

Data analysis of continuous gravitational wave: All sky search and study of templates

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Abstract

We have studied the problem of all sky search in reference to continuous gravitational wave particularly for such sources whose wave-form are known in advance. We have made an analysis of the number of templates required for matched filter analysis as applicable to these sources. We have employed the concept of *fitting factor (FF)*; treating the source location as the parameters of the signal manifold and have studied the matching of the signal with templates corresponding to different source locations. We have investigated the variation of FF with source location and have noticed a symmetry in template parameters, θ_T and ϕ_T . It has been found that the two different template values in source location, each in θ_T and ϕ_T , have same FF. We have also computed the number of templates required assuming the noise power spectral density $S_n(f)$ to be flat. It is observed that higher FF requires exponentially increasing large number of templates.

1 Introduction

Gravitational wave (GW) Laser Interferometer antennas are essentially omni - directional with their response better than 50% of the average over 75% of the whole sky (Grishchuk et al., 2000). Hence the data analysis systems will have to carry out all sky searches for its sources. We know that the amplitude of intense GW believed bathing the earth is very small, as compared to the sensitivity of GW detectors, and is further masked by the dominant noise. In these circumstances, continuous gravitational wave (CGW) sources are of

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prime importance because for such sources we can achieve enhanced signal-to-noise ratio (SNR) by investigating longer observation data set. However, a long observation time introduces modulation effects, arising due to the relative motion of the detector and the source. As a consequence, there results redistribution of power in the forest of side bands resulting into the reduction of the expected power due to amplitude modulation (AM). The problem of all sky search gains another dimension in view of the fact that there are reasons to believe the presence of intense GW sources whose locations and even frequencies are not known. Amongst such sources pulsars occupy an important position. Similar to all sky search one will also have to do all frequency search. All sky all frequency search is the holy grail of gravitation pulsar astronomy. In this paper we confine ourselves to the problem of all sky search.

Search of CGW without a priori knowledge appears to be computationally quite demanding even by the standard computers expected to be available in the near future. For example, in the case of bandwidth 10^3 Hz, observation time 10^7 sec. and star's minimum decay time of 100 years one would require 10^{14} *Tflops* computer (Frasca, 2000). Very fast computer and large memories with ample amount of disk space seems inevitable. However, a choice of optimal data processing and a clever programming is also integral part of a solution to this problem. Amongst these the pre-correction of time series due to the Doppler modulation before the data is processed may be a method, which will reduce the computational requirements. In reference to this, Schutz (1991) has introduced the concept of patch in the sky as the region of space throughout which the required Doppler correction remains the same. He has also demonstrated that the number of patches required for 10^7 sec. observation data set and one KHz signal would be about 1.3×10^{13} . However, the size of the patch would also depend on the data analysis technique being employed.

Matched filtering is the most suitable technique for the detection of signals from sources viz., pulsars whose wave form is known. The wave forms are used to construct a bank of templates, which represent the expected signal wave form with all possible ranges of its parameters. The time of arrival, source location, frequency of the signal, ellipticity of the source and its spin down represent important parameters of GW emitted by a pulsar. For detection of GW we check if the cross correlation of the templates with the corresponding data set exceeds the preassigned threshold. We introduce in the next section the criterion of the *fitting factor (FF)* (Apostolatos, 1995) applicable to such analysis. We consider the source location as parameters of the signal manifold and investigate the matching of the waveforms corresponding to different source locations. In section 3 we compute the number of templates required for all sky search. A discussion of the results is provided in the section 4.

2 Matched filter analysis: Templates

The bank of templates is matched, in practice, to only a discrete set of signals from among the continuum of possible signals. Consequently, it is natural that all the signals will not

get detected with equal efficiency. However, it is possible to choose judiciously the set of templates so that all the signals of a given amplitude are detected with a given minimum detection loss. The standard measure for deciding what class of wave form is good enough is the FF . It quantitatively describes the closeness of the true signals to the template manifold in terms of the reduction of SNR arising due to the cross correlation of a signal outside the manifold with the best matching templates lying inside the manifold. If the FF of a template family is unity the signal lies in the manifold. If the FF is less than unity the signal lies outside manifold.

Even if the signal discrete templates lie within the template manifold it would be unlikely that any of the actual templates used would correspond to the signal. The parameters describing the search template (source location, ellipticity, etc.) can vary continuously through out a finite range of values. The set of templates characterised by the continuously varying parameters is of-course infinite. However, in practice the interferometer output must be cross correlated with a finite subset of the templates whose parameter values vary in discrete steps from one template to the next. This subset (“the discrete template family”) has measure zero on the manifold of the full set of possible templates (“the continuous template family”), so the template which most closely match a signal will generally lie in between the signal and the nearest of the discrete template family. The mismatch between the signal and the nearest of the discrete templates will cause some reduction in SNR. This would mean that the members of the discrete