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Conformal fluctuations in a quantum universe with a scalar field

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Quantum conformal fluctuations are analyzed using the path integral approach. Conformal fluctuations are introduced in a classical Robertson-Walker universe with a massless scalar field as a source. It is shown that the conformal fluctuations diverge at the classical singularity. However, the effective metric still retains the singularity. Some other aspects of introducing the scalar field in this quantum gravity model are briefly discussed.

I. INTRODUCTION

Path integrals have been used in the recent past to provide some insight into quantum gravity models.[1–6] The technique, essentially, consists of concentrating on some selected degrees of freedom, (e.g., the conformal factor) and leads to the following result: When the space-time metric has a system of noninteracting dust particles as its source, the quantum conformal fluctuations diverge at the classical singularity.[3] This method also leads to the concept of stationary states for the geometry and introduces a lower bound on the length scale.[6]

It is important to understand the nature of the quantum fluctuations when the source consists of fields rather than dust. We investigate here a simple model with a massless scalar field as the source, and show that the fluctuations again diverge at the classical singularity. This suggests that the result is of general validity. We also investigate the nature of the effective metric (for a discussion of this concept see Ref. 4) for this particular case. It turns out that the effective metric retains the singular behavior, in sharp contrast with universes containing dust as source. Thus the effective metric crucially depends on the nature of the source, and must be interpreted with care. Some other aspects of a scalar field as a source are discussed at the end.

II. CLASSICAL SOLUTION

Consider a model universe described by the Robertson-Walker (RW) line element written as

$$ds^2=S^2(t)\left[-dt^2+\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\right] . \tag{1}$$

We consider the source to be a massless scalar field described by the action

$$J=-\tfrac{1}{2}\int \partial_\mu\phi\partial^\mu\phi\sqrt{-g}\,d^4x . \tag{2}$$

Because of the symmetries of the RW metric, it is clear that ϕ can depend only on time t. Using the energy-momentum tensor from Eq. (2) as the source, one can write Einstein's equations as (a dot denotes differentiation with respect to time t)

$$\frac{3\dot S^2}{S^2}+3k=4\pi G\dot\phi^2 , \tag{3}$$

$$\frac{2\ddot S}{S}-\frac{\dot S^2}{S^2}+k=-4\pi G\dot\phi^2 . \tag{4}$$

It is, however, easier to work with the "equation of motion" for ϕ, which reads

$$\frac{d}{dt}(S^2\dot\phi)=0 , \tag{5}$$

leading to

$$\dot{\phi}=\frac{\alpha}{S^2}\alpha=\text{constant} . \tag{6}$$

Of course, Eqs. (3), (4), and (5) are related by Bianchi identities. Thus Eq. (6) can be put into Eq. (3) [or (4)] and integrated to give the metric function $S(t)$. For $k > 0$ this leads to the solution

$$S^2(t)=\left[\frac{A}{4k}\right]^{1/2}\sin 2\sqrt{k}\,(t+t_0) ,\qquad A=\left[\frac{16\pi G\alpha^2}{3}\right] , \tag{7}$$

and for $k < 0$ it is given by

$$S^2(t)=\left[\frac{A}{4|k|}\right]^{1/2}\sinh 2\sqrt{|k|}(t+t_0) \tag{8}$$

The scalar field $\phi(t)$ has the form

$$\phi(t)=\frac{\alpha}{\sqrt{A}}\ln\{\tan[\sqrt{k}\,(t+t_0)]\}+\phi_0 . \tag{9}$$

Here t_0, ϕ_0 are integration constants to be fixed by the initial conditions. For $k=0$, the solutions read

$$S^2(t)=\sqrt{A}\,(t+t_0) ,\qquad \phi(t)=\frac{\alpha}{\sqrt{A}}\ln|t+t_0|+\phi_0 . \tag{10}$$

We shall consider conformal fluctuations in the closed model with $k > 0$. Similar results hold for the other cases.

The space-time described by Eqs. (1) and (7) has a singularity in the past and future [when $\sqrt{2}$ $k(t+t_0)=0$ as well as $2\sqrt{k}(t+t_0)=\pi$]. At both these events $\phi(t)$ diverges (similar to the density of dust particles). We are interested in the conformal fluctuations near these points.

III. QUANTUM CONFORMAL FLUCTUATIONS

We shall not describe here the rationale and details of this approach, which may be found in the references cited previously.

