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On the Brown-York quasilocal energy, gravitational charge, and black hole event horizons

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Abstract

We prove the recently proposed identity for certain black hole spacetimes that relates the difference of the Brown-York quasilocal energy and the Komar charge at the event horizon of the hole to the total energy of the spacetime. We prove this identity for the Kerr-Newman family of black hole spacetimes and for non-static (cosmological) spherically symmetric shear-free perfect fluid solutions of general relativity that contain black hole event horizons. We explicitly demonstrate its validity by applying it to several asymptotically flat as well as non-flat black hole solutions, including the case of a black hole with a global monopole charge.

I. INTRODUCTION

The concept of a black hole horizon, ever since its birth with Lemaitre's demonstration of the non-singularity of the Schwarzschild horizon, has played a monumental role in the understanding of the causal structure of several spacetimes. It has been associated with many general relativistic theorems and laws of import, eg., the singularity theorems [1], the black hole area theorem [2], and the classical laws of black hole mechanics. It plays an important role in Hawking's semiclassical calculation on the evaporation of a black hole [3] and is also related to its entropy [4]. Finally, although, the horizon does not have any special significance in the frame of a freely falling observer, to an asymptotic inertial observer it behaves very much like a physical membrane (see, eg., ref. [5]).

An apparent horizon is a local construct that is topologically defined to be the outer boundary of the union of trapped surfaces in a spacetime. On the other hand the event horizon is a surface from beyond which null rays cannot escape to future null infinity without

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violating causality. These definitions implicitly depend on the behavior of families of null geodesics in a given spacetime. Moreover, it is known that the properties of such a family are affected by the matter stress tensor through the Raychaudhuri equation [6]. This prompts us to ask if there exists a direct characterization of the black hole horizon in terms of the quasilocal energy or gravitational "charge" of bounded regions embedded in such spacetimes. To seek such a characterization will be the main aim of this paper.

It is well known that there are inherent difficulties in defining energy in general relativity (GR), essentially owing to its non-localizibility. So far, considerable effort has been put in to formulate a satisfactory definition. There exist various expressions for it in the literature, such as those based on tensors defined on a non-dynamical background metric [7], pseudo-tensors [8–12], integral expressions of energy of the type defined by Komar [13] and Hawking [14], and spinor constructions [15–20]. Every definition has some motivating physical/ geometrical considerations and emphasizes one or the other characteristic of the field. Also, the total energy of an isolated system has been defined in terms of the behavior of the gravitational field at large distances from the system [21]. More recently, Brown and York have introduced in Ref. [22] (henceforth referred to as BY) a way to define the quasilocal energy of a spatially bounded system in general relativity in terms of the total mean curvature of the boundary. Finally, the BY formalism has been extended to the case of a generic scalar-tensor theory of gravity in spacetime dimensions greater than two [23–25].

In most physical theories, there are separate measures for the mass (or energy) of a particle and its charge (which determines its strength of coupling to the field). By contrast, in GR there is a single measure for both. Besides, the gravitational field of a particle itself contributes to its gravitational charge since it contains energy. It is this aspect of the gravitational field that has been the main cause of ambiguity in attempts at distinguishing charge from energy in GR. However, one possibility for defining gravitational charge would be to suitably adopt the Gauss theorem to the case of GR [12]. Note that the measure of field strength at a point would be the proper acceleration relative to an asymptotic observer, defined by $\mathbf{g} = -N\nabla(\ln N)$, where N is the lapse function. Its flux through a closed 2-surface would give the gravitational charge according to the Gauss theorem. It is in fact equivalent to the Komar integral [13]:

$$M_c = -\frac{1}{8\pi}\oint_{\mathsf{B}} \epsilon_{abcd}\nabla^c\xi^b \quad , \tag{1.1}$$

where ξ^a is a timelike Killing vector and ϵ_{abcd} is the volume element on the spacetime. Equivalently it can also be given by the usual Gauss integral [12]

$$M_c = \frac{1}{4\pi}\oint_{\mathsf{B}} \mathbf{g}.\mathbf{ds} \quad , \tag{1.2}$$

where $\mathbf{g} = -N\nabla(\ln N)$, N is the lapse function and the integral is taken over the closed 2-surface B. The norm of $\mathbf{g}$ evaluated at the horizon gives the surface gravity of the hole. This is how gravitational charge is intimately related to surface gravity (and, therefore, the temperature) of a black hole. Equation (1.1) defines a conserved (both in space and time) gravitational charge when the spacetime is vacuum and admits a timelike Killing vector. When either of these conditions is relaxed, M_c depends upon the location of B. In such a case it describes the *quasilocal* charge associated with the spatial volume bounded by B.

