{-2} dt^2 + dr^2 + r^2 d\Omega^2 \quad (17)$$

which represents a solution with a null naked singularity at $r = 0$. The naked singularity at $r = 2\eta$ has shifted to $r = 0$ because we have taked the $\sigma = \eta \rightarrow 0$ limit.

As is done for the electrically charged black holes, we can now use the electric-magnetic duality symmetry to obtain the corresponding magnetically charged solution in the string frame.

The magnetically charged solution is related to the electrically charged one through the relation :

$$ds_{mag.}^2 = \left(\exp(-4\tilde{\phi}) \right) ds_{elec}^2 \quad (18)$$

which leads to :

$$ds_{mag.}^2 = - \left(1 - \frac{2\eta}{r} \right)^{\frac{m-\sigma}{\eta}} dt^2 + \left(1 - \frac{2\eta}{r} \right)^{\frac{-m-\sigma}{\eta}} \left(1 + \left\{ \left(1 - \left(1 - \frac{2\eta}{r} \right)^{\frac{m+\sigma}{\eta}} \right) \sinh^2 \alpha \right\} \right)^2 dr^2 \\ + \left(1 - \frac{2\eta}{r} \right)^{1-\frac{m+\sigma}{\eta}} r^2 \left(1 + \left\{ \left(1 - \left(1 - \frac{2\eta}{r} \right)^{\frac{m+\sigma}{\eta}} \right) \sinh^2 \alpha \right\} \right)^2 d\Omega^2 \quad (19)$$

with the magnetic field given by $F_{\theta\phi} = Q_E \sin \theta d\theta \wedge d\phi$ and the new dilaton related to the old one only through a change of sign.

For $\sigma = 0$ and $m = \eta$ we can see that the solution reduces to the standard magnetically charged black hole with the following redefinitions :

$$M = \eta \cosh^2 \alpha \quad ; \quad Q = \sqrt{2}\eta \sinh \alpha \cosh \alpha \quad ; \quad \bar{r} = r + 2\eta \sinh^2 \alpha$$

The metric takes the usual form :

$$ds^2 = \frac{1 - \frac{2M}{r}}{1 - \frac{Q^2}{M\bar{r}}} dt^2 + \frac{d\bar{r}^2}{\left(1 - \frac{2M}{r} \right) \left(1 - \frac{Q^2}{M\bar{r}} \right)} + r^2 d\Omega^2 \quad (20)$$

The other two limits $m = 0, \eta = \sigma$ and $m = 0, \eta = -\sigma$ turn out to yield the following two metrics :

(1) $m = 0 \quad , \quad \eta = \sigma :$

$$ds^2 = - \left(1 - \frac{2\eta}{r} \right)^{-1} dt^2 + \left(1 + \frac{2\eta}{r} \sinh^2 \alpha \right) \left[\frac{dr^2}{1 - \frac{2\eta}{r}} + r^2 d\Omega^2 \right] \quad (21)$$

(2) $m = 0 \quad , \quad \eta = -\sigma :$

$$ds^2 = - \left(1 - \frac{2\eta}{r} \right) dt^2 + \frac{\left(1 - \frac{2\eta}{r} \cosh^2 \alpha \right)^2}{1 - \frac{2\eta}{r}} dr^2 + \left(r - 2\eta \sinh^2 \alpha \right)^2 d\Omega^2 \quad (22)$$

Lastly, setting $m = 0$ and then allowing $\sigma \rightarrow 0, \alpha \rightarrow \infty$ with $M = \sigma \sinh^2 \alpha$ fixed yields the usual non-singular infinite throat solution given as :

$$ds^2 = -dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} + r^2 d\Omega^2 \quad (23)$$

This is the sort of *extremal* limit we obtain for this class of solutions. The only difference of this solution (as well as the extremal limit of the electrically charged hole) with the extremal limits of the solutions obtained by boosting the Schwarzschild is that m is now replaced by σ and therefore $M = \sigma \sinh^2 \alpha$. The various solutions and their limiting cases are tabulated below in Table I.