Consider the conformally perturbed metric of the form

$$ds_F{}^2=\Omega^2(t)S^2(t)\left[-dt^2+\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\right] , \tag{11}$$

where $S(t)$ is the function solved for (in the previous section), while $\Omega(t)$ is a quantum variable. The principle of path integrals states that the probability amplitude for $\Omega(t)$ to go from a value Ω_1 at time t_1 to a value Ω_2 at time t_2 is given by the functional integral

$$K[\Omega_2 t_2;\Omega_1 t_1]=\int \mathscr{D}\Omega(t)\exp\left\{\frac{i}{\hbar}J[\Omega(t)]\right\} , \tag{12}$$

where J is the total action for the system, given by

$$J=\frac{1}{16\pi G}\int R_F\sqrt{-g_F}d^4x-\frac{1}{2}\int (g_F^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)\sqrt{-g_F}\,d^4x , \tag{13}$$

where the subscript F denotes the quantities evaluated for the metric in Eq. (11). However, the gravitational part of this action involves the second derivatives of $\Omega(t)$. This can be taken care of by adding necessary surface terms, explained in Refs. 1 and 2. When these modifications are made, the action J in Eq. (13) reads [we have also used Eq. (6)]

$$J=\frac{V}{16\pi G}\int_{t_1}^{t_2}\left[6S^2\left[k+\frac{\ddot{S}}{S}\right]\Omega^2-6S^2\dot{\Omega}^2\right]dt+\frac{V\alpha^2}{2}\int_{t_1}^{t_2}\frac{\Omega^2}{S^2}dt , \tag{14}$$

where V is the proper volume in which the fluctuations are studied (for a closed universe we can take it to be the total space integral.) A rearrangement leads to

$$\frac{A}{2V\alpha^2}J=\int_{t_1}^{t_2}dt\left\{\left[S^2\left[k+\frac{\ddot{S}}{S}\right]+\frac{A}{4S^2}\right]\Omega^2-S^2\dot{\Omega}^2\right\} \tag{15}$$

[where $A = 16\pi G\alpha^2/3$; see Eq. (7)].

We shall show that the coefficient of Ω^2 vanishes. (This is a result of the relation $R=-8\pi GT$.) From Eqs. (3) and (4) one gets

$$\frac{\dot{S}^2}{S^2}+\frac{\ddot{S}}{S}+2k=0 . \tag{16}$$

Using (3) one can remove the $\dot{S}^2/S^2$ term, giving

$$\frac{4\pi G}{3}\dot{\phi}^2+\frac{\ddot{S}}{S}+2k=0 . \tag{17}$$

This, on using Eqs. (6) and (8), leads to

$$\frac{A}{4S^2}+S^2\left[\frac{\ddot{S}}{S}+k\right]=0 , \tag{18}$$

which is what has to be proved. Thus [also using the explicit form for $S(t)$] the action takes the simple form

$$J=\int_{t_1}^{t_2} dt f(t)\dot{\Omega}^2 \ ,$$
$$f(t)=-\frac{V\alpha^2}{(Ak)^{1/2}}\sin 2\sqrt{k}\,(t+t_0) \ . \tag{19}$$

This is the action for which the functional integral has to be evaluated. This can be now trivially done because the action is quadratic in $\Omega(t)$. The kernel is simply

$$K[\Omega_2 t_2;\Omega_1 t_1]=\bar{F}(t_2 t_1)\exp\left[\frac{i}{\hbar}J\right] \ , \tag{20}$$

where J is the value of action evaluated for the classical path, which is simply

$$J=\frac{(\Omega_2-\Omega_1)^2}{F(t_2)-F(t_1)} \ ,$$

where

$$F(t)=\int^t \frac{dt}{f(t)} \ . \tag{21}$$

In our case

$$F(t)=-\frac{\sqrt{A}}{2V\alpha^2}\ln[\tan\sqrt{k}\,(t+t_0)] \ , \tag{22}$$

giving

$$K[\Omega_2 t_2;\Omega_1 t_1]=\bar{F}(t_2 t_1)\exp\left[-\frac{2iV\alpha^2}{\hbar\sqrt{A}}\frac{(\Omega_2-\Omega_1)^2}{\ln\left|\dfrac{\tan\sqrt{k}\,(t_2+t_0)}{\tan\sqrt{k}\,(t_1+t_0)}\right|}\right] . \tag{23}$$

Consider a wave packet which at time $t = t_1$ is peaked about $\Omega_1 = 1$ in the form

$$\psi(\Omega_1)=\left[\frac{1}{2\pi\Delta^2(t_1)}\right]^{1/4}\exp\left[-\frac{(\Omega_1-1)^2}{4\Delta^2(t_1)}\right] . \tag{24}$$