In general, the measures of quasilocal charge and energy will be different. In what follows, by the energy and charge of a spacetime region, we shall mean the gravitational quasilocal energy and charge (references to the electric charge of that region will be made explicitly).

For quasilocal energy, we shall here adopt the definition given by Brown and York [22], which explicitly brings out contribution due to field energy. The BY derivation of the quasilocal energy can be summarised as follows. The system under consideration is a spatial three-surface Σ bounded by a two-surface B in a spacetime region that can be decomposed as a product of a spatial three-surface and a real line interval representing time. The time evolution of the two-surface boundary B is the timelike three-surface boundary ${}^3\mathsf{B}$. They then obtain a surface stress-tensor on the boundary by taking the functional derivative of the action with respect to the three-metric on ${}^3\mathsf{B}$. The energy surface density is the projection of the surface stress tensor normal to a family of spacelike two-surfaces like B that foliate ${}^3\mathsf{B}$. The integral of the energy surface density over such a two-surface B is the quasilocal energy associated with a spacelike three-surface Σ whose *orthogonal* intersection with ${}^3\mathsf{B}$ is the two-boundary B. This yields the following expression for the quasilocal energy:

$$E = \frac{1}{8\pi} \int d^2x \sqrt{\sigma}(K - K_0) \tag{1.3}$$

where σ is the determinant of the 2-metric on B, K is the trace of the extrinsic curvature of B, and the subscript 0 refers to a reference spacetime, not necessarily flat. It is clear from the above definition of charge that it is the lapse function that determines it. On the other hand the BY quasilocal energy is not at all sensitive to it and is instead determined entirely by the spatial metric.

Dadhich [26,27] has recently proposed a novel energetics characterization of the event horizon of a black hole in spherically symmetric spacetimes. He proposed that on a given spatial slice, its location is at that curvature radius, r, at which the gravitational field energy equals the gravitational charge. That is, at the horizon the following identity holds:

$$E_H - E_\infty = M_H \quad , \tag{1.4}$$

where E_H is the quasilocal energy at the horizon, E_∞ is the total energy of the spacetime, and M_H is the gravitational charge at the horizon. The physical interpretation of this identity is as follows. The gravitational charge is essentially the measure of a body's ability to produce gravitational pull (the usual Newtonian aspect), whereas the field energy produces curvature in space [28] to constrain free particle motion (the relativistic non-linear effect arising from the field energy density). The above identity implies that when these two contributions become equal the body turns into a black hole.

The layout of this paper is as follows. In section II, we prove the identity (1.4) characterizing a black hole event horizon for the Kerr-Newman family and for certain cosmological black holes. In section III, we explicitly verify the validity of this identity by applying it to several examples of black hole solutions. We end with a brief discussion on the implications of this formula in section IV.

II. PROVING THE IDENTITY

In this section we prove the identity (1.4) and explicitly state the assumptions under which it holds. The main ingredient in our proof is a local relation between the covariant derivative of the trace of the extrinsic curvature of the timelike three-boundary $^3\mathsf{B}$ and certain scalars of the Ricci tensor. We first obtain this relation from the Gauss-Codacci embeddability conditions. Note that these conditions are automatically satisfied because of the assumption that the boundary $^3\mathsf{B}$ is embeddable in the spacetime.

The first relation we require is the decomposition of the four-dimensional (4D) Ricci scalar into spatial and timelike components

$$\mathcal{R} = \gamma^{\mu\nu}\gamma^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + 2n^\mu n^\nu \mathcal{R}_{\mu\nu} \quad , \tag{2.1}$$

where $\gamma_{\mu\nu}$ is the 3-metric on $^3\mathsf{B}$ and n_μ is its spacelike normal with unit norm. The Gauss-Codacci relation for the projection of the Riemann tensor onto $^3\mathsf{B}$ gives the first term above:

$$\gamma^{\mu\nu}\gamma^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} = R - \Theta^2 + \Theta_{\mu\nu}\Theta^{\mu\nu} \quad , \tag{2.2}$$

where R is the 3D Ricci scalar associated with $^3\mathsf{B}$ and Θ is its extrinsic curvature. On using the Ricci identity, $\mathcal{R}_{\mu\alpha\nu\beta}n^\beta = 2\nabla_{[\mu}\nabla_{\alpha]}n_\nu$, the second term of Eq. (2.1) gives

$$n^\mu n^\nu \mathcal{R}_{\mu\nu} = \Theta^2 - \Theta_{\mu\nu}\Theta^{\mu\nu} + \nabla_\mu(\Theta n^\mu + b^\mu) \quad , \tag{2.3}$$

where $b^\mu \equiv n^\nu \nabla_\nu n^\mu$. Using the above Eqs. (2.3) and (2.2) in the decomposition formula (2.1), we obtain the relation we need

$$\nabla_\mu(\Theta n^\mu + b^\mu) + R = \mathcal{R} - n^\mu n^\nu \mathcal{R}_{\mu\nu} \, . \tag{2.4}$$