TABLE I

| m | σ | η | α | Solution |
|--|--|---|---|--|
| Nonzero | Nonzero | Nonzero | 0 | JNW (Wyman) |
| Nonzero | 0 | $\eta = \pm m$ | 0 | Schwarzschild |
| 0 | Nonzero | $\eta = \pm \sigma$ | 0 | $m = 0$ JNW (Wyman) |
| Nonzero | Nonzero | Nonzero | Nonzero | (a) Naked Elec. JNW (b) Dual Mag. JNW |
| 0 | Nonzero | $\eta = \pm \sigma$ | Nonzero | (a) $m = 0$ Elec. JNW (b) $m = 0$ Dual Mag. JNW |
| Nonzero | 0 | $\eta = \pm m$ | Nonzero | (a) Elec. GHS (b) Dual Mag. GHS |
| 0 | $\sigma \rightarrow 0,$ $\sigma \sinh^2 \alpha \rightarrow M$ | $\eta \rightarrow 0, \frac{\sigma}{\eta} \rightarrow 1$ | $\alpha \rightarrow \infty,$ $\sigma \sinh^2 \alpha \rightarrow M$ | (a) 'Extr.' Elec JNW (b) Dual 'Extr.' Mag. JNW |
| $m \rightarrow 0,$ $m \cosh^2 \alpha \rightarrow M$ | 0 | $\eta \rightarrow 0, \frac{m}{\eta} \rightarrow 1$ | $\alpha \rightarrow \infty,$ $m \cosh^2 \alpha \rightarrow M$ | (a) Extr. Elec GHS (b) Dual Extr. Mag. GHS |

In a sense, therefore these solutions are generalisations of the black hole solutions due to Garfinkle, Horowitz and Strominger. Their construction starts off with a Schwarzschild solution and uses the T-duality transformation. On the other hand, if one starts out with a solution in Einstein-massless scalar theory (which represents a naked singularity) the duality transformations provide a whole class of nakedly singular solutions which violate the tenets of cosmic censorship. Setting the dilaton charge to zero we get back the black holes with regular event horizons. One might be tempted to say, at this point that the presence of the dilaton somehow forbids the existence of regular horizons in this class of solutions and thereby acts as a parameter which controls the existence-non-existence of naked singularities!

B. Solutions using the Buscher formulae

One can also obtain new solutions by using the Buscher formulae. For the case when the antisymmetric tensor field is zero and with no Maxwell field present, we have the simple relations :

$$\bar{g}_{00} = \frac{1}{g_{00}} \quad ; \quad \bar{\phi} = \phi - \frac{1}{2} \ln(-g_{00}) \quad (24)$$

where $(\bar{g}_{\mu\nu}, \bar{\phi})$ is the new solution.

For the string frame metric given earlier, we therefore have the new solution given as :

$$ds_{str.}^2 = - \left(1 - \frac{2\eta}{r}\right)^{-\frac{m-\sigma}{\eta}} dt^2 + \left(1 - \frac{2\eta}{r}\right)^{\frac{\sigma-m}{\eta}} dr^2 + \left(1 - \frac{2\eta}{r}\right)^{1+\frac{\sigma-m}{\eta}} r^2 d\Omega^2 \quad (25)$$

with the dilaton as :

$$\bar{\phi} = -\frac{m}{2\eta} \ln \left(1 - \frac{2\eta}{r}\right) \quad (26)$$

Notice that this solution goes over to the Schwarzschild (negative or positive mass depending on the sign of η) for $m = 0$. For $\sigma = 0$ we end up with a metric given as :

$$ds^2 = \left(1 - \frac{2\eta}{r}\right)^{\mp 1} [-dt^2 + dr^2 + r(r - 2\eta)d\Omega^2] \quad (27)$$

with $m = \pm\eta$. Note that these two solutions are exactly the same as the ones obtained earlier with $m = 0, \sigma = \pm\eta$. The Buscher transformations, in a sense, interchange the roles of m and σ . The general metrics (for arbitrary m and σ) are also mapped onto each other under the interchange :

$$m \rightarrow -\sigma \quad ; \quad \sigma \rightarrow -m$$

Additionally, one should note that one can use the $O(2,1)$ transformation and the electric–magnetic duality symmetry to obtain newer solutions from the one obtained above. In particular, all the previously stated charged solutions will be mapped onto newer ones by the simple use of the correspondence between m and σ .

III. PROPERTIES OF THE SOLUTIONS

We shall now embark upon enumerating the various properties of each of these solutions in somewhat more detail. More precisely, we focus on the the kind of singularities present and analyse the role of the *energy conditions* (i.e. their violation/conservation) . A list of all the Riemann tensor components, Ricci tensors and the Ricci scalar for a class of metrics which contains the metrics under discussion here as special cases is given in the Appendix. We shall make use of these results in this section.