The wave function $\psi(\Omega)$ at any later time can be found by direct integration,

$$\psi(\Omega_2)=\int d\Omega_1 K[\Omega_2 t_2;\Omega_1 t_1]\psi(\Omega_1) \ . \tag{25}$$

The analysis is identical to that of spreading wave packets in ordinary quantum mechanics. Thus the dispersion $\Delta(t)$ at any later time can be written as

$$\Delta(t)=\Delta(t_1)\left[1+\frac{\beta\hbar^2}{\Delta^4(t_1)}\times\left[\ln\frac{\tan\sqrt{k}\,(t+t_0)}{\tan\sqrt{k}\,(t_1+t_0)}\right]\right]^{1/2} , \tag{26}$$

where β is a constant formed from various parameters. Quite clearly, at both the classical singularities, the dispersion $\Delta(t)$ diverges logarithmically, leading to infinite uncertainty. Thus the previous result that conformal fluctuations diverge at the classical singularity holds for this case also.

The "effective metric" (as introduced in Ref. 5), is defined as

$$\begin{aligned} g_{\mu\nu}^{\text{eff}} &= \int g_{\mu\nu}(\Omega)\,|\psi(\Omega)|^2 d\Omega \\ &= g_{\mu\nu}^{\text{old}}\int \Omega^2\,|\psi(\Omega)|^2 d\Omega \\ &= [1+\Delta^2(t)]g_{\mu\nu}^{\text{old}} \ . \end{aligned} \tag{27}$$

Thus, in our case, the effective metric has the expansion factor

$$S_{\text{eff}}^2(t)=[1+\Delta^2(t)]S_{\text{old}}^2(t) \ . \tag{28}$$

Near the singularity $\Delta^2(t) \to \infty$, $S_{\text{old}}^2 \to 0$. Whether S_{eff}^2 is singular or not depends on which effect predominates. By using the explicit forms we can see that, near the singularity,

$$S_{\text{eff}}^2 \sim \lim_{\epsilon\to 0}[\sqrt{\epsilon}\,\ln\epsilon]^2 \to 0 \ . \tag{29}$$

Thus the effective metric retains the singular behavior. An explicit calculation of the curvature constants confirms the result.

It was shown previously that in a RW universe with dust particles as a source the effective metric remains nonsingular, and the collapse of the dust ball is stopped at the Planck length. This can be understood in terms of the stationary states of the closed Friedmann model which has a lower bound at the Planck length.[6]

It is clear from the Lagrangian in Eq. (19) that no such lower bound can arise in the present case. The Schrödinger equation, for the present case, will read

$$i\hbar\frac{\partial\psi}{\partial\tau}=-\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial\Omega^2} \text{ with } \tau=\int\frac{dt}{f(t)}=F(t)\ . \tag{30}$$

A physically reasonable solution has the form

$$\psi(\tau,\Omega)=\sqrt{2k}\,e^{+iE\tau/\hbar}e^{-k\Omega} \text{ with } k^2=2E/\hbar^2\ . \tag{31}$$

The expectation value for Ω^2 has no (nonzero) lower bound. It is conceivable that the absence of a ground state with lower limit on $\langle\Omega^2\rangle$ leads to the singular behavior of the effective matrix. Further investigation is necessary before this connection can be established rigorously.

IV. SOME FURTHER REMARKS

In the above discussion we have treated the scalar field as a classical distribution and dealt with the quantum fluctuations of gravity. For complete understanding (especially for the discussion of stationary states) it is necessary to treat both the scalar field and the conformal factor as quantum variables. The state would be described by a complex amplitude $\psi(\phi,\Omega,t)$. A detailed investigation leads to the following possibilities (the results will be published later).

(i) The nature of the stationary states depends crucially on the source. For example, the $k=0$ RW universe has exponentially decaying wave functions for stationary states. However, if a scalar field is introduced as a quantum variable, the wave functions are described by modified Bessel functions.

(ii) The conformal factor appears in the action [see Eq. (14)] with a negative sign in the kinetic energy term. Because of this one can generate a scalar field from the vacuum state (by suitable quantum fluctuations) without any apparent violations of energy conservation. Essentially, the conformal factor acts as a negative-energy field. A detailed investigation is required before one can decide whether the solution would be of steady-state or of big-bang nature.

We wish to make one more concluding remark. It seems safe to conjecture, based on this and previous works, that "quantum conformal fluctuations *always* diverge at the classical space-time singularity."

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