We now prove the identity generically for (a) asymptotically flat spherically symmetric static (SS) spacetimes and for (b) non-static (cosmological) spherically symmetric shear-free perfect fluid spacetimes containing black hole event horizons. In the subsequent section we will explicitly verify the identity for the Reissner-Nordstrom family of black hole spacetimes, which belong to case (a) above, and the McVittie spacetimes, which belong to case (b). We will prove the identity to hold for the Kerr-Newman black holes in a separate calculation below.

A. Asymptotically flat spherically symmetric spacetimes

The metric of any spherically symmetric spacetime can be written as

$$ds^2 = -N^2 dt^2 + \mu^2 dt dr + \lambda^{-2} dr^2 + r^2\Gamma^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad , \tag{2.5}$$

where N, μ, λ, and Γ are dependent on r and t only. Here, the $t = const.$ hypersurfaces may contain trapped surfaces. Outside the (outer) event horizon these are taken to be spacelike.

We shall assume that the following condition on the Ricci tensor holds in the spacetimes of interest

$$u^\mu u^\nu \mathcal{R}_{\mu\nu} = \mathcal{R} - n^\mu n^\nu \mathcal{R}_{\mu\nu} \quad , \tag{2.6}$$

where u^μ is a unit timelike normal to the spacelike slice Σ. (This above identity holds for the Kerr-Newman family.) Then the above equation becomes

$$\nabla_\mu(\Theta n^\mu + b^\mu) = u^\mu u^\nu \mathcal{R}_{\mu\nu} - R \quad , \tag{2.7}$$

where $b^\mu \equiv n^\nu \nabla_\nu n^\mu$.

Let $\mathcal{H}$ denote the spatial two-surface at the intersection of a $t = const.$ hypersurface with an event horizon. Let $\Sigma_\mathcal{H}$ denote the region on such a hypersurface that is bounded from the 'interior' by $\mathcal{H}$ and from the 'exterior' by a two-sphere at infinity. On multiplying both sides of Eq. (2.7) by the lapse N and integrating over the volume of Σ_H, we obtain for spherically symmetric metric (2.5):

$$\left[\lambda^{-1} r^2 \Gamma^2(r) \Theta\right]_H^\infty + \int dr \Gamma^2(r) r^2 R = \kappa \int_{\Sigma_H} d^3x \sqrt{h} N u^\mu u^\nu \mathcal{R}_{\mu\nu} \quad , \tag{2.8}$$

where h is the determinant of the three-metric on Σ_H. Above, we have used the identity (2.6). If the boundary ${}^3\mathsf{B}$ is assumed to be the time evolution of a spatial two-sphere, then $R = dk_0/dr$, where k_0 is the trace of the extrinsic curvature of the two-sphere as embedded isometrically (with respect to the black hole spacetime) in *flat* spacetime. If the spacetime is asymptotically flat, then this is the appropriate term to be added to the unreferenced BY quasilocal energy to obtain the (non-divergent) physical quasilocal energy. Furthermore, the term on the rhs above is nothing but the Komar integral. Thus, it contributes $(-M_H + E_\infty)$, where M_H is the gravitational charge at the horizon. Also using the fact that for a spherically symmetric spacetime

$$\Theta = -n_\mu a^\mu + k = \Theta^t_t + (\Theta^\theta_\theta + \Theta^\varphi_\varphi) = -\frac{N'\lambda}{N} - \frac{2\lambda}{r} \quad , \tag{2.9}$$

where, $a^\mu \equiv u^\nu \nabla_\nu u^\mu$ is the acceleration of the timelike hypersurface normal u^μ [22] and k is the trace of the extrinsic curvature of the two-boundary B, i.e., $k = (\Theta^\theta_\theta + \Theta^\varphi_\varphi)$. Using the definition of the Komar integral in terms of the radial derivative of the lapse, we find that the left-hand side of (2.8) is equal to $M_H - 2E_H + 3E_\infty$. Using this in Eq. (2.8) yields the identity (1.4). Note that we have nowhere assumed the spacetime to be a solution of general relativity. However, the association of k with the quasilocal energy has a nice justification [22,25] provided the quasilocal two-surface B is taken to be embedded in such a solution.

