EINSTEIN-CARTAN THEORY OF GRAVITATION: KINEMATICAL PARAMETERS AND MAXWELL EQUATIONS

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Abstract. In the space-time manifold of Einstein-Cartan Theory (ECT) of gravitation, the expressions for the time-like kinematical parameters are derived and the propagation equation for expansion is obtained. It has been observed that when the spin tensor is u-orthogonal the spin of the gravitating matter has no influence on the propagation equation of expansion while it has influence when it is not u-orthogonal. The usual formula for the curl of gradient of a scalar function is not zero in ECT. So is the case with the divergence of the curl of a vector. Their expressions on the space-time manifold of ECT are derived. A new derivative operator d_* is introduced to develop the calculus on space-time manifold of ECT. It is obtained by taking the covariant derivative of an associated tensor of a form with respect to asymmetric connections. We have used this differential operator to obtain the form of the Maxwell's equations in the ECT of gravitation. Cartan's equations of structure are also derived through the new derivative operator. It has been shown that unlike the consequences of exterior derivative in Einstein space-time, the repetition of d_* on a form of any degree is not zero.

1. Introduction

Einstein's General Theory of Relativity is the most successful theory of gravitation amongst all gravitation theories. It generalizes special relativity and Newton's theory of gravitation that incorporates gravity and acceleration of particles. The success of general relativity has stimulated interest in several generalizations of Einstein's original general relativity. The modified theories of gravitation have received considerable attention due to many reasons, such as the understanding of Mach's principle, adaptability with quantum physics, the link of gravitation with other interactions, incorporation of intrinsic spin of matter etc. Einstein-Cartan theory of gravitation is one such modified theory of gravitation in which the spin-an intrinsic feature of gravitating matter is introduced. It is a viable theory of gravitation that differs very slightly from the Einstein theory. This modified theory of gravitation was put forward in 1923 by Cartan [1].It is motivated by the desire to provide a simple description of the influence of spin of gravitating matter. This is achieved by taking a space-time as a four dimensional differential manifold endowed with a Riemannian metric in which the connections are asymmetric. In recent years the Einstein-Cartan Theory (ECT) of gravitation has been developed by Hehl. et.al [5].A brief account of some of the work performed recently on the ECT was presented by Trautman [13]. This theory postulates that the spin of matter is the

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source of torsion of space-time geometry. While singularities are inevitable in Einstein's theory, the new theory raised the hopes of avoiding singularities through the spin of matter. However, interest in Einstein-Cartan theory has been renewed in recent years. The theory is still considered viable and remains an active topic of researchers. The geometry of the space-time manifold of ECT of gravity does not remain Riemannian due to asymmetric connections, rather it becomes non-Riemannian. This non-Riemannian character of the space-time is introduced through affine connections Γ^l_{ij} which are defined by $\Gamma^l_{ij} = \left\{ {k \atop ij} \right\} - K_{ij}{}^k$ where $K_{ij}{}^k$ is the contortion tensor satisfying $K_{i(jk)} = 0$ and $\left\{ {k \atop ij} \right\} = \left\{ {k \atop ji} \right\}$ are the symmetric Christoffel symbols. The relation between the torsion tensor and the contortion tensor is given by

$$Q_{ij}{}^k = -\frac{1}{2}\left(K_{ij}{}^k - K_{ji}{}^k\right). \tag{1.1}$$

It also follows from the equation (1.1)that $K_{ijk} = -Q_{ijk} + Q_{jki} - Q_{kij}$. The relevant field equations of ECT are given by Hehl et.al [5]. The variation with respect to the metric tensor g^{ij} yields the equation

$$R_{ij} - \frac{1}{2}Rg_{ij} = -Kt_{ij}, \tag{1.2}$$

where R_{ij} is the Ricci tensor, t_{ij} is the canonical energy momentum tensor. One should note that the above equation is not the same as the Einstein's field equation because the Ricci tensor is no longer symmetric but instead contains information about the torsion tensor as well. The right hand side of the equation cannot be symmetric either, so t_{ij} must also contain information about the spin tensor. Similarly, the variation of the action with respect to the torsion tensor $Q_{ij}{}^k$ yields a new equation

$$Q_{ij}{}^k + \delta^k_i Q_{jl}{}^l - \delta^k_j Q_{il}{}^l = KS_{ij}{}^k, \tag{1.3}$$

where $S_{ij}{}^k$ is the spin angular momentum tensor. The relation between $S_{ij}{}^k$ and t_{ij} is defined by

$$t^{ij} = T^{ij} + (\nabla_k + 2Q_{kl}{}^l)(S^{ijk} - S^{jki} - S^{kij}), \tag{1.4}$$

T^{ij} is the stress energy momentum tensor. Hehl et. al[6] have split up the spin angular momentum tensor in to spin tensor in the form

$$S_{ij}{}^k = S_{ij}u^k. \tag{1.5}$$

The spin tensor is anti-symmetric and u-orthogonal. The u-orthogonality condition gives $S_{ij}u^i = 0$.This condition is referred as Franckel's condition. Investigations in the first four sections of the paper are made on the assumption that the spin tensor satisfies the franckel's condition. However, this additional restriction on the spin tensor, we feel to be redundant. Hence in the last Section 5, we represent our results of the first four sections by considering the spin tensor is not u-orthogonal.

Now contracting k and j in the field equation (1.3) and using the u-orthogonality condition of the spin tensor, we get $Q_{ik}{}^k = 0$. Consequently, the ECT field equation becomes

$$Q_{ij}{}^k = KS_{ij}u^k. \tag{1.6}$$

The six independent components of the spin tensor can be expressed in terms of its only three complex tetrad components as follows:

$$\begin{aligned} s_0 &= S_{13} = S_{ij} l^i m^j, \\ s_1 &= \frac{1}{2}(S_{12} + S_{43}) = \frac{1}{2} S_{ij}(l^i n^j + \bar{m}^i m^j), \\ s_2 &= S_{32} = S_{ij} m^i n^j, \end{aligned} \tag{1.7}$$

where l^i, n^i, m^i and $\bar{m}^i$ are null vector fields of the Newman-Penrose tetrad. The condition

$$S_{ij} u^i = 0 \Rightarrow s_0 = s_2, s_1 = -\bar{s}_1. \tag{1.8}$$

Reader may refer to the equation (5.8)for clarification. The investigations in the paper are arranged as follows: In the Section 2 we define the kinematical parameters of time like vector field in the ECT of gravitation and the role of spin on the propagation equation of expansion scalar is examined.In the Section 3, we introduce a new derivative operator d_* operated on a form.This converts p- form to $(p+1)$- form. It is obtained by taking the covariant derivative of an associate tensor of a form with respect an asymmetric connections. The Cartan equations of structure are also depicted through the new differential operator. It has been shown that this new differential operator when operated on a differential form of any degree does not satisfy a very important property that the repetition of exterior derivative of a form vanishes identically. In the study of electromagnetic field the Maxwell equations play very important role. Therefore, to study electromagnetic field in the ECT of gravitation,the form of the Maxwell equations are derived in the Section 4 by applying the new operator d_* on a 2 form when associated tensor is electromagnetic field tensor.We consider the u-orthogonality of the spin tensor is an unnecessary restriction. The results of the paper are summarised in the Section 5 by assuming the spin tensor is not u-orthogonal. In this case we have shown that the spin influences the propagation equation for expansion.

2. Kinematical Parameters in Einstein-Cartan Theory of Gravitation

A study of kinematical parameters of a null vector field, time like vector field and a space-like vector field is well known in the literature Sachs[12], Greenberg[4]. Kinematical parameters of a time-like vector field as well as space-like vector field and their propagation equations are studied by many authors, Greenberg[4] Ellis[3], Palle [11]. The propagation equation for expansion is the well-known Raychaudhary equation in the literature. Our aim is introduce the kinematical parameters of the time like vector field in Einstein-Cartan theory of gravitation and study the role of the spin tensor on their propagation equations. The calculus on a geometry where in which the connections Γ^k_{ij} are asymmetric is described by the covariant derivative denoted by a semi-colon (;) and for a vector field u_i it is defined as

$$u_{i;j} = \frac{\partial u_i}{\partial x^j} - u_k \Gamma^k_{ji},$$

where the asymmetric connections Γ^k_{ji} are defined as $\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} - K_{ij}{}^k$. Thus we have

$$u_{i;j} = u_{i/j} + K_{jil}u^l, \tag{2.1}$$

where

$$u_{i/j} = \frac{\partial u_i}{\partial x^j} - u_k \left\{ {k \atop ij} \right\} \tag{2.2}$$

is the covariant derivative with respect to the symmetric connections in the Einstein space-time. Similarly for the contravariant vector field u^i it is defined as

$$u^i{}_{;j} = u^i{}_{/j} - K_{jl}{}^i u^l, \tag{2.3}$$

Contracting the indices in the equation (2.3) and defining the expansion parameters $\theta = u^i{}_{;i}$ and $\hat{\theta} = u^i{}_{/i}$ respectively in the ECT of gravitation and Einstein theory of gravitation we obtain the relation

$$\theta = \hat{\theta} - 2Q_{ki}{}^i u^k. \tag{2.4}$$

2.1. **Shear Tensor in ECT of gravitation.** The shear tensor field in the ECT of gravitation is defined as

$$\sigma_{ij} = u_{(i;j)} - \dot{u}_{(i}u_{j)} - \frac{1}{3}\theta h_{ij}, \tag{2.5}$$

where $h_{ij} = g_{ij} - u_i u_j$ is the three dimensional projection operator and $\dot{u}_i$ is the acceleration vector in the ECT and is defined as $\dot{u}_i = u_{i;j}u^j$ and

$$\dot{u}_i = \hat{\dot{u}}_i + 2Q_{ijk}u^j u^k, \tag{2.6}$$

where $\hat{\dot{u}}_i = u_{i/j}u^j$ is the acceleration vector in Einstein theory. Consequently, the equation (2.5) becomes

$$\sigma_{ij} = \hat{\sigma}_{ij} - (Q_{kij} + Q_{kji})u^k - (Q_{ilk}u_j + u_i Q_{jlk})u^l u^k + \frac{2}{3}Q_{kl}{}^l u^k h_{ij}. \tag{2.7}$$

where

$$\hat{\sigma}_{ij} = \hat{u}_{(i/j)} - \hat{\dot{u}}_{(i}u_{j)} - \frac{1}{3}\hat{\theta} h_{ij}, \tag{2.8}$$

is the shear tensor in Einstein's theory of gravitation.