A. Singularities

The Riemann tensor components, after substitution of the various functional forms for $f(r)$, $g(r)$ and $h(r)$ in the expressions given in the Appendix turn out to have an inverse dependence on the quantity $(r - 2\eta)$. This indicates a divergence at $r = 2\eta$. Note also that the geometry has a horizon at $r = 2\eta$ (by virtue of the fact that a zero in g_{00} indicates a horizon in the geometry for static, spheri–symmetric metrics). Thus, we have a spacetime which virtually ends at $r = 2\eta$ where it has a singular horizon. It is easy to see that the Riemann tensor components for the charged solutions also possess a similar divergence at $r = 2\eta$ and therefore constitute solutions with naked singularities.

B. Energy Conditions

We shall be concerned with the Null Energy Condition (NEC) and the Averaged Null Energy Condition (ANEC) for the string frame metrics and matter fields given above.

The NEC and ANEC are stated as follows :

NEC : For all null k^μ we must have $T_{\mu\nu}k^\mu k^\nu \geq 0$. In the case of a diagonal $T_{\mu\nu}$ with components (ρ, τ, p, p) we therefore need to prove :

$$\rho + \tau \geq 0 \quad ; \quad \rho + p \geq 0 \quad (28)$$

Physically, the NEC implies the positivity of matter energy density in all frames of reference. It should be noted that there are other local Energy Conditions (such as the Weak Energy Condition, the Strong Energy Condition and the Dominant Energy Condition) which also appear as assumptions in the proof of the Hawking–Penrose singularity theorems. We choose the NEC because it is one of the weakest amongst all.

ANEC : For all null k^μ we must have :

$$\int_{\lambda_1}^{\lambda_2} T_{\mu\nu}k^\mu k^\nu d\lambda \geq 0 \quad (29)$$

where the integration is along a null curve with the parameter λ being the generalised affine parameter in the sense of Hawking and Ellis [15]). For null geodesics, λ is the usual affine parameter.

It is important to note that for radial null geodesics, the integrand will contain $\rho + \tau$ only, apart from the usual exponential of the combination of the dilaton and the redshift function.

As for the local conditions there also exist global versions of the Weak Energy Condition where the null curve along which the integration is performed in the ANEC are replaced by a non–spacelike one.

The global energy conditions constitute valid assumptions under which the singularity theorems can be proved.

We now discuss each of the cases separately :

Case 1 : Pure dilatonic solution and its Buscher dual

The NEC here reduces to the following two inequalities :

$\rho + \tau \geq 0$ Inequality

$$\left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}-2} \left[\frac{4\sigma(r-\eta+\sigma)}{r^4} \right] \geq 0 \quad (30)$$

$\rho + p \geq 0$ Inequality :

$$\left(1 - \frac{2\eta}{r}\right)^{\frac{m-\sigma}{\eta}-2} \left[\frac{2\sigma(-r+2m+\eta)}{r^4} \right] \geq 0 \quad (31)$$

The first of these holds for all $r \geq 2\eta$ while the second one is violated beyond $r = 2m + \eta$. For the Buscher dual solution we make a replacement $m \rightarrow -\sigma, \sigma \rightarrow -m$. This results in a violation of the first inequality but a conservation of the second one. One can therefore see the reflection of duality in the violation/conservation of the NEC. The ANEC evaluated along radial null geodesics, by virtue of being a global inequality will hold good for the first solution but will be violated for the second one.

Case 2 : The electric solutions

The NEC inequalities for the electric solutions turn out to be :

$\rho + \tau \geq 0$ Inequality :

$$2 \left(1 - \frac{2\eta}{r}\right)^{\frac{m-\sigma}{r}} \frac{\cosh^2 \alpha [2\sigma(r+\sigma-\eta)] + \sinh^2 \alpha \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} [2m(r-m-\eta)]}{r^2(r-2\eta)^2 \left(\cosh^2 \alpha - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} \sinh^2 \alpha \right)} \geq 0 \quad (32)$$