Finally, we note that the above identity holds for more general electrovac spacetimes, such as the Kerr-Newman family. We discuss that case in the next section.

B. Non-static (cosmological) spherically symmetric shear-free perfect fluid spacetimes

The metric for such spacetimes can be given as [30]

$$ds^2 = -\dot{\lambda}^2 e^{-2f(t)} dt^2 + e^{2\lambda(r,t)}(dr^2 + r^2 d\Omega^2) \quad , \tag{2.10}$$

where the isotropy condition implies that

$$e^{\lambda}(\lambda'' - \lambda'^2 - \lambda'/r) = b(r) \quad , \tag{2.11}$$

$b(r)$ being a function of r only.

Before embarking on a general proof for the validity of the identity (1.4), we note that unlike the previous case, here the metric lacks a timelike Killing vector. In its absence, by gravitational charge, we shall mean the same expression as the Komar integral with ξ^a replaced by $(\partial/\partial t)^a$, where t is the comoving time in the above metric. In the given form, Eq. (2.10) is manifestly spherically symmetric. By repeating calculations similar to the previous case, it is straightforward to show that the lhs of (2.8) is once again $M_H - 2E_H + 3E_\infty$. However, the volume integral of the product of the terms on the rhs of (2.4) and the lapse is non-trivial. Nevertheless, for the spacetimes under consideration, Eq. (2.10) shows that the Komar integral obeys:

$$-\frac{3}{8\pi}\int \nabla_{[l}\{\epsilon_{mn]cd}\nabla^c\xi^d\} = \frac{1}{4\pi}\int d^3x\sqrt{h}[-\dot{\lambda}'' + \dot{\lambda}'\lambda' + 2\dot{\lambda}'/r]e^{-2\lambda - f(t)} \quad , \tag{2.12}$$

which is exactly the volume integral obtained from the rhs of (2.4) for such spacetimes. This proves the identity.

III. SOME EXAMPLES

A. Static black hole spacetimes

Consider the Reissner-Nordstrom (RN) family of black hole spacetimes. The corresponding metric and the electromagnetic field can be given as (see, eg., [29]):

$$ds^2 = [C_\theta Z_r - C_r Z_\theta]\left\{\frac{dr^2}{\Delta_r} + \frac{\sin^2\theta}{\Delta_\theta}\right\} + \frac{\Delta_\theta[C_r dt - Z_r d\varphi]^2 - \Delta_r[C_\theta dt - Z_\theta d\varphi]^2}{[C_\theta Z_r - C_r Z_\theta]} \quad , \tag{3.1a}$$

$$F = \frac{2Q}{r^2} dr \wedge dt - 2P\sin\theta d\theta \wedge d\varphi \quad , \tag{3.1b}$$

where Q and P are the electric and magnetic monopole charges, respectively, of the hole. Also, t and r are the curvature coordinates.

The electromagnetic stress tensor is

$$\kappa T_{ab} = (E^2 + B^2)\{\omega_a^{(0)}\omega_b^{(0)} + \omega_a^{(3)}\omega_b^{(3)} + \omega_a^{(2)}\omega_b^{(2)} - \omega_a^{(1)}\omega_b^{(1)}\} \quad , \tag{3.2}$$

where $E = Q/r^2$ and $B = P/r^2$ and the tetrad of forms are

$$\omega^{(0)} = \sqrt{\frac{\Delta_r}{C_\theta Z_r - C_r Z_\theta}}[C_\theta dt - Z_\theta d\varphi] \quad , \tag{3.3a}$$

$$\omega^{(1)} = \sqrt{\frac{C_\theta Z_r - C_r Z_\theta}{\Delta_r}} dr \quad , \tag{3.3b}$$

$$\omega^{(2)} = \sqrt{\frac{C_\theta Z_r - C_r Z_\theta}{\Delta_\theta}} \sin\theta d\theta \quad , \tag{3.3c}$$

$$\omega^{(3)} = \sqrt{\frac{\Delta_\theta}{C_\theta Z_r - C_r Z_\theta}} [C_r dt - Z_r d\varphi] \, . \tag{3.3d}$$

For RN spacetimes $C_\theta = 1$, $C_r = 0$, $Z_r = r^2$, $Z_\theta = 0$, $\Delta_r = r^2 - 2Mr + Q^2 + P^2$, and $\Delta_\theta = \sin^2\theta$ Furthermore,

$$\Delta = r^2 - 2Mr + Q^2 + P^2 \, , \qquad \Delta_\theta = \sin^2\theta \, . \tag{3.4}$$