2.2. **Rotation Tensor in ECT of Gravitation:** The rotation tensor in the ECT of gravitation is defined as

$$\omega_{ij} = u_{[i;j]} - \dot{u}_{[i}u_{j]} - \frac{1}{3}\theta h_{ij}. \tag{2.9}$$

On simplifying this equation we obtain

$$\omega_{ij} = \hat{\omega}_{ij} + Q_{ijk}u^k - (Q_{ilk}u_j - u_i Q_{jlk})u^l u^k. \tag{2.10}$$

where

$$\hat{\omega}_{ij} = \hat{u}_{[i/j]} - \hat{\dot{u}}_{[i}u_{j]}, \tag{2.11}$$

is the rotation tensor in Einstein theory of gravitation. Hence we can decompose the covariant derivative of the time like vector field in terms of its kinematical parameters as

$$u_{i;j} = \sigma_{ij} + \omega_{ij} + \frac{1}{3}\theta h_{ij} + \dot{u}_i u_j. \tag{2.12}$$

On using the ECT field equations (1.6) and (1.8), we find from the equations (2.4),(2.6),(2.7), and (2.10)that

$$\theta = \hat{\theta},$$
$$\dot{u}_i = \hat{\dot{u}}_i,$$
$$\sigma_{ij} = \hat{\sigma}_{ij},$$
$$\omega_{ij} = \hat{\omega}_{ij} + KS_{ij}. \tag{2.13}$$

The equation (2.12) reduces to the form

$$u_{i;j} = \hat{\sigma}_{ij} + \hat{\omega}_{ij} + \frac{1}{3}\hat{\theta}h_{ij} + \hat{\dot{u}}_i u_j + KS_{ij}. \tag{2.14}$$

This is equivalent to the definition (2.1) for the time-like vector field u_i.

$$u_{i;j} = u_{i/j} + KS_{ij}. \tag{2.15}$$

where,

$$u_{i/j} = \hat{\sigma}_{ij} + \hat{\omega}_{ij} + \frac{1}{3}\hat{\theta}h_{ij} + \hat{\dot{u}}_i u_j. \tag{2.16}$$

However, for any general vector field A_i we have

$$A_{i;j} = A_{i/j} + K(S_{ij}u_l + S_{il}u_j + S_{jl}u_i)A^l. \tag{2.17}$$

Similarly, the covariant derivative of any second rank tensor A_{ij} is defined as

$$A_{ij;k} = A_{ij/k} + KA_{lj}(S_i{}^l u_k - S_{ki}u^l - S^l{}_k u_i) + KA_{il}(S_j{}^l u_k - S_{kj}u^l - S^l{}_k u_j). \tag{2.18}$$

We see that the only term which is affected by the spin tensor is the rotation tensor and all other kinematical parameters have the invariant characteristic. Now if f is any scalar function of coordinates then its covariant derivative with respect to the symmetric Christoffel symbols is the same as its partial derivative. It turns out that the second order covariant derivative of a scalar is commutative. However, in ECT of gravitation we obtain from the equation (2.17) by replacing the vector A_i by the gradient of the scalar function $f_{;i}$ we obtain

$$f_{;ij} - f_{;ji} = 2Kf_{;l}u^l S_{ij}. \tag{2.19}$$

We see a very important property of the covariant derivative of a scalar function that it is commutative in Einstein space-time is no longer true in the ECT space-time.Also by interchanging the indices i and j in the equation (2.17) and subtracting the result from the equation (2.17) we obtain the expression for the curl of a vector A_i as

$$curl A_i = (\widehat{curl} A_i) + 2KS_{ij}u^l A_l.$$

If the vector A_i is the time-like vector field u_i then the above equation becomes

$$curl u_i = (\widehat{curl} u_i) + 2KS_{ij}, \tag{2.20}$$

where the term on the left hand side $curl u_i = u_{i;j} - u_{j;i}$ represents the curl of a vector u_i in the ECT space-time, while the first term on the right hand side with hatch on the head i.e.,$(\widehat{curl} u_i) = u_{i/j} - u_{j/i}$ represents the curl of a vector in the Einstein space-time.It follows from the equation that

$$curl(grad f) = 2f_{,k}Q_{ij}^{\ \ k}. \tag{2.21}$$

However, the divergence of an arbitrary vector is obtained by contracting its covariant derivative. Since the spin tensor is u-orthogonal, it follows from the equation (2.17) that $A^i{}_{;i} = A^i{}_{/i}$.

2.3. **Auto-parallel Curves:** Auto-parallel curve in ECT space-time is a curve such that the tangent vector field to the curve is parallely transported along the curve.If $t^i = \frac{dx^i}{ds}$ is the unit tangent vector field to the curve, then auto-parallel curve is defined as

$$t^i_{;j} t^j = 0.$$

Using the equation (2.3) we obtain the differential equations of auto-parallel curve in the ECT of gravitation in the form

$$\frac{d^2x^i}{ds^2} + \left\{ {i \atop jk} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} + 2KS^i_{\ j} u_k t^j t^k = 0. \tag{2.22}$$

If the 4-velocity vector field $u^i = \frac{dx^i}{ds}$ is the unit tangent vector field to the curve, then the equations of auto-parallel curve reduce to the equations in Einstein theory.

2.4. **Ricci Identity in ECT of Gravity:** Let A^i be any arbitrary vector field in Einstein-Cartan space-time, then the Ricci identity is given by

$$A^i{}_{;jk} - A^i{}_{;kj} = -A^l R_{kjl}{}^i - 2A^i{}_{;l} Q_{kj}{}^l\ , \tag{2.23}$$

where

$$R_{kji}{}^h = \hat{R}_{kji}{}^h + K_{ji}{}^h{}_{;k} - K_{ki}{}^h{}_{;j} - K_{li}{}^h (K_{kj}{}^l - K_{jk}{}^l) + K_{ji}{}^l K_{kl}{}^h - K_{jl}{}^h K_{ki}{}^l, \tag{2.24}$$

is the Riemann curvature tensor in ECT and $\hat{R}_{kji}{}^h$ is the Riemann curvature tensor in Einstein theory. We see from the equation (2.24) that the Riemann curvature tensor satisfies the following properties.

$$\begin{aligned} &R_{hijk} = -R_{ihjk} = -R_{hikj}, \\ &R_{hijk} \neq R_{jkhi}, \\ &R_{hijk} + R_{hjki} + R_{hkij} \neq 0. \end{aligned} \tag{2.25}$$

Expressing the contortion tensor into torsion tensor and then using the equation (1.6) we obtain the expression for the Riemannian curvature tensor as

$$\begin{aligned} R_{kji}{}^h \;=\; & \hat{R}_{kji}{}^h + 2K \left[S_{i[j;k]} u^h + u^h{}_{;[j} S_{k]i} - S_i{}^h{}_{;[j} u_{k]} + S_i{}^h u_{[j;k]} - S^h_{[j;k]} u_i + u_{i;[j} S^h{}_{k]} \right] + \\ & + 2K^2 S_i{}^h S_{kj} + K^2 \Big[S_{ji} S^h_{\ k} - S^h_{\ j} S_{ki} - S_i{}^l S_{kl} u^h u_j + S_i{}^l S_{jl} u^h u_k - S^l_{\ j} S_l{}^h u_i u_k + \\ & + S^l_{\ k} S_l{}^h u_i u_j \Big]. \end{aligned} \tag{2.26}$$

The cyclic property of the Riemann curvature tensor in ECT space-time becomes

$$R_{kji}{}^h + R_{jik}{}^h + R_{ikj}{}^h = 2(Q_{ij}{}^h{}_{;k} + Q_{jk}{}^h{}_{;i} + Q_{ki}{}^h{}_{;j}) - 4(Q_{ij}{}^l Q_{kl}{}^h + Q_{ki}{}^l Q_{jl}{}^h + Q_{jk}{}^l Q_{il}{}^h). \tag{2.27}$$

Using equation (1.6) the equation(2.27) reduces to

$$R_{kji}{}^h + R_{jik}{}^h + R_{ikj}{}^h = 2K[(S_{ij}u^h)_{;k} + (S_{ki}u^h)_{;j} + (S_{jk}u^h)_{;i}]. \tag{2.28}$$

Contracting the index k with h in the equation (2.26) we obtain the expression for the Ricci tensor as

$$\begin{aligned} R_{ij} &= \hat{R}_{ij} - K\dot{S}_{ij} - KS_{ij}\theta + Ku_iS_j{}^k{}_{;k} + Ku_{i;k}S_j{}^k + KS^k_{i;k}u_j + KS^k_iu_{j;k} - \\ &\quad -2K^2S^k_iS_{kj} + 2K^2S^2u_iu_j. \end{aligned} \tag{2.29}$$