$\rho + p \geq 0$ Inequality :

$$\frac{\left[2\sigma \cosh^4 \alpha A(r) + \sinh^2 \alpha \cosh^2 \alpha \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} B(r) + 2m \sinh^4 \alpha \left(1 - \frac{2\eta}{r}\right)^{\frac{2(m+\sigma)}{\eta}} C(r) \right]}{r^2(r-2\eta)^2 \left(\cosh^2 \alpha - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} \sinh^2 \alpha \right)^2 \left(1 - \frac{2\eta}{r}\right)^{\frac{\sigma-m}{\eta}}} \geq 0 \quad (33)$$

where,

$$A(r) = 2m - r + \eta ; B(r) = 2m(4m - r + \eta) + 2\sigma(4\sigma + 4m - 2\eta + r)$$

$$C(r) = r - \eta + 2\sigma$$

The L.H. S of these two inequalities are shown in Fig. 1(a) and 1(b). The values of σ, m and hence η are as follows : (a) $\sigma = 2m = 1, \eta = \frac{\sqrt{5}}{2}$ (dashed curve) (b) $m = 2\sigma = 1, \eta = \frac{9\sqrt{5}}{2}$ (dot-dashed curve) (c) $\sigma = m = \frac{1}{\sqrt{2}}, \eta = 1$ (solid curve). We choose $\cosh^2 \alpha = 2$. Notice that the $\rho + \tau \geq 0$ inequality is always satisfied while the second inequality is certainly violated beyond a certain value of $r > 2\eta$. Therefore the local NEC is essentially violated beyond a certain value of $r > 2\eta$.

Case 3 : The magnetic solutions

The NEC inequalities are given as :

$\rho + \tau \geq 0$ Inequality :

$$2 \frac{\left[-\sigma \cosh^2 \alpha X(r) + \cosh^2 \alpha \sinh^2 \alpha \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} Y(r) + \sinh^4 \alpha \left(1 - \frac{2\eta}{r}\right)^{\frac{2(m+\sigma)}{\eta}} Z(r) \right]}{r^3 \left(1 - \frac{2\eta}{r}\right)^2 \left(\cosh^2 \alpha - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} \sinh^2 \alpha \right)^2} \geq 0 \quad (34)$$

where

$$X(r) = 1 - \frac{\eta+\sigma}{r}, Y(r) = 2\sigma \left(1 - \frac{\eta+\sigma}{r}\right) + (m+\sigma) \left(-1 + \frac{3\sigma+m+\eta}{r}\right)$$

$$Z(r) = -\sigma \left(1 - \frac{\eta+\sigma}{r}\right) - (m+\sigma) \left(-1 + \frac{\sigma-m+\eta}{r}\right)$$

$\rho + p \geq 0$ Inequality :

$$\begin{aligned}
& \frac{\left[-\sigma \cosh^2 \alpha X(r) + \cosh^2 \alpha \sinh^2 \alpha \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} Y(r) + \sinh^4 \alpha \left(1 - \frac{2\eta}{r}\right)^{\frac{2(m+\sigma)}{\eta}} Z(r) \right]}{r^3 \left(1 - \frac{2\eta}{r}\right)^2 \left(\cosh^2 \alpha - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} \sinh^2 \alpha \right)^2} \\
& + \frac{\left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} \left[r(r-2\eta) - (r-m-\sigma-\eta) + r(r-2\eta)P(r) \right]^2}{r^2(r-2\eta)^2 \left(\cosh^2 \alpha - \left(1 - \frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} \sinh^2 \alpha \right)^2} \\
& (m-\sigma) \frac{2(m-r+\eta) + r(r-2\eta)P(r)}{r^2(r-2\eta)^2} \geq 0 \tag{35}
\end{aligned}$$

where $P(r) = -\frac{2(m+\sigma)\left(1-\frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}-1} \sinh^2 \alpha}{\cosh^2 \alpha - \left(1-\frac{2\eta}{r}\right)^{\frac{m+\sigma}{\eta}} \sinh^2 \alpha}$

These L.H.S of the inequalities are plotted in Figures 2 (a), 2 (b) and 2 (c) respectively. Fig 2(b) and 2(c) plot the same quantity but in different domains of r . The values of σ, m, η and $\cosh^2 \alpha$ are the same as before.

It is quite clear from the figures that the first inequality is violated everywhere—infact the violation is infinite in value as one approaches the point $r = 2\eta$. Therefore, there will also be a violation in the ANEC along radial null geodesics. The second inequality is satisfied beyond a value $r = r_0 \geq 2\eta$. Note that the behaviour of the two inequalities is somewhat opposite to those for the electric solutions. Recall that for the electric solutions the first inequality is satisfied for values of $r \geq 2\eta$ whereas the second one is violated in a similar domain ($r > 2\eta$). In the magnetic case, the first inequality is violated for all $r \geq 2\eta$ while the second one is conserved beyond a certain $r > 2\eta$. Therefore, it is worth noting once again that dual solutions do exhibit a duality in the behaviour of the energy condition inequalities. In particular a duality is clearly seen for the ANEC along radial null geodesics.