The Ricci tensor is given by

$$\mathcal{R}_{ab} = \kappa T_{ab} \quad , \tag{3.5}$$

which is just Einstein's equation for $\mathcal{R} = 0$. It is clear from Eq. (3.2) that

$$\hat{t}^\mu \hat{t}^\nu \mathcal{R}_{\mu\nu} = -\hat{r}^\mu \hat{r}^\nu \mathcal{R}_{\mu\nu} \, . \tag{3.6}$$

Moreover, the metric (3.1a), when applied to RN spacetimes, has the same form as Eq. (2.5). Hence the proof presented in the preceding section remains valid for such spacetimes.

We now verify the identity (1.4) for such black holes. For the SSS metric (2.5), we find that the gravitational charge is

$$M_c(r) = \frac{r^2 N'}{h} \quad , \tag{3.7}$$

and the quasilocal energy is

$$E(R) = \left[r\left(\frac{1}{h_0} - \frac{1}{h}\right) \right]_{r=R} \quad , \tag{3.8}$$

which is evaluated at some curvature radius R. Here a prime denotes derivative with respect to r. Note that the E is entirely determined by the metric function h, while in M_c, h only enters in writing the norm of $\mathbf{g}$.

For the RN black hole, we obtain the gravitational charge and the gravitational charge to be

$$M_c(r) = M - Q^2/r \tag{3.9}$$

$$E(r) = r[1 - (1 - 2M/r + Q^2/r^2)^{1/2}] \simeq M + \frac{M^2 - Q^2}{2r} \quad , \tag{3.10}$$

respectively. Thus, the horizon defining identity (1.4) leads to the equation:

$$r - (r^2 - 2Mr + Q^2)^{1/2} - M = M - Q^2/r \quad , \tag{3.11}$$

which on simplifying gives

$$(2Mr - Q^2)(r^2 - 2Mr + Q^2) = 0 \, . \tag{3.12}$$

We thus either have $r^2 - 2Mr + Q^2 = 0$ defining the horizon $r_+ = M + \sqrt{M^2 - Q^2}$ for $M^2 \geq Q^2$ or $r = Q^2/2M$, the hard core radius of naked singularity for $M^2 < Q^2$.

It may be noted that the horizon characterizing equation (3.11) also follows from equipartition of the quasilocal energy E into matter and non-matter energy. Here the matter energy would be $M - Q^2/2r$ and hence the equipartition would mean $E = 2(M - Q^2/2R)$, thus again yielding Eq. (3.11).

The characterizing relation (1.4) should hold good in all coordinates, and let us in particular verify it for an electrically charged hole in isotropic coordinates. The corresponding metric is given by

$$ds^2 = -[\frac{1 - \alpha^2/4r^2}{1 + M/r + \alpha^2/4r^2}]^2 dt^2 + (1 + M/r + \alpha^2/4r^2)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \quad (3.13)$$

where $\alpha^2 = M^2 - Q^2$. While evaluating the energy integral (1.3), we should be careful in choosing the proper asymptotic reference flat space that isometrically embeds the 2-surface of the above metric. For the metric (3.13) we get energy and charge expressions as follows:

$$E = M + \alpha^2/2r \quad (3.14)$$

and

$$M_c = \frac{M + \alpha^2/r + M\alpha^2/4r^2}{1 + M/r + \alpha^2/4r^2}. \quad (3.15)$$

Hence, the relation (1.4) once again gives the horizon to be at $r = \alpha/2$, showing that it is valid for isotropic coordinates as well. Note that $E(\alpha/2) = M + \sqrt{M^2 - Q^2}$. This calculation can be generalized to the case of magnetically charged black holes in a straightforward manner.

We now prove that the identity (1.4) holds also for the Kerr-Newman black holes. These are axisymmetric spacetimes and, therefore, do not lie in the domain of "category (a)" spacetimes discussed in the previous section. The metric for such a spacetime is given by Eq. (3.1a) with $C_\theta = 1$, $C_r = 0$, $Z_r = r^2 + a^2$, $Z_\theta = a\sin^2\theta$, $\Delta_r = (r^2 - 2Mr + a^2 + Q^2 + P^2)$, and $\Delta_\theta = \sin^2\theta$ in the Boyer-Lindquist coordinates. The electromagnetic vector potential is

$$A = \frac{Qr[dt - a\sin^2\theta d\varphi] + P\cos\theta[adt - (r^2 + a^2)d\varphi]}{r^2 + a^2\cos^2\theta}, \quad (3.16)$$

where a is the total angular momentum per unit mass of the spacetime.