2.5. **Raychaudhari Equation in ECT of Gravity:** To see the effect of spin tensor on the Raychaudhary equation, we start with the Ricci identity for a time like vector field u_i given by

$$u_{i;jk} - u_{i;kj} = u_hR_{kji}{}^h - 2u_{i;h}Q_{kj}{}^h. \tag{2.30}$$

Multiplying this equation by u^k we obtain

$$(u_{i;j})^{\cdot} = u_{i;kj}u^k + R_{kjih}u^hu^k - 2u_{i;h}Q_{kj}{}^hu^k, \tag{2.31}$$

where

$$\dot{u}_i = u_{i;k}u^k.$$

This implies that

$$\dot{u}_{i;j} = u_{i;k;j}u^k + u_{i;k}u^k_{;j}. \tag{2.32}$$

Eliminating the term $u_{i;kj}u^k$ from the equation (2.31) we get

$$(u_{i;j})^{\cdot} = \dot{u}_{i;j} - u_{i;k}u^k_{;j} + R_{kjih}u^hu^k - 2u_{i;h}Q_{kj}{}^hu^k. \tag{2.33}$$

Using the equation (2.12) in the equation (2.33), then contracting the indices i and j and simplifying the equation by using the definitions given below

$$\begin{aligned} &\hat{\sigma}_{ij}\hat{\sigma}^{ij} = 2\hat{\sigma}^2,\ \ \hat{\sigma}_{ij}u^i = 0,\ \ \sigma_{ij}h^{ij} = 0, \\ &\hat{\omega}_{ij}\hat{\omega}^{ij} = 2\hat{\omega}^2,\ \ \hat{\omega}_{ij}u^i = 0,\ \ \hat{\omega}_{ij}g^{ij} = 0, \\ &S_{ij}S^{ij} = 2S^2,\ \ \hat{\sigma}_{ij}g^{ij} = 0,\ \ \hat{\omega}_{ij}\hat{\sigma}^{ij} = 0, \\ &\dot{u}_iu^i = 0, h_{ij}u^i = 0, \end{aligned} \tag{2.34}$$

we obtain the equation

$$\dot{\theta} = \dot{u}^i_{\ i} - 2\left(\hat{\sigma}^2 - \hat{\omega}^2 - K\hat{\omega}_{ij}S^{ij} - K^2S^2 + \frac{1}{6}\hat{\theta}^2\right) + R_{ij}u^iu^j. \tag{2.35}$$

From the equation (2.29) on using the equation (2.12) we find

$$R_{ij}u^iu^j = \hat{R}_{ij}u^iu^j - 2K\hat{\omega}_{ij}S^{ij} - 2K^2S^2. \tag{2.36}$$

Substituting this value in the equation (2.35) we get

$$\dot{\theta} = \dot{u}^i_{\ ;i} - 2(\hat{\sigma}^2 - \hat{\omega}^2) - \frac{1}{3}\hat{\theta}^2 + \hat{R}_{ij}u^iu^j. \tag{2.37}$$

This is the Raychaudhuri equation in ECT space-time. We see that though the fact that the shear tensor induces contraction of the flow vector, and the rotation induces expansion, the spin tensor has no influence on flow vector and the Raychaudhuri equation remains invariant in its form even in the ECT of gravitation. The propagation equations for other parameters are published by the author [9].

3. Calculus in Einstein-Cartan Theory:

In Einstein's theory of gravitation the calculus of differential forms is developed by exterior derivative. The exterior differentiation is effected by an operator d and applied to forms. It is an operation that converts r-form to $r+1$ form obtained by taking either the partial derivative or covariant derivative (immaterial which) of the associated rth order tensor in the Einstein space-time. Mathematically, it is defined as

$$d : \bigwedge^{r} T_p^* \rightarrow \bigwedge^{r+1} T_p^*. \tag{3.1}$$

If $\tilde{\omega} \in \bigwedge^r T_p^*$ is any r-form then $d\tilde{\omega} \in \bigwedge^{r+1} T_p^*$ is $r+1$-form, where the operator d is linear which implies that $d(a\tilde{\omega} + b\tilde{\sigma}) = ad\tilde{\omega} + bd\tilde{\sigma}$ for any two r-forms and a, b are constants. The exterior derivative also obeys the Leibniz rule

$$d(fg) = gdf + fdg, \tag{3.2}$$

where f and g are functions of coordinates . In the algebra of differential forms, the multiplication is defined by the wedge product and is skew symmetric. That is

$$dx^i \wedge dx^j = -dx^j \wedge dx^i. \tag{3.3}$$

One consequence of this is that $dx^i \wedge dx^i = 0$. The derivative operator satisfies the following properties when applied to different forms.

$$\begin{aligned} &df = f_{;i}dx^i, for \ \ 0 - form f,\\ &d(\tilde{\omega} \wedge \tilde{\sigma}) = d\tilde{\omega} \wedge \tilde{\sigma} + (-1)^{deg.of\tilde{\omega}} \ \ \tilde{\omega} \wedge d\tilde{\sigma},\\ &d(f\tilde{\omega}) = df \wedge \tilde{\omega} + fd\tilde{\omega}, \ \ f \in R,\\ &d(d\tilde{\omega}) = 0, for \ any \ form \ \tilde{\omega}. \end{aligned} \tag{3.4}$$

In Einstein-Cartan theory of gravitation it is convenient to use the techniques of differential form as it gives a more conceptual and uniform way of understanding curvature and also provides an important tool for computation. However, in the ECT space-time, the covariant derivative is defined with respect to the asymmetric affine connections. Therefore, even for a scalar function the covariant derivative of second order is not commutative. Hence in ECT space-time we introduce a new operator d_* applied to any form p. It converts p- form to $(p+1)$- form and is obtained by taking the covariant derivative of an associated p^{th}order skew symmetric tensor with respect to the asymmetric connections. For a scalar function f we have always

$$f_{;i} = f_{/i} = f_{,i}. \tag{3.5}$$

It follows that for any coordinate function x^i, $d_*x^i = dx^i$.Operating d_* on the function (d_*f) we get

$$d_*(d_*f) = \frac{1}{2}(f_{;ij} - f_{;ji})dx^j \wedge dx^i. \tag{3.6}$$

This can be written as

$$d_*(d_*f) = [\frac{1}{2}(f_{/ij} - f_{/ji}) + f_{;k}Q_{ij}^{\ \ k}]dx^j \wedge dx^i.$$

That is

$$d_*(d_*f) = d(df) + f_{;k}Q_{ij}^{\ \ k}dx^j \wedge dx^i, \tag{3.7}$$

where we have $d(df) = \frac{1}{2}(f_{/ij} - f_{/ji})dx^j \wedge dx^i$. As repetition of exterior derivative on any form is identically zero. Hence we have therefore,

$$d_*(d_*f) = curl(gradf) = f_{;k}Q_{ij}^{\ \ k}dx^j \wedge dx^i \tag{3.8}$$

We observe that $d_*^2 \neq 0$. Now if $\tilde{\omega} = \omega_i dx^i$ is any 1-form then its derivative is 2-form given by

$$d_*\tilde{\omega} = \frac{1}{2}(\omega_{i;j} - \omega_{j;i})dx^j \wedge dx^i. \tag{3.9}$$

It can be written as

$$d_*\tilde{\omega} = [\frac{1}{2}(\omega_{i,j} - \omega_{j,i}) + \omega_k Q_{ij}^{\ \ k}]dx^j \wedge dx^i,$$

$$d_*\tilde{\omega} = d\tilde{\omega} - \omega_k Q_{ij}^{\ \ k}dx^i \wedge dx^j, \tag{3.10}$$

where

$$d\tilde{\omega} = \frac{1}{2}(\omega_{i,j} - \omega_{j,i})dx^j \wedge dx^i.$$

Operating d_* on the equation (3.9) we obtain

$$d_*(d_*\tilde{\omega}) = \frac{1}{2}(\omega_{i;jk} - \omega_{j;ik})dx^k \wedge dx^j \wedge dx^i. \tag{3.11}$$

By cyclic permutation of the indices i, j, k twice in turn in the equation(3.11) we get two more equations. Adding these equations to the equation(3.11) we get

$$d_*(d_*\tilde{\omega}) = -\frac{1}{6}[(\omega_{i;jk} - \omega_{i;kj}) - (\omega_{j;ik} - \omega_{j;ki}) + (\omega_{k;ij} - \omega_{k;ji})]dx^i \wedge dx^j \wedge dx^k. \tag{3.12}$$

Using the Ricci identities

$$\omega_{i;jk} - \omega_{i;kj} = \omega_h R_{kji}^{\ \ \ h} - 2\omega_{i;h}Q_{kj}^{\ \ h},$$

we obtain

$$d_*(d_*\tilde{\omega}) = -\frac{1}{6}[\omega_h(R_{kji}^{\ \ \ h} + R_{ikj}^{\ \ \ h} + R_{jik}^{\ \ \ h}) - 2(\omega_{i;h}Q_{kj}^{\ \ h} - \omega_{j;h}Q_{ki}^{\ \ h} + \omega_{k;h}Q_{ji}^{\ \ h})]dx^i \wedge dx^j \wedge dx^k, \tag{3.13}$$

By using the cyclic property of curvature tensor in the Einstein-Cartan space-time we obtain

$$\begin{aligned} d_*(d_*\tilde{\omega}) \ &= \ -\frac{1}{3}[\omega_h(Q_{ij}^{\ \ h}{}_{;k} + Q_{jk}^{\ \ h}{}_{;i} + Q_{ki}^{\ \ h}{}_{;j}) - 2\omega_h(Q_{ij}^{\ \ l}Q_{kl}^{\ \ h} + Q_{ki}^{\ \ l}Q_{jl}^{\ \ h} + Q_{jk}^{\ \ l}Q_{il}^{\ \ h}) - \\ &\quad -(\omega_{i;h}Q_{kj}^{\ \ h} - \omega_{j;h}Q_{ki}^{\ \ h} + \omega_{k;h}Q_{ji}^{\ \ h})]dx^i \wedge dx^j \wedge dx^k, \end{aligned} \tag{3.14}$$