IV. CONCLUSIONS

We first summarize the results obtained.

(i) Beginning with the known solutions of the Einstein–scalar system we first construct a variety of solutions by utilising the duality properties of the theory. Specifically, we obtain the T-dual partner of the well-known JNW solution, the electric and magnetically charged metrics are also obtained by using the electric-magnetic duality symmetry. The relations between various charges and masses are also written down.

(ii) Apart from the solutions obtained we analyse the status of the Null Energy Condition and find it to be violated in most cases. However, the behaviour of the local inequalities exhibit a sort of ‘approximate duality’ symmetry. However, for the ANEC evaluated along radial null geodesics we find a clear evidence of conservation for the electric solution and a violation for the magnetic solution. Thus, as has been discussed before in the stringy black hole context [16] as well as for cosmologies [17] we discover a ‘duality’ in the conservation/ violation of the ANEC for the whole class of metrics parametrized by three quantities Q, M, σ .

(iii) The solutions which satisfy the ANEC seem to be a very new class of spacetimes – such examples are not abundant in the literature—and seem to suggest an extension of the Cosmic Censorship Conjecture to accommodate spacetimes which satisfy such inequalities.

It remains to be seen whether the reflection of the duality symmetries of string theory in the energy conditions or their averaged versions is a generic feature of stringy spacetimes—whether they be black holes, cosmologies or naked singularities. The first two have been discussed earlier—this paper concludes the sequence of examples by dealing with the case of naked singularities as well. Work towards an understanding of a general proof of the relation between energy conditions and the duality symmetries of string theory are in progress and will be communicated in due course.

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APPENDIX

In this appendix, we list the various Riemann, Ricci tensors and the Ricci scalar for a general class of metrics. These are used in various parts of the paper.

The line element we consider is given as :

$$ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + h^2(r)d\Omega^2 \quad (36)$$

We choose the one-form basis (static observer's basis) to evaluate these quantities. This is given as :

$$e^0 = f(r)dt \quad ; \quad e^1 = g(r)dr \quad ; \quad e^2 = h(r)d\theta \quad ; \quad e^3 = h(r)\sin\theta d\phi \quad (37)$$

The nonzero components of the Riemann tensor are given as :

$$R_{110}^0 = \frac{1}{g^2} \left(\frac{f''}{f} - \frac{f'g'}{fg} \right) \quad ; \quad R_{220}^0 = R_{330}^0 = \frac{1}{g^2} \frac{f'h'}{fh} \quad (38)$$

$$R_{112}^2 = R_{113}^3 = \frac{1}{g^2} \left(\frac{h''}{h} - \frac{h'g'}{hg} \right) \quad ; \quad R_{232}^3 = \frac{1}{h^2} \left(1 - \left(\frac{h'}{g} \right)^2 \right) \quad (39)$$

The Ricci tensor components are given as :

$$R_{00} = \frac{1}{g^2} \left(\frac{f''}{f} - \frac{f'g'}{fg} + 2\frac{f'h'}{fh} \right) \quad (40)$$

$$R_{11} = -\frac{1}{g^2} \left(\frac{f''}{f} - \frac{f'g'}{fg} \right) - \frac{2}{g^2} \left(\frac{h''}{h} - \frac{h'g'}{hg} \right) \quad (41)$$

$$R_{22} = R_{33} = -\frac{1}{g^2} \left(\frac{h''}{h} - \frac{h'g'}{hg} + \frac{f'h'}{fh} \right) + \frac{1}{h^2} \left(1 - \left(\frac{h'}{g} \right)^2 \right) \quad (42)$$

The Ricci scalar is given as :

$$R = -\frac{2}{g^2} \left(\frac{f''}{f} - \frac{f'g'}{fg} \right) - \frac{4}{g^2} \left(\frac{h''}{h} - \frac{h'g'}{hg} \right) - \frac{4}{g^2} \frac{f'h'}{fh} + \frac{2}{h^2} \left(1 - \left(\frac{h'}{g} \right)^2 \right) \quad (43)$$

Given these expressions for $R_{\mu\nu}$ one can then evaluate the quantities $\rho + \tau = R_{00} + R_{11}$ and $\rho + p = R_{00} + R_{22}$ which comprise the L.H.S. of the null energy condition inequalities.