Note that the Ricci tensor for such a spacetime is

$$\kappa R_{ab} = \frac{Q^2 + P^2}{r^2 + a^2\cos^2\theta}\{\omega_a^{(0)}\omega_b^{(0)} + \omega_a^{(3)}\omega_b^{(3)} + \omega_a^{(2)}\omega_b^{(2)} - \omega_a^{(1)}\omega_b^{(1)}\}. \quad (3.17)$$

With such a form, it is easy to verify that the condition (2.6) holds for such spacetimes.

To prove the identity (1.4), one proceeds by integrating Eq. (2.7) over the volume of a spatial section on $t = const.$, the same way as was done for SSS spacetimes. Here t is the Boyer-Lindquist time coordinate. Such a procedure gives rise to boundary terms as on the lhs of Eq. (2.8). Unlike in the cases discussed above, here the topology of $t = const.$ slices is that of a Wheeler wormhole, $R \times S^2$. Therefore, one needs two disjoint surfaces at different "radii" to constitute a complete boundary B. However, the integral of the lhs of (2.7) gives rise to a difference in the boundary terms at infinity and at the outer horizon, $r = r_+$, each boundary contributing two surfaces. Thus, choosing one of the surfaces to be common to these two boundaries, gives a null contribution from that surface to this difference. Then the difference in the boundary terms is just the difference between two surface terms, one at infinity and the other at the outer horizon. Since $(\partial/\partial t)^a$ is timelike Killing field in this metric, one defines the Komar integral in terms of this field. As before, the acceleration term $-n_\mu a^\mu$ contributes $M_H - E_\infty$. The contribution from the k term at the horizon vanishes, as in the previous cases. Only after one subtracts a 'reference term' contribution, arising from the R term, does one get an extra contribution of $4E_\infty - 2E_H$ from the rhs. The lhs being the Komar integral, it once again equals $E_\infty - M_H$. This proves the identity (1.4) for all black hole spacetimes of Kerr-Newman family.

As in the case of RN spacetimes, one can explicitly verify the identity (1.4) in the Kerr-Newman case. Here, although it is not straightforward to evaluate charge (1.1) and energy (1.3) due to disjoint surfaces constituting a boundary around the black hole, nevertheless, one can compute difference of terms occurring in the identity, as mentioned above. Furthermore, the horizon does close, thus, allowing the computation of M_H as the horizon Komar charge. In particular, Eqns. (1.1) and (1.3) give $M_c(r_+) = M_H = \sqrt{M^2 - a^2}$, and $E(r_+) = E_H = r_+$. These agree with (1.4). This concludes the demonstration of its validity as the horizon characterizing equation for all asymptotically flat black holes, (electrically/ magnetically) charged and/or rotating. That is, $E_H = r_+$ is always true.

Equipartition of field energy and charge, and equivalently of matter and non-matter energy, is a general principle for characterization of horizon and hence must be true in general for all black holes. It is easy to deduce from the above expressions for the energy and charge that the identity (1.4) holds for a Schwarzschild black hole ($Q = 0$) as well. For an extremal hole, it follows from Eqns. (3.10) and (3.14) that $E = M$ everywhere indicating that it is conserved. The charge M_c is on the other hand conserved for $Q = 0$, i.e., for the Schwarzschild hole. This means an extremal hole can never result from collapse of dispersed energy distribution because there is no driving force available to bring particles together. This is exactly what the third law of black hole dynamics say [31]. It is interesting that here this conclusion follows directly from simple energetics.

It is remarkable that the two limits, $Q = 0$ or $Q = M$ characterize conservation of charge or energy, respectively. It is the gravitational charge density (defined by $\rho_c = T_0^0 - T_\alpha^\alpha = (1/4\pi)R_{ik}u^i u^k, u_i u^i = -1$) that determines gravitational charge. This is obviously different from energy density $T_0^0 = \rho$. In the case of an electrically charged black hole note that $\rho = Q^2/2r^4$, while $\rho_c = Q^2/r^4$. Thus, E will be the sum of ρ and interaction field energy of putting together an (electrically charged) fluid ball. It will hence include all the contributions of energy required to create particles, bring them together and arrange them in a given configuration. Thus charge and energy would in general be different. The quasilocal energy E can be interpreted as the energy enclosed by the radius R. M is the total energy that includes all contributions, localizable as well as non-localizable. The component of energy that lies outside R will be $Q^2/2R$ due to electric field and $-M^2/2R$, in the Newtonian limit, due to gravity. Hence the energy enclosed by R will be $M - (Q^2/2R - M^2/2R)$. This is what E is as given by (3.10).