From equations (3.8)and (3.14) we see that

$$d_*^2(f) = curl(gradf) \neq 0. \tag{3.15}$$

$$d_*^2(\tilde{\omega}) = div(curlA_i) \neq 0. \tag{3.16}$$

Similarly, if $\tilde{\phi} = F_{ij}dx^i \wedge dx^j$ is any 2-form, where F_{ij} are the skew symmetric components of 2-form,then its derivative is 3-form given by

$$d_*\tilde{\phi} = d\tilde{\phi} - \frac{2}{3}(F_{lj}Q_{ki}^{\ \ l} + F_{il}Q_{kj}^{\ \ l} + F_{lk}Q_{ij}^{\ \ l})dx^i \wedge dx^j \wedge dx^k, \tag{3.17}$$

where

$$d\tilde{\phi} = \frac{1}{3}(F_{ij,k} + F_{jk,i} + F_{ki,j})dx^i \wedge dx^j \wedge dx^k. \tag{3.18}$$

3.1. **Tetrad Formalism:** We use the Newman-Penrose [10] tetrad formalism and its extension by Jogia and Griffiths [7] which is suitably adopted for dealing with certain problems in ECT. We summarize the formalism here briefly.At each point of a 4-dimensional non-Riemannian space-time of ECT we introduce a tetrad $e^i_{(\alpha)} = (l^i, n^i, m^i, \bar{m}^i)$ consisting of four null basis vectors. The tetrad of the dual basis vector fields is given by $e^{(\alpha)}_i = (n_i, l_i, -\bar{m}_i, -m_i)$. Of these basis vector fields l_i, n_i are real null vector fields, and m_i, $\bar{m}_i$ are complex null vector fields satisfying the conditions $l_i n^i = 1$ and $\bar{m}_i m^i = -1$ and all other inner products are zero. The tensor components of the metric tensor g_{ij} and its tetrad components $\eta_{\alpha\beta}$ are related by the equation $g_{ij} = \eta_{\alpha\beta} e^{(\alpha)}_i e^{(\beta)}_j$, where $\eta_{12} = \eta_{21} = -\eta_{34} = -\eta_{43} = 1$ and all other components are zero. The tetrad indices of a tensor can be raised or lowered down by the tetrad components of the metric tensor $\eta_{\alpha\beta}$ while the tensor indices are raised and lowered by the metric tensor g_{ij}. Following the notations of Jogia and Griffiths[7], the ECT field equation (1.6) reduces to its tetrad components as

$$\begin{aligned}
&\sigma_1 = \lambda_1 = \pi_1 = \tau_1 = 0,\\
&\kappa_1 = \bar{\nu}_1 = -\sqrt{2}Ks_0, \;\; \bar{\alpha}_1 = \beta_1 = -\frac{1}{\sqrt{2}}Ks_0,\\
&\rho_1 = \mu_1 = -\sqrt{2}Ks_1, \;\; \gamma_1 = \epsilon_1 = -\frac{1}{\sqrt{2}}Ks_1.
\end{aligned} \tag{3.19}$$

We record below the tetrad components of the Ricci's coefficients of rotations defined by $\gamma_{\alpha\beta\sigma} = -e_{(\alpha)i;j} e^i_{(\beta)} e^j_{(\sigma)}$ for the ready references

$$\begin{aligned}
&\gamma_{121} = -(\epsilon^0 + \bar{\epsilon}^0), \qquad \gamma_{132} = -\tau^0,\\
&\gamma_{122} = -(\gamma^0 + \bar{\gamma}^0), \qquad \gamma_{133} = -\sigma^0,\\
&\gamma_{343} = -(\bar{\alpha}^0 - \beta^0), \qquad \gamma_{231} = \bar{\pi}^0,\\
&\gamma_{134} = -\rho^0 + \sqrt{2}Ks_1, \qquad \gamma_{233} = \bar{\lambda}^0,\\
&\gamma_{131} = -\kappa^0 + \sqrt{2}Ks_0, \qquad \gamma_{232} = \bar{\nu}^0 - \sqrt{2}Ks_0,\\
&\gamma_{123} = -(\bar{\alpha}^0 + \beta^0) + \sqrt{2}Ks_0,\\
&\gamma_{341} = \epsilon^0 - \bar{\epsilon}^0 - \sqrt{2}Ks_1,\\
&\gamma_{342} = \gamma^0 - \bar{\gamma}^0 - \sqrt{2}Ks_1,\\
&\gamma_{234} = \bar{\mu}^0 + \sqrt{2}Ks_1,
\end{aligned} \tag{3.20}$$

where $\gamma_{\alpha\beta\gamma} = \gamma^0_{\alpha\beta\gamma} + K_{\gamma\alpha\beta}$ and $K_{\gamma\alpha\beta}$ are the tetrad components of the torsion tensor.As an application of the new derivative operator d_* we apply it to the tetrad basis of 1-form $\theta^\alpha = e^{(\alpha)}_i dx^i$ and obtain

$$d_*\theta^{(\alpha)} = \frac{1}{2}(e^{(\alpha)}_{i/j} - e^{(\alpha)}_{j/i})dx^j \wedge dx^i - e^{(\alpha)}_k Q_{ij}{}^k dx^i \wedge dx^j \tag{3.21}$$

Replacing the coordinate basis 1-forms dx^i by the tetrad basis of 1-forms $\theta^{(\alpha)}$ we get

$$d_*\theta^{(\alpha)} = \frac{1}{2}\eta^{\gamma\sigma}(\gamma^0_{\sigma\alpha\beta} - \gamma^0_{\sigma\beta\alpha} - 2Q_{\alpha\beta\sigma})\theta^\alpha \wedge \theta^\beta. \tag{3.22}$$

By giving different values to the Greek indices from 1 to 4 we obtain the following equations

$$\begin{aligned}
d_*\theta^{(1)} &= (\gamma^0 + \bar{\gamma}^0 + \gamma_1 + \bar{\gamma}_1)\theta^{12} - (\mu^0 - \bar{\mu}^0 + \mu_1 - \bar{\mu}_1)\theta^{34} + \\
&\quad +[(\bar{\alpha}^0 + \beta^0 - \bar{\pi}^0 + \bar{\alpha}_1 + \beta_1 - \bar{\pi}_1)\theta^{13} - (\bar{\nu}^0 + \bar{\nu}_1)\theta^{23}] + [c.c], \\
d_*\theta^{(2)} &= (\epsilon^0 + \bar{\epsilon}^0 + \epsilon_1 + \bar{\epsilon}_1)\theta^{12} - (\rho^0 - \bar{\rho}^0 + \rho_1 - \bar{\rho}_1)\theta^{34} + \\
&\quad +[(\kappa^0 + \kappa_1)\theta^{13} + (\tau^0 - \bar{\alpha}^0 - \beta^0 + \tau_1 - \bar{\alpha}_1 - \beta_1)\theta^{23}] + [c.c], \\
d_*\theta^{(3)} &= -[(\bar{\tau}^0 + \pi^0 + \bar{\tau}_1 + \pi_1)\theta^{12} + (\bar{\rho}^0 + \epsilon^0 - \bar{\epsilon}^0 + \bar{\rho}_1 + \epsilon_1 - \bar{\epsilon}_1)\theta^{13} + (\bar{\sigma}^0 + \bar{\sigma}_1)\theta^{14} - \\
&\quad -(\mu^0 - \gamma^0 + \bar{\gamma}^0 + \mu_1 - \gamma_1 + \bar{\gamma}_1)\theta^{23} - (\lambda^0 + \lambda_1)\theta^{24} - \\
&\quad -(\alpha^0 - \bar{\beta}^0 + \alpha_1 - \bar{\beta}_1)\theta^{34}].
\end{aligned} \tag{3.23}$$

where c.c denotes the complex conjugate of the preceding term. These equations are equivalent to

$$\begin{aligned}
d_*\theta^{(1)} &= d\theta^{(1)} + (\gamma_1 + \bar{\gamma}_1)\theta^{12} - (\mu_1 - \bar{\mu}_1)\theta^{34} + [(\bar{\alpha}_1 + \beta_1 - \bar{\pi}_1)\theta^{13} - \bar{\nu}_1\theta^{23}] + [c.c], \\
d_*\theta^{(2)} &= d\theta^{(2)} + (\epsilon_1 + \bar{\epsilon}_1)\theta^{12} - (\rho_1 - \bar{\rho}_1)\theta^{34} + [\kappa_1\theta^{13} + (\tau_1 - \bar{\alpha}_1 - \beta_1)\theta^{23}] + [c.c], \\
d_*\theta^{(3)} &= d\theta^{(3)} - [(\bar{\tau}_1 + \pi_1)\theta^{12} + (\bar{\rho}_1 + \epsilon_1 - \bar{\epsilon}_1)\theta^{13} + \bar{\sigma}_1\theta^{14} - (\mu_1 - \gamma_1 + \bar{\gamma}_1)\theta^{23} - \lambda_1\theta^{24} - \\
&\quad -(\alpha_1 - \bar{\beta}_1)\theta^{34}],
\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
d\theta^{(1)} &= (\gamma^0 + \bar{\gamma}^0)\theta^{12} - (\mu^0 - \bar{\mu}^0)\theta^{34} + [(\bar{\alpha}^0 + \beta^0 - \bar{\pi}^0)\theta^{13} - (\bar{\nu}^0)\theta^{23}] + [c.c], \\
d\theta^{(2)} &= (\epsilon^0 + \bar{\epsilon}^0)\theta^{12} - (\rho^0 - \bar{\rho}^0)\theta^{34} + [\kappa^0\theta^{13} + (\tau^0 - \bar{\alpha}^0 - \beta^0)\theta^{23}] + [c.c], \\
d\theta^{(3)} &= -[(\bar{\tau}^0 + \pi^0)\theta^{12} + (\bar{\rho}^0 + \epsilon^0 - \bar{\epsilon}^0)\theta^{13} + \bar{\sigma}^0\theta^{14} - (\mu^0 - \gamma^0 + \bar{\gamma}^0)\theta^{23} - \\
&\quad -\lambda^0\theta^{24} - (\alpha^0 - \bar{\beta}^0)\theta^{34}].
\end{aligned} \tag{3.25}$$

The expression for $d_*\theta^{(4)}$ $(d\theta^{(4)})$is obtained by taking the complex conjugate of the terms and interchanging 3 and 4 in the expression $d_*\theta^{(3)}$ $(d\theta^{(3)})$.