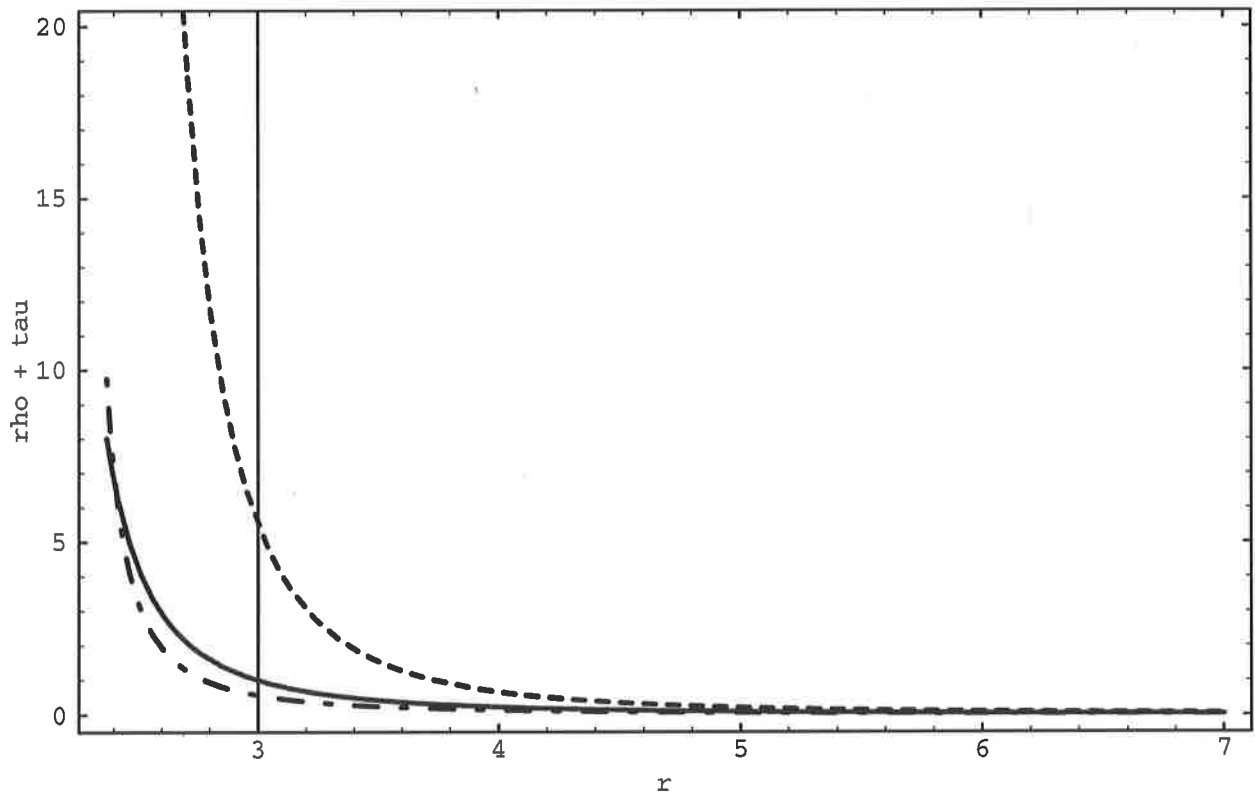
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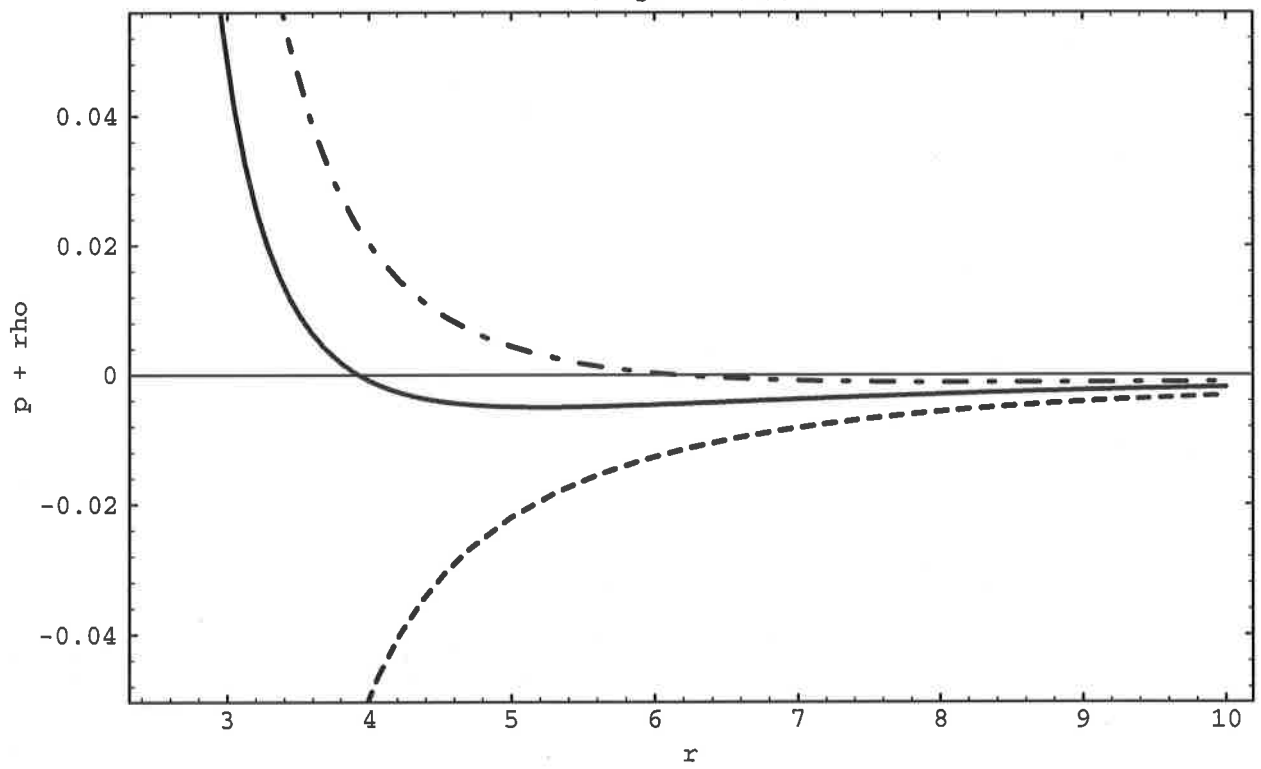
FIGURE CAPTIONS