B. Black hole in FRW model

In reality a black hole always sits in a cosmological background. It is therefore desirable to consider a black hole spacetime embedded in the standard FRW homogeneous and expanding model of the Universe. The metric describing an electrically charged black hole in the FRW expanding model is given by [33,32],

$$ds^2 = -\frac{F^2}{G^2}dt^2 + S(t)^2 G^2 H^{-2}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{3.18}$$

where

$$F = 1 - H\alpha^2/(4r^2S^2) \tag{3.19a}$$
$$H = 1 + kr^2/4,\ \alpha^2 = M^2 - Q^2 \tag{3.19b}$$
$$G = 1 + H^{1/2}M/(rS) + H\alpha^2/(4r^2S^2)\ , \tag{3.19c}$$

where $k = \pm 1, 0$ is the space curvature parameter and $S(t)$ is the expansion factor. Above, we have chosen to write the metric in isotropic coordinates, which in this case also happens to the comoving.

To obtain the QE in such spacetimes, one first evaluates the trace of the extrinsic curvature of the boundary two-sphere of the spherical region described above. Since the metric (3.18) is asymptotically FRW, we choose the FRW spacetime (with the corresponding value of k) to be the reference spacetime. We then scale the isotropic coordinate r in the reference spacetime appropriately so that the two-sphere boundaries embedded in both the black hole spacetime and the reference spacetime are isometric to eachother. Computing k_0 with the resulting background metric and using it in Eq. (1.3) gives

$$E = -Sr^2G'/H\ , \tag{3.20}$$

in the isotropic gauge. Using the expression for G given in Eq. (3.19a), we get

$$E = MH^{-3/2} + \alpha^2/(2rSH)\ , \tag{3.21}$$

where H and α are defined above.

Here $E(\infty) = MH^{-3/2}$ for $rS \to \infty$. When we switch off expansion, i.e., set $S = const.$, we get the energy for an electrically charged black hole in the Einstein universe, and for $S = 1, k = 0$, we retrieve (3.14) the energy in the isotropic coordinates. Thus the above expression gives the result as expected.

Gravitational charge defined by Eq. (1.1) be given by

$$M_c = \alpha^2/2rSH + (MH^{-3/2} + \alpha^2/2rSH)F/G. \tag{3.22}$$

Note that $M_c(\infty) = E(\infty) = MH^{-3/2}$ and Eq. (1.4) again defines the horizon at $rS = \alpha H^{1/2}/2$. Thus the black hole characterization (1.4) holds good for an electrically charged black hole sitting in an FRW expanding universe. If we expand M_c for large rS,

$$M_c \simeq M/H^{3/2} - Q^2/2rSH \tag{3.23}$$

which again reduces to Eq. (3.22) when the expansion and curvature parameters are switched off.

C. Black hole with global monopole charge

Global monopoles are supposed to be created in phase transitions in the early Universe when global O(3) symmetry is spontaneously broken into $U(1)$. What global charge does is to produce stresses of the kind $T_0^0 = T_1^1 = \eta^2/r^2$, the rest being zero [34,35]. Recently, one of us considered resolution of the Riemann curvature relative to a timelike vector and have shown that the Schwarzschild black hole is dual to a black hole with global monopole charge [36]. By duality we mean interchange of active and passive electric parts of the field. Further it turns out that flat spacetime is dual to the massless global monopole (uniform relativistic potential) spacetime and to the FRW model with the equation of state $\rho + 3p = 0$, the characteristic condition of global texture [37].

It is possible to put a global monopole charge on a static black hole. The metric for the spacetime dual to the Schwarzschild black hole incorporating global monopole charge is given by

$$ds^2 = -\left(1 - 8\pi\eta^2 - \frac{2M}{r}\right) dt^2 + \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (3.24)$$

where η represents the global monopole charge. When $\eta = 0$, the Schwarzschild solution follows. It has horizon at $r = 2M/(1 - 8\pi\eta^2)$. The spacetime is obviously not vacuum. The physical effects of global monopole charge have been studied in Refs. [35,38]. It turns out that the Schwarzschild field essentially remains undisturbed except for the Hawking temperature is decreased by the factor $(1 - 8\pi\eta^2)^2$ while the perihelion shift and the bending of light are increased by $(1 - 8\pi\eta^2)^{-3/2}$ [38]. For $\eta^2 \ll 1$, it is however not significant.