3.2. **Cartan's Equations of structure:** There are two equations of structure due to Cartan. They play a very crucial role in deriving the components of Riemann Curvature tensor. The essence of Riemannian geometry is studied through the equations of structure. Their forms and their uses are well known in the Riemannian geometry.Katkar [8] has obtained these equations in ECT of gravitation and claimed the essence of non-Riemannian geometry can be summarised by exploiting these equations. These equations are derived here through the new differential operator d_* for the use in the subsequent work. We take the derivative of $\theta^\alpha = e^{(\alpha)}_i dx^i$ and find

$$d_*\theta^\alpha = e^{(\alpha)}_{i;j} dx^j \wedge dx^i. \tag{3.26}$$

This after using the definition of Ricci's coefficient of rotation yields

$$d_*\theta^\alpha = \gamma^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma,$$

$$d_*\theta^\alpha = -\omega^\alpha_\beta \wedge \theta^\beta, \tag{3.27}$$

where

$$\omega^\alpha_\beta = \gamma^\alpha_{\beta\gamma}\theta^\gamma, \tag{3.28}$$

$$\gamma^\alpha_{\beta\gamma} = \gamma^{0\alpha}_{\beta\gamma} - K_{\gamma\beta}{}^\alpha, \tag{3.29}$$

Thus we have

$$\omega^\alpha_\beta = \omega^{0\alpha}_\beta - K_{\gamma\beta}{}^\alpha\ \theta^\gamma. \tag{3.30}$$

This gives the relation between the connection 1-forms with respect to the ECT space-time and the Einstein space-time; the additional term is due to the contribution of the contortion tensor. The equation (3.27) is called the Cartan's first equation of structure in the ECT space-time. The equation (3.27)can also be written as

$$d_*\theta^\alpha = d\theta^\alpha + K_{\gamma\beta}{}^\alpha\theta^\gamma \wedge \theta^\beta, \tag{3.31}$$

where

$$d\theta^\alpha = -\omega^{0\alpha}_\beta \wedge \theta^\beta, \tag{3.32}$$

is the Cartan's first equation of structure in Einstein theory of gravitation. Now to derive the second equation of structure we take the derivative of the equation (3.28) and obtain

$$\begin{aligned} d_*\omega^\alpha_\beta &= (\gamma^\alpha{}_{\beta\sigma} e^{(\sigma)}_i)_{;j} dx^j \wedge dx^i, \\ d_*\omega^\alpha_\beta &= \frac{1}{2}[(\gamma^\alpha{}_{\beta\sigma} e^{(\sigma)}_i)_{;j} - (\gamma^\alpha{}_{\beta\sigma} e^{(\sigma)}_j)_{;i}] dx^j \wedge dx^i, \end{aligned} \tag{3.33}$$

where we have

$$e^{(\alpha)}_{k;ij} e^{\ k}_{(\beta)} = -(\gamma^\alpha{}_{\beta\sigma} e^{(\sigma)}_i)_{;j} + \gamma^\alpha_{\rho\sigma}\gamma^\rho{}_{\beta\epsilon} e^{(\sigma)}_i e^{(\epsilon)}_j. \tag{3.34}$$

Using this in the immediate above equation we derive the equation

$$d_*\omega^\alpha_\beta = -\frac{1}{2} e^{\ k}_{(\beta)}(e^{(\alpha)}_{k;ij} - e^{(\alpha)}_{k;ji}) dx^j \wedge dx^i + \gamma^\alpha{}_{\rho\sigma}\gamma^\rho{}_{\beta\epsilon}\theta^\epsilon \wedge \theta^\sigma. \tag{3.35}$$

Using the Ricci identity (2.30) we obtain after simplifying the terms of the equation

$$d_*\omega^\alpha_\beta = -\frac{1}{2} R_{\delta\sigma\beta}{}^\alpha \theta^\delta \wedge \theta^\sigma + (\gamma^\alpha{}_{\rho\sigma}\gamma^\rho{}_{\beta\epsilon} - \gamma^\alpha{}_{\beta\rho} Q_{\epsilon\sigma}{}^\rho)\theta^\epsilon \wedge \theta^\sigma. \tag{3.36}$$

Define the term

$$\Omega^\alpha_\beta = -\frac{1}{2} R_{\delta\sigma\beta}{}^\alpha\ \theta^\delta \wedge \theta^\sigma, \tag{3.37}$$

where Ω^α_β are the tetrad components of curvature 2-form. Using the equations (1.6) and (3.28), the equation (3.36) yields

$$\Omega^\alpha_\beta = d_*\omega^\alpha_\beta + \omega^\alpha_\rho \wedge \omega^\rho_\beta - K\gamma^\alpha{}_{\beta\rho} S_{\epsilon\sigma} u^\rho \theta^\epsilon \wedge \theta^\sigma. \tag{3.38}$$

This is the Cartan's second equation of structure. This can also be put in the following form

$$\begin{aligned} \Omega_{\alpha\beta} \;=\; & \Omega^0{}_{\alpha\beta} + [d_* K_{\epsilon\alpha\beta} + \eta^{\gamma\mu}(K_{\gamma\beta\alpha}\omega^0{}_{\mu\epsilon} + K_{\gamma\beta\alpha}K_{\epsilon\sigma\mu}\theta^\sigma - K_{\epsilon\beta\mu}\omega^0{}_{\alpha\gamma} + \\ & + K_{\epsilon\gamma\alpha}\omega^0{}_{\mu\beta} + K_{\sigma\gamma\alpha}K_{\epsilon\beta\mu}\theta^\sigma)] \wedge \theta^\epsilon - K(\gamma^0_{\alpha\beta\rho} + K_{\rho\alpha\beta}) S_{\epsilon\sigma} u^\rho \theta^\epsilon \wedge \theta^\sigma, \end{aligned} \tag{3.39}$$

where

$$\Omega^{0\alpha}{}_\beta = d\omega^{0\alpha}{}_\beta + \omega^{0\alpha}{}_\rho \wedge \omega^{0\rho}{}_\beta. \tag{3.40}$$

is the Cartan's equation of structure in the Einstein space-time geometry. We will derive the following expression for the repetition of d_* of the tetrad basis 1-form.It is evident from the Cartan's first equation of structure (3.27) that

$$d_*^2\theta^\alpha = -(d_*\omega^\alpha_{\ \beta} \wedge \theta^\beta - \omega^\alpha_{\ \beta} \wedge d_*\theta^\beta). \tag{3.41}$$

A simple but straight forward calculations show that

$$d_*^2\theta^\alpha = [dK_{\gamma\beta}^{\ \ \alpha} + K_{\gamma\beta}^{\ \ \sigma}\omega^{0\alpha}_{\ \ \sigma} + K_{\gamma\sigma}^{\ \ \alpha}\omega^{0\sigma}_{\ \ \beta} - K_{\sigma\beta}^{\ \ \alpha}\omega^{0\sigma}_{\ \ \gamma} - K_{\gamma\sigma}^{\ \ \alpha}K_{\epsilon\beta}^{\ \ \sigma}\theta^\epsilon + K_{\sigma\beta}^{\ \ \alpha}K_{\epsilon\gamma}^{\ \ \sigma}\theta^\epsilon] \wedge \theta^\gamma \wedge \theta^\beta. \tag{3.42}$$

In the absence of the contortion tensor the result (3.42) reduces to $d^2\theta^\alpha = 0$.

4. Maxwell Equations in Einstein-Cartan Theory

Let F_{ij} be a skew-symmetric electromagnetic tensor field. The tensor form of the Maxwell's equations can have the same form in both the Einstein theory of gravitation and the Einstein-Cartan theory of gravitation. The difference lies in the definition of covariant derivative. In the Einstein space time it is with respect to the symmetric Christoffel symbols while in the ECT space-time it is with respect to the asymmetric connections. The source free Maxwell's equations in the ECT space-time are given by

$$F^{ij}_{\ \ ;j} = 0, \quad F_{[ij;k]} = 0. \tag{4.1}$$

where

$$F_{[ij;k]} = \frac{1}{3}(F_{ij;k} + F_{jk;i} + F_{ki;j}). \tag{4.2}$$

In Einstein theory of gravitation, tetrad components of the Maxwell's equations were given by Debney and Zund (1971). We determine the form of these equations through the new derivative operator d_* in the ECT of gravitation. We convert the equation (3.17)in to the tetrad components as

$$d_*\tilde{\phi} = d\tilde{\phi} + \frac{1}{3}\eta^{\epsilon\sigma}[F_{\sigma\beta}(K_{\gamma\alpha\epsilon} - K_{\alpha\gamma\epsilon}) + F_{\alpha\sigma}(K_{\gamma\beta\epsilon} - K_{\beta\gamma\epsilon}) + F_{\sigma\gamma}(K_{\alpha\beta\epsilon} - K_{\beta\alpha\epsilon})]\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma, \tag{4.3}$$

where

$$d\tilde{\phi} = \frac{1}{3}[F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} - \eta^{\epsilon\sigma}[F_{\sigma\beta}(\gamma^0_{\epsilon\alpha\gamma} - \gamma^0_{\epsilon\gamma\alpha}) + F_{\alpha\sigma}(\gamma^0_{\epsilon\beta\gamma} - \gamma^0_{\epsilon\gamma\beta}) + + F_{\sigma\gamma}(\gamma^0_{\epsilon\beta\alpha} - \gamma^0_{\epsilon\alpha\beta})]\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma]. \tag{4.4}$$