- Figure 1(a)** : $\rho + \tau$ versus r for the electric solution.
- Figure 1(b)** : $\rho + p$ versus r for the electric solution.
- Figure 2(a)** : $\rho + \tau$ versus r for the magnetic solution.
- Figure 2(b)** : $\rho + p$ versus r for the magnetic solution.
- Figure 2(c)** : $\rho + p$ versus r for the magnetic solution in a different domain of r .

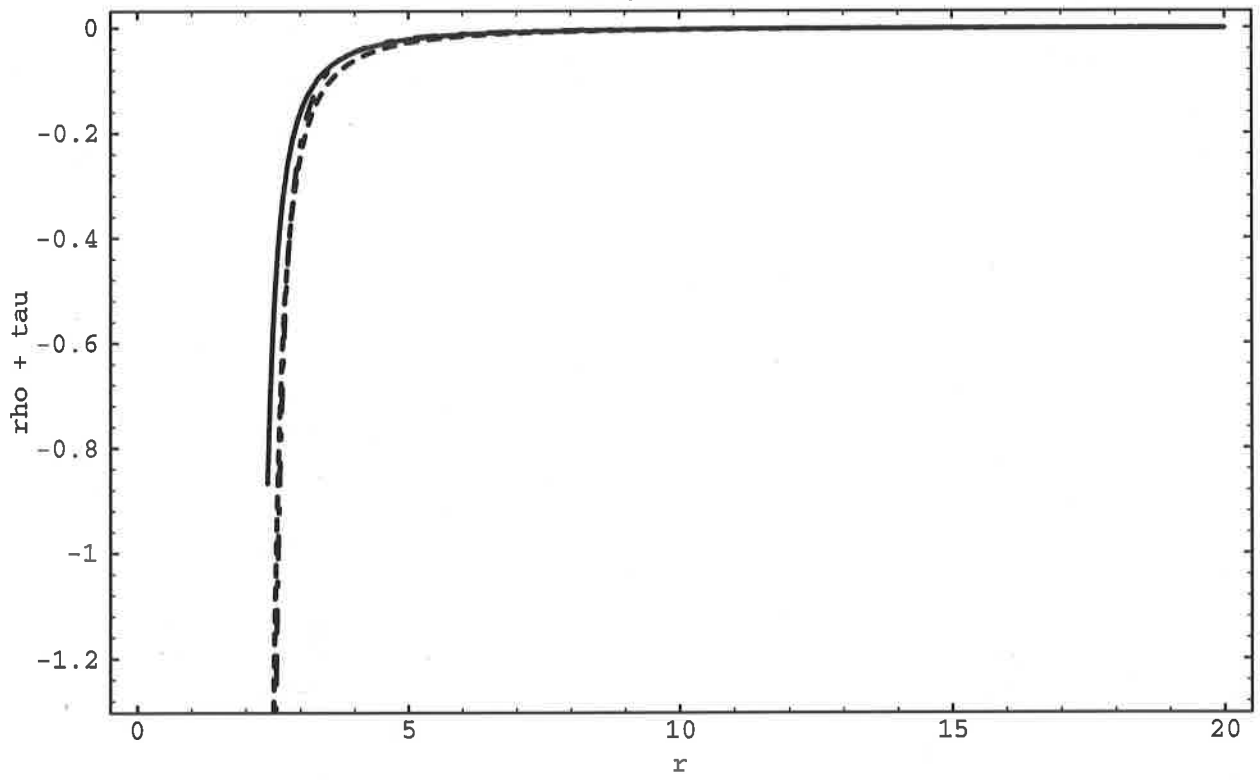
{Figure 1(a)}



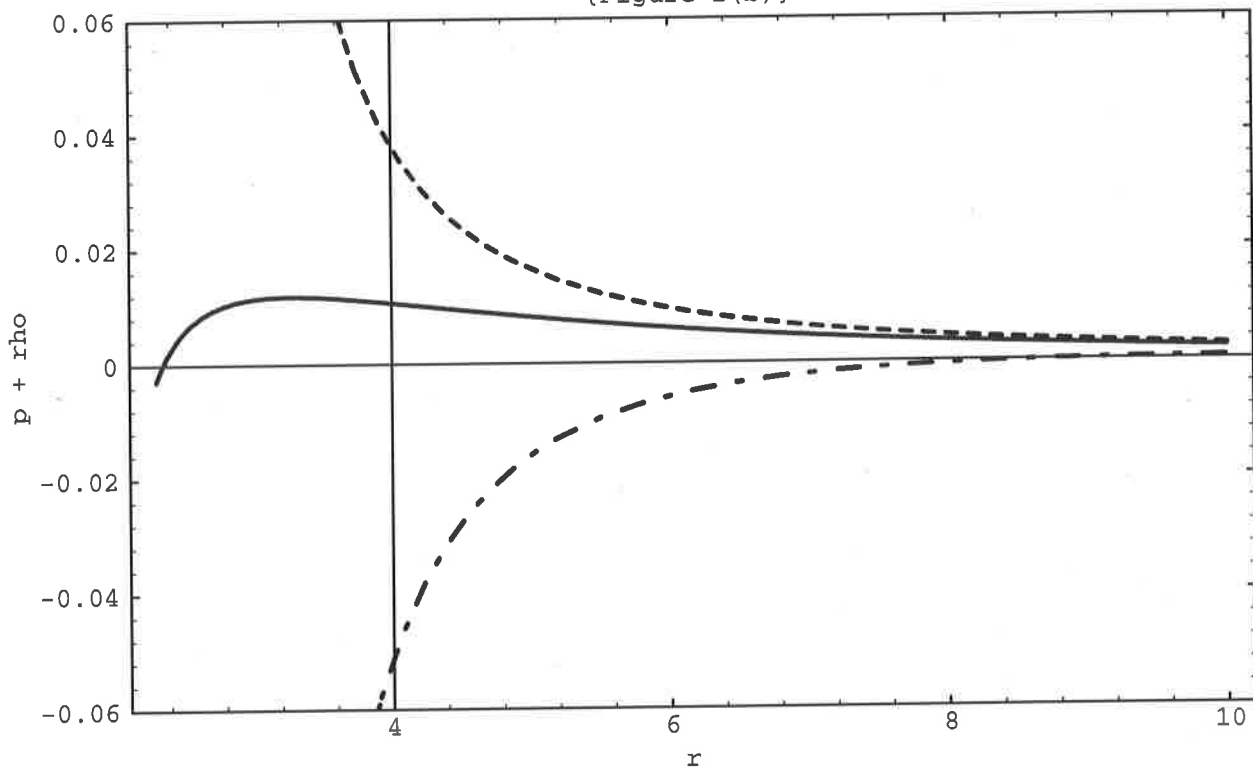
{Figure 1(b)}



{Figure 2(a)}



{Figure 2(b)}



{Figure 2(c)}