The spacetime is no longer asymptotically flat [28] . But it is a solution of the dual-vacuum equation [38]. We would like to compute the quasilocal energy of a black hole with global monopole charge. Recently Hawking and Horowitz [39] have also computed energy of a particle embedded in the anti-de Sitter universe by employing a similar definition. For the metric (3.24), let us bring out the factor $1 - 8\pi\eta^2$ and absorb it by redefining t so as to agree with that of the asymptotic observer. Note that $h_0 = (1 - 8\pi\eta^2)^{-1/2}$ for the asymptotic reference spacetime, and Eq. (3.8) will therefore give

$$E = r\left[(1 - 8\pi\eta^2)^{1/2} - \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)^{1/2}\right] \quad (3.25)$$

$$\simeq (1 - 8\pi\eta^2)^{-1/2}\left[M + \frac{M^2}{2R(1 - 8\pi\eta^2)}\right]. \quad (3.26)$$

Here $E(\infty) = M(1 - 8\pi\eta^2)^{-1/2}$, $E(hor) = 2E(\infty)$. In view of choosing proper t coordinate, gravitational charge will also scale to give $M_c(\infty) = E(\infty)$. Again it is obvious that Eq. (1.4) will define the horizon. The reference spacetime is in this case not flat but is dual-flat in the same sense as the metric (3.24) is dual-vacuum. Dual spacetimes share the basic characteristic with the original, for instance radial free fall does not distinguish between them. Dual-flat spacetime could in this sense be considered as "minimally curved" as it is free of gravity at the Newtonian level [28]. The dual-flatness, which is characterized by covariant tensor equation like the original field equation, may be considered as a proper condition for minimal curvature. As mentioned earlier that the FRW model with the equation of state $\rho + 3p = 0$, the characteristic of global texture, is also dual flat. This is the expanding and homogeneous solution of the dual-flat equation, while the zero-mass monopole, which corresponds to uniform relativistic potential, is the static solution. In both cases, gravitational charge, which produces field at the Newtonian level, vanishes. Intuitively, it is quite appealing that such spacetimes should be minimally curved, and in particular uniform potential should, if at all, produce minimal curvature [26,27].

IV. CONCLUSION

We have thus shown that the horizon characterizing equation (1.4) as well as the expressions for gravitational charge and quasilocal energy are covariant. The former is however defined in terms of the latter two, and their covariance therefore implies the covariance of the black hole defining relation. Thus quasilocal energy and charge are in general related only at the horizon. It may be noted that the former is anchored on the non-Newtonian relativistic while the latter on the Newtonian aspect of the field. At the horizon the two turn equal.

It is remarkable that the definitions of quasilocal energy and charge apply for non-static non-asymptotically flat spacetime of a black hole in the FRW expanding model and give physically acceptable sensible results. The energy expressions (3.14) and (3.21) in the isotropic coordinates are more direct than (3.10) in the curvature coordinates. It may however be mentioned that while computing quasilocal energy for the metric (3.13), one should be rather careful in 'nullifying' the cosmological effect.

We shall now turn to a slightly detailed discussion of properties of black hole in relation to energy and charge.

The gravitational charge of a black hole $M_c = (\kappa/4\pi)A$, which relates it intimately to the surface gravity (temperature) κ of the hole. Here A is area of the horizon. The third law of black hole dynamics states that it is impossible to reduce gravitational charge of hole to zero by a finite sequence of physical processes [31]. In view of the relation (9), we could as well say that the field energy, E_F cannot be reduced to zero in a finite sequence of physical interactions. Temperature of the hole is zero for the extremal limit $M^2 = Q^2$. What happens is that as extremality is approached the allowed window for infalling energy and radiation pinches off. And thus the extremality is never attained. That is a non- extremal black hole can never turn extreme. Recent quantum field theoretic and topological considerations seem to suggest that the converse may also be true; i.e. extremal can also not turn into non-extremal (see Refs. [39,40]). Then extremal and non-extremal black holes will bear a good analogy with zero and non-zero mass particles, respectively, the gravitational charge being the analogue of mass [27].

Finally it is always welcome to gain some insight into the very difficult and ambiguous concept of energy in GR. Most of the definitions refer to quasilocal energy (they generally include contribution of matter energy density), while gravitational charge is essentially defined through the Komar integral and its generalization leading to the formulation of the

Gauss theorem for stationary spacetimes [12]. Here we have extended these definitions to non-stationary and non-asymptotically flat spacetimes as well.

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