The tetrad components of the electromagnetic field tensor F_{ij} are given by

$$F_{\alpha\beta} = F_{ij}e^i_{(\alpha)}e^j_{(\beta)}. \tag{4.5}$$

Debney and Zund [2] have given the expression for the electromagnetic field tensor in the form

$$F_{ij} = -4Re\phi_1 l_{[i}n_{j]} - 4\,i\,Im\phi_1\bar{m}_{[i}m_{j]} + [2\phi_2 l_{[i}m_{j]} + 2\phi_0\bar{m}_{[i}n_{j]}] + [c.c], \tag{4.6}$$

where

$$\begin{aligned} F_{12} &= 2\,Re\phi_1, \\ F_{13} &= \phi_0, \\ F_{23} &= -\bar{\phi}_2, \end{aligned}$$

$$F_{34} \quad = \quad -2\, i\, Im\phi_1. \tag{4.7}$$

We see that the equation $d_*\tilde{\phi} = 0$ is equivalent to the following set of Maxwell's equations in the ECT of gravitation.

$$\begin{aligned}
&D\phi_1 - \bar{\delta}\phi_0 = (\pi^0 - 2\alpha^0 + \sqrt{2}K\bar{s_0})\phi_0 + 2(\rho^0 - \sqrt{2}Ks_1)\phi_1 - (\kappa^0 - \sqrt{2}Ks_0)\phi_2,\\
&\delta\phi_2 - \Delta\phi_1 = -(\nu^0 - \sqrt{2}Ks_0)\phi_0 + 2(\mu^0 - \sqrt{2}Ks_1)\phi_1 - (\tau^0 - 2\beta^0 + \sqrt{2}Ks_0)\phi_2,\\
&D\phi_2 - \bar{\delta}\phi_1 = -\lambda^0\phi_0 + 2\pi^0\phi_1 - (\rho^0 - 2\epsilon^0)\phi_2,\\
&\delta\phi_1 - \Delta\phi_0 = (\mu^0 - 2\gamma^0)\phi_0 + 2\tau^0\phi_1 - \sigma^0\phi_2.
\end{aligned} \tag{4.8}$$

We notice that in the absence of torsion tensor all these results reduce to the results of the Einstein space-time.

4.1. **Commutator relations.** : We express the tensor components of the covariant derivative of a scalar function f in terms of its tetrad components and vice versa as

$$f_{;i} = f_{;\alpha}e_i^{(\alpha)}, \tag{4.9}$$

and

$$f_{;\alpha} = f_{;i}e^i_{(\alpha)}. \tag{4.10}$$

Using these relations, we obtain

$$f_{;ij}e^i_{(\alpha)}e^j_{(\beta)} = f_{;\alpha\beta} - f_{;\sigma}\gamma^\sigma_{\alpha\beta}. \tag{4.11}$$

Interchanging the indices i and j in the above equation and subtracting the result from it we obtain the equation

$$(f_{;ij} - f_{;ji})e^i_{(\alpha)}e^j_{(\beta)} = f_{;\alpha\beta} - f_{;\beta\alpha} - f_{;\sigma}(\gamma^\sigma_{\alpha\beta} - \gamma^\sigma_{\beta\alpha}). \tag{4.12}$$

Replacing u_i by $f_{;i}$ in the definition of covariant derivative (2.1) we obtain the equation

$$(f_{;ij} - f_{;ji})e^i_{(\alpha)}e^j_{(\beta)} = f_{;\sigma}(K_{\alpha\beta}{}^{\sigma} - K_{\beta\alpha}{}^{\sigma}). \tag{4.13}$$

From equations (4.12) and (4.13)we readily obtain the equation

$$f_{,\alpha\beta} - f_{,\beta\alpha} = f_{,\sigma}\eta^{\sigma\epsilon}(\gamma^0_{\epsilon\alpha\beta} - \gamma^0_{\epsilon\alpha\beta}). \tag{4.14}$$

where we have used the relation $\gamma^\sigma_{\alpha\beta} = \gamma^{0\sigma}_{\alpha\beta} - K_{\beta\alpha}{}^{\sigma}$.
By giving different values to the Greek indices from 1 to 4 in the equation (4.14) we derive the following commutator relations.

$$\begin{aligned}
&f_{,12} - f_{,21} = [\Delta, D]f = (\gamma^0 + \bar{\gamma}^0)Df + (\epsilon^0 + \bar{\epsilon}^0)\Delta f - (\bar{\tau}^0 + \pi^0)\delta f - (\tau^0 + \bar{\pi}^0)\bar{\delta}f,\\
&f_{,13} - f_{,31} = [\delta, D]f = (\bar{\alpha}^0 + \beta^0 - \bar{\pi}^0)Df + \kappa^0\Delta f - (\bar{\rho}^0 + \epsilon^0 - \bar{\epsilon}^0)\delta f - \sigma^0\bar{\delta}f,\\
&f_{,23} - f_{,32} = [\delta, \Delta]f = -\bar{\nu}^0Df + (\tau^0 - \bar{\alpha}^0 - \beta^0)\Delta f + (\mu^0 - \gamma^0 + \bar{\gamma}^0)\delta f + \bar{\lambda}^0\bar{\delta}f,\\
&f_{,34} - f_{,43} = [\bar{\delta}, \delta]f = -(\mu^0 - \bar{\mu}^0)Df - (\rho^0 - \bar{\rho}^0)\Delta f + (\alpha^0 - \bar{\beta}^0)\delta f - (\bar{\alpha}^0 - \bar{\beta}^0)\bar{\delta}f.
\end{aligned} \tag{4.15}$$

5. Frankel Condition

For the classical description of the spin of gravitating matters Hehl et.al.[5] have decomposed the spin tensor described in the equation (1.5) with an additional restriction that it is u-orthogonal which is unnecessary. Throughout this section we assume unless otherwise mentioned,

$$S_{ij}u^i \neq 0. \tag{5.1}$$

To study its consequences, we express the torsion tensor $Q_{ij}^{\ \ k}$ in terms of its tetrad components as

$$Q_{ij}^{\ \ k} = Q_{\alpha\beta}^{\ \ \gamma} e_i^{(\alpha)} e_j^{(\beta)} e_{(\gamma)}^k, \tag{5.2}$$

where

$$Q_{\alpha\beta}^{\ \ \gamma} = -\frac{1}{2}(K_{\alpha\beta}^{\ \ \gamma} - K_{\beta\alpha}^{\ \ \gamma}). \tag{5.3}$$

Following the notations of Jogia and Griffiths [7],we obtain the expression

$$\begin{aligned} Q_{ij}^{\ \ k} \ = \ & (\mu_1 - \bar{\mu}_1)\bar{m}_{[i}m_{j]}l^k + (\rho_1 - \bar{\rho}_1)\bar{m}_{[i}m_{j]}n^k + (\epsilon_1 + \bar{\epsilon}_1)l_{[i}n_{j]}n^k + (\gamma_1 + \bar{\gamma}_1))l_{[i}n_{j]}l^k - \\ & -[\bar{\nu}_1 l_{[i}\bar{m}_{j]}l^k + \kappa_1\bar{m}_{[i}n_{j]}n^k - \lambda_1 l_{[i}m_{j]}m^k - \sigma_1\bar{m}_{[i}n_{j]}\bar{m}^k + (\bar{\pi}_1 + \tau_1)l_{[i}n_{j]}\bar{m}^k + \\ & +(\alpha_1 + \bar{\beta}_1 - \bar{\tau}_1)l_{[i}m_{j]}n^k + (\bar{\alpha}_1 + \beta_1 - \bar{\pi}_1)\bar{m}_{[i}n_{j]}l^k - (\bar{\mu}_1 + \gamma_1 - \bar{\gamma}_1)l_{[i}m_{j]}\bar{m}^k - \\ & -(\bar{\rho}_1 + \epsilon_1 - \bar{\epsilon}_1)\bar{m}_{[i}n_{j]}m^k - (\bar{\alpha}_1 - \beta_1)\bar{m}_{[i}m_{j]}\bar{m}^k] - [c.c]. \end{aligned} \tag{5.4}$$

Contracting the index $j$with k in the equation (5.4),we get

$$Q_{ik}^{\ \ k} = -\frac{1}{2}[(\mu_1+\bar{\mu}_1-\gamma_1-\bar{\gamma}_1)l_i-(\rho_1+\bar{\rho}_1-\epsilon_1-\bar{\epsilon}_1)n_i+(\alpha_1-\bar{\beta}_1-\pi_1+\bar{\tau}_1)m_i+(\bar{\alpha}_1-\beta_1-\bar{\pi}_1+\tau_1)\bar{m}_i]. \tag{5.5}$$

Using the equations (5.4)and (5.5) in the field equation (1.3), we obtain

$$\begin{aligned} KS_{ij}^{\ \ k} \ = \ & (\mu_1 + \bar{\mu}_1)l_{[i}n_{j]}l^k + (\rho_1 + \bar{\rho}_1)l_{[i}n_{j]}n^k + (\mu_1 - \bar{\mu}_1)\bar{m}_{[i}m_{j]}l^k + (\rho_1 - \bar{\rho}_1)\bar{m}_{[i}m_{j]}n^k - \\ & -[\bar{\nu}_1 l_{[i}\bar{m}_{j]}l^k + (2\alpha_1 - \pi_1)l_{[i}m_{j]}n^k + \kappa_1\bar{m}_{[i}n_{j]}n^k + (2\bar{\beta}_1 - \bar{\tau}_1)m_{[i}n_{j]}l^k + \\ & +(\mu_1 - 2\gamma_1)l_{[i}m_{j]}\bar{m}^k + (\rho_1 - 2\epsilon_1)\bar{m}_{[i}n_{j]}m^k + (\pi_1 + \bar{\tau}_1)l_{[i}n_{j]}m^k + (\pi_1 - \bar{\tau}_1)\bar{m}_{[i}m_{j]}m^k - \\ & -\lambda_1 l_{[i}m_{j]}m^k - \bar{\sigma}_1 m_{[i}n_{j]}m^k] + c.c. \end{aligned} \tag{5.6}$$

Similarly, the spin tensor S_{ij} can be expressed in terms of its tetrad components as,

$$S_{ij} = S_{\alpha\beta} e_i^{(\alpha)} e_j^{(\beta)}, \tag{5.7}$$

The tetrad components of the spin tensor are defined in the equation (1.7). Using these components we write equation (5.7) as

$$S_{ij} = -2[(s_1 + \bar{s}_1)l_{[i}n_{j]} + (s_1 - \bar{s}_1)\bar{m}_{[i}m_{j]} - (s_0\bar{m}_{[i}n_{j]} + \bar{s}_2 l_{[i}m_{j]}) - c.c.] \tag{5.8}$$

Theorem: The following statements are equivalent.

$$\begin{aligned} (i)\ \ S_{ij}^{\ \ k} \ &= \ S_{ij}u^k \\ (ii)\ \ \rho_1 \ &= \ \mu_1 = 2\epsilon_1 = 2\gamma_1 = -\sqrt{2}Ks_1, \\ \nu_1 \ &= \ 2\alpha_1 = -\sqrt{2}K\bar{s}_2, \\ \kappa_1 \ &= \ 2\beta_1 = -\sqrt{2}Ks_0, \end{aligned}$$

$$\pi_1 = \tau_1 = \lambda_1 = \sigma_1 = 0. \tag{5.9}$$

Proof: We define the time-like vector u^i as $u^i = \frac{1}{\sqrt{2}}(l^i + n^i)$ such that $u_i u^i = 1$. Multiplying the equation(5.8) by u^i we get

$$S_{ij}u^k = -\sqrt{2}[(s_1+\bar{s}_1)l_{[i}n_{j]} + (s_1-\bar{s}_1)\bar{m}_{[i}m_{j]} - (s_0\bar{m}_{[i}n_{j]} + \bar{s}_2 l_{[i}m_{j]}) - c.c.](l^k+n^k). \tag{5.10}$$

If $S_{ij}{}^k = S_{ij}u^k$,then the corresponding coefficients of the equations (5.6) and (5.10)must be identical. Hence equating the corresponding coefficients,we obtain the relations

$$\begin{aligned}
&(\rho_1 + \bar{\rho}_1) = (\mu_1 + \bar{\mu}_1) = -\sqrt{2}K(s_1 + \bar{s}_1),\\
&(\rho_1 - \bar{\rho}_1) = (\mu_1 - \bar{\mu}_1) = -\sqrt{2}K(s_1 - \bar{s}_1),\\
&\kappa_1 = 2\beta_1 - \tau_1 = -\sqrt{2}Ks_0,\\
&\bar{\mu}_1 = 2\bar{\alpha}_1 - \bar{\pi}_1 = -\sqrt{2}Ks_2,\\
&\mu_1 - 2\gamma_1 = 0, \rho_1 - 2\epsilon_1 = 0, \pi_1 + \bar{\tau}_1 = 0,\\
&\pi_1 - \bar{\tau}_1 = 0, \lambda_1 = 0, \sigma_1 = 0.
\end{aligned} \tag{5.11}$$

Solving these equations we obtain the required results. In fact the set of equations (5.9) is the consequence of the field equation (1.3). One can independently derive these equations from the field equation. By virtue of the equation (5.9) the equation (5.7)becomes

$$KS_{ij} = \sqrt{2}[(\rho_1 + \bar{\rho}_1)l_{[i}n_{j]} + (\rho_1 - \bar{\rho}_1)\bar{m}_{[i}m_{j]} - (\kappa_1\bar{m}_{[i}n_{j]} + \bar{\nu}_1 l_{[i}\bar{m}_{j]}) - c.c.]. \tag{5.12}$$

Contracting the field equation (1.3),we obtain

$$Q_i = 2Q_{ik}{}^k = -KS_{ik}{}^k = -KS_{ik}u^k. \tag{5.13}$$

Hence we have

$$Q_{ij}{}^k = \frac{1}{2}K[2S_{ij}{}^k + \delta_i^k S_{jl}{}^l - \delta_j^k S_{il}{}^l]. \tag{5.14}$$

Equivalently,we write

$$Q_{ij}{}^k = \frac{1}{2}K[2S_{ij}u^k + \delta_i^k S_{jl}u^l - \delta_j^k S_{il}u^l]. \tag{5.15}$$

This on using (5.12) we have

$$\begin{aligned}
Q_{ij}{}^k = &\frac{1}{2}[(\rho_1 + \bar{\rho}_1)(l_{[i}n_{j]}(l^k + n^k) + l_{[i}\bar{m}_{j]}m^k + \bar{m}_{[i}n_{j]}m^k + l_{[i}m_{j]}\bar{m}^k + m_{[i}n_{j]}\bar{m}^k) +\\
&+2(\rho_1 - \bar{\rho}_1)\bar{m}_{[i}m_{j]}(l^k + n^k) - [\nu_1(l_{[i}m_{j]}(2l^k + n^k) + m_{[i}n_{j]}l^k + \bar{m}_{[i}m_{j]}m^k) +\\
&+\kappa_1(l_{[i}\bar{m}_{j]}n^k + \bar{m}_{[i}n_{j]}(l^k + 2n^k) + \bar{m}_{[i}m_{j]}\bar{m}^k)] - c.c.].
\end{aligned} \tag{5.16}$$

However, the Frankel's condition $S_{ij}u^j = 0$ implies that $s_0 = s_2$ and $s_1 + \bar{s}_1 = 0$. Consequently the conditions (5.9) reduce to

$$\begin{aligned}
\pi_1 &= \tau_1 = \lambda_1 = \sigma_1 = 0,\\
\kappa_1 &= \bar{\nu}_1 = 2\beta_1 = 2\bar{\alpha}_1 = -\sqrt{2}Ks_0,\\
\rho_1 &= \mu_1 = 2\epsilon_1 = 2\gamma_1 = -\sqrt{2}Ks_1.
\end{aligned} \tag{5.17}$$

These are the same set of equations cited in the equation (3.19). The study of kinematical parameters has proved to be vital in astrophysical applications. Raychaudhary equation describes the effects of shear and rotation in the expansion and contraction of the universe. For any vector of the tetrad $e_{(\alpha)_j}$ we have

$$e_{(\alpha)i;j} = e_{(\alpha)i/j} + K(u_i S_{jk} + u_j S_{ik} + u_k S_{ij} + g_{ij} S_{kh} u^h - g_{jk} S_{ih} u^h) e^k_{(\alpha)}. \tag{5.18}$$

For $\alpha = 1$, we obtain

$$\begin{aligned} l_{i;j} &= l_{i/j} + \frac{1}{2}[(\rho_1 + \bar{\rho}_1)(l_i l_j + l_i n_j) + 2\rho_1 \bar{m}_i m_j + 2\bar{\rho}_1 m_i \bar{m}_j - \\ &\quad -\nu_1 l_i m_j - \bar{\nu}_1 l_i \bar{m}_j - \bar{\kappa}_1 (l_i m_j + 2 m_i n_j) - \kappa_1 (l_i \bar{m} j + 2\bar{m}_i n_j)]. \end{aligned} \tag{5.19}$$

On using equations (5.9), we obtain the equation

$$\begin{aligned} l_{i;j} &= l_{i/j} - \frac{K}{\sqrt{2}}[(s_1 + \bar{s}_1)(l_i l_j + l_i n_j) + s_1 \bar{m}_i m_j + \bar{s}_1 m_i \bar{m}_j - \bar{s}_2 l_i m_j - s_2 l_i \bar{m}_j - \\ &\quad -\bar{s}_0 (l_i m_j + 2 m_i n_j) - s_0 (l_i \bar{m} j + 2\bar{m}_i n_j)], \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} l_{i/j} &= (\gamma^0 + \bar{\gamma}^0) l_i l_j - (\alpha^0 + \bar{\beta}^0) l_i m_j - (\bar{\alpha}^0 + \beta^0) l_i \bar{m}_j + (\epsilon^0 + \bar{\epsilon}^0) l_i n_j - \bar{\tau}^0 m_i l_j + \\ &\quad + \bar{\sigma}^0 m_i m_j + \bar{\rho}^0 m_i \bar{m}_j - \bar{\kappa}^0 m_i n_j - \tau^0 \bar{m}_i l_j + \rho^0 \bar{m}_i m_j + \sigma^0 \bar{m}_i \bar{m}_j - \\ &\quad -\kappa^0 \bar{m}_i n_j. \end{aligned} \tag{5.21}$$

For $\alpha = 2, 3, 4$, one can obtain similar expressions for other null vectors of the tetrad.These are cited in the appendix for the ready reference. Thus the kinematical parameters and the auto-parallel curves in this case become

$$\begin{aligned} &\theta = \hat{\theta}, \\ &\dot{u}_i = \hat{\dot{u}}_i + K S_{ij} u^j, \\ &\sigma_{ij} = \hat{\sigma}_{ij}, \\ &w_{ij} = \hat{w}_{ij} + K(S_{ij} + u_i S_{jk} - S_{ik} u_j) u^k, \\ &\frac{d^2 x^i}{ds^2} + \left\{ {}^{\,i}_{jk} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} + K g^{il}[2 S_{lj} u_k + g_{kl} S_{jh} u^h - g_{jk} S_{lh} u^h] \frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \end{aligned} \tag{5.22}$$

In the case when $S_{ij} u^j \neq 0$ the expressions for the curl of a vector, curl of gradient of a scalar function and divergence of a vector are respectively given by

$$\begin{aligned} &(Curl u_i) = (C\hat{url} u_i) + K[2 S_{ij} + u_i S_{jk} u^k - u_j S_{ik} u^k], \\ &Curl(grad f) = K[2 S_{ij} f_{,k} u^k + f_{,i} S_{jl} u^l - f_{,j} S_{il} u^l], \\ &(div A_i) = (di\hat{v} A_i) + K A^k S_{ki} u^i. \end{aligned} \tag{5.23}$$

However, if an arbitrary vector A_i be taken as the time like vector field u^i, then we have

$$(div u_i) = (di\hat{v} u_i).$$

Also from the first equation of the set (5.23)we obtain on using the equation (5.18), the curl of the null vector field l_i as

$$\begin{aligned} l_{i;j} - l_{j;i} &= l_{i/j} - l_{j/i} - \sqrt{2} K[(s_1 + \bar{s}_1) l_{[i} n_{j]} + 2(s_1 - \bar{s}_1)\bar{m}_{[i} m_{j]} - \\ &\quad -(2 s_0 \bar{m}_{[i} n_{j]} + (s_0 + s_2) l_{[i} \bar{m}_{j]}) - c.c]. \end{aligned} \tag{5.24}$$

Similarly, from the third equation of the set(5.23)we find the divergence of the null vector field l^i as

$$l^i_{\ ;i} = l^i_{\ /i} + \frac{K}{\sqrt{2}}(s_1 + \bar{s}_1). \tag{5.25}$$

The divergence of the vector field l^i viz., $l^i_{\ /i}$ in the Einstein theory of gravitation can be obtained from the equation (5.21). It is evident that the equations (5.24) and (5.25) also follow from the equation (5.20). The curl and divergence of other vector fields of the tetrad are similarly obtained.These results are presented in the appendix. Now to find the Raychaudhary equation when the spin tensor is not u-orthogonal, we obtain the equation for the Ricci tensor in the form

$$\begin{aligned} R_{ij} &= \hat{R}_{ij} + K[-\dot{S}_{ij} - S_{ij}\theta + u_i S_j^{\ k}{}_{;k} + u_{i;k}S_j^{\ k} + S_i^{\ k}{}_{;k}u_j + S_i^{\ k}u_{j;k} + g_{ij}(S^k_{\ h;k}u^h + S^k_{\ h}u^h_{\ ;k})] + \\ &+K^2[2S^2u_iu_j - 2S_{ik}S_j^{\ k} - g_{ij}S^h_{\ k}S_{ht}u^ku^t - S_j^{\ h}S_{hk}u_iu^k - S_{ih}S^h_{\ k}u_ju^k]. \end{aligned} \tag{5.26}$$

Following the process explained in the Section 2, we find the Raychaudhari equation (2.37)in the form

$$\dot{\theta} = \dot{u}^i_{\ i} - 2(\hat{\sigma}^2 - \hat{\omega}^2 + \frac{1}{3}\hat{\theta}^2) + \hat{R}_{ij}u^iu^j + 2K^2S^2 + K\hat{\omega}_{ik}S^{ik} + KS_{ik}\hat{\dot{u}}^iu^k - KS^i_{\ k;i}u^k. \tag{5.27}$$

We see that the spin of the gravitating matter affects the propagation equation of the expansion. Similarly, we record below the corresponding Maxwell's equations (4.8) when spin tensor is not u-orthogonal.

$$\begin{aligned} &D\phi_1 - \bar{\delta}\phi_0 = (\pi^0 - 2\alpha^0 + \sqrt{2}K\bar{s}_2)\phi_0 + 2(\rho^0 - \sqrt{2}Ks_1)\phi_1 - (\kappa^0 - \sqrt{2}Ks_0)\phi_2, \\ &\delta\phi_2 - \Delta\phi_1 = -(\nu^0 - \sqrt{2}K\bar{s}_2)\phi_0 + 2(\mu^0 - \sqrt{2}Ks_1)\phi_1 - (\tau^0 - 2\beta^0 + \sqrt{2}Ks_0)\phi_2, \\ &D\phi_2 - \bar{\delta}\phi_1 = -\lambda^0\phi_0 + 2\pi^0\phi_1 - (\rho^0 - 2\epsilon^0)\phi_2, \\ &\delta\phi_1 - \Delta\phi_0 = (\mu^0 - 2\gamma^0)\phi_0 + 2\tau^0\phi_1 - \sigma^0\phi_2. \end{aligned} \tag{5.28}$$

The equations obtained in the paper are the basic results in the ECT of gravitation. We hope that these results can be exploited in the subsequent study in the Einstein-Cartan theory of gravitation.

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Appendix:

$$\begin{aligned} n_{i;j} &= n_{i/j} + \frac{K}{\sqrt{2}}[(s_1 + \bar{s}_1)(n_il_j + n_in_j) - (\bar{s}_0 + \bar{s}_2)n_im_j - (s_0 + s_2)n_i\bar{m}_j - 2\bar{s}_2m_il_j + \\ &+2s_1m_i\bar{m}_j - 2s_2\bar{m}_il_j + 2\bar{s}_1\bar{m}_im_j], \\ m_{i;j} &= m_{i/j} - \frac{K}{\sqrt{2}}[2s_2l_il_j - 2\bar{s}_1l_im_j + (s_1 - \bar{s}_1)m_il_j - (\bar{s}_2 - \bar{s}_0)m_im_j + (s_2 - s_0)m_i\bar{m}_j + \\ &+(s_1 - \bar{s}_1)m_in_j + 2s_1n_im_j - 2s_0n_in_j], \end{aligned}$$

where

$$\begin{aligned} n_{i/j} &= -(\gamma^0 + \bar{\gamma}^0)n_il_j + (\alpha^0 + \bar{\beta}^0)n_im_j + (\bar{\alpha}^0 + \beta^0)n_i\bar{m}_j - (\epsilon^0 + \bar{\epsilon}^0)n_in_j + \nu^0m_{il_j-} \\ &-\lambda^0m_im_j - \mu^0m_i\bar{m}_j + \pi^0m_in_j + \nu^0\bar{m}_il_j - \mu^0\bar{m}_im_j + \lambda^0\bar{m}_i\bar{m}_j - -\bar{\pi}^0\bar{m}_in_j, \end{aligned}$$

and

$$\begin{aligned} m_{i/j} &= \bar{\nu}^0 l_i l_j - \bar{\mu}^0 l_i m_j - \bar{\lambda}^0 l_i \bar{m}_j + \bar{\pi}^0 l_i n_j + (\gamma^0 - \bar{\gamma}^0) m_i l_j - (\alpha^0 - \bar{\beta}^0) m_i m_j + \\ &\quad + (\bar{\alpha}^0 - \beta^0) m_i \bar{m}_j + (\epsilon^0 - \bar{\epsilon}^0) m_i n_j - \tau^0 n_i l_j + \rho^0 n_i m_j + \sigma^0 n_i \bar{m}_j - \kappa^0 n_i n_j. \end{aligned}$$

Expressions for the Curl of n_i and m_i:

$$\begin{aligned} n_{i;j} - n_{j;i} &= n_{i/j} - n_{j/i} - \sqrt{2}K[(s_1 + \bar{s}_1) l_{[i} n_{j]} + 2(s_1 - \bar{s}_1)\bar{m}_{[i} m_{j]} - \\ &\quad -[(s_0 + s_2)\bar{m}_{[i} n_{j]} + 2s_2 l_{[i}\bar{m}_{j]}] - c.c]. \\ m_{i;j} - m_{j;i} &= m_{i/j} - m_{j/i} + \sqrt{2}K[(s_1 + \bar{s}_1)(l_{[i} m_{j]} + m_{[i} n_{j]}) + (s_2 - s_0)\bar{m}_{[i} m_{j]}]. \end{aligned}$$

Expressions for the Divergence of n_i and m_i:

$$\begin{aligned} n^i{}_{;i} &= n^i{}_{/i} - \frac{K}{\sqrt{2}}(s_1 + \bar{s}_1). \\ m^i{}_{;i} &= m^i{}_{/i} + \frac{K}{\sqrt{2}}(s_2 - s_0). \end{aligned}$$

If however, the spin tensor is u-orthogonal then on using the equation (1.8) it is evident from these equations that

$$\begin{aligned} l_{i;j} - l_{j;i} &= l_{i/j} - l_{j/i} - 2\sqrt{2}K[(2s_1\bar{m}_{[i} m_{j]} - s_0(l_{[i}\bar{m}_{j]} + (\bar{m}_{[i} n_{j]}) - c.c]. \\ n_{i;j} - n_{j;i} &= n_{i/j} - n_{j/i} - 2\sqrt{2}K[2s_1\bar{m}_{[i} m_{j]} - (s_0(l_{[i}\bar{m}_{j]} + \bar{m}_{[i} n_{j]}) - (c.c)]. \\ m_{i;j} - m_{j;i} &= m_{i/j} - m_{j/i}. \\ l^i{}_{;i} &= l^i{}_{/i}, \\ n^i{}_{;i} &= n^i{}_{/i}, \\ m^i{}_{;i} &= m^i{}_{/i}. \end{aligned}$$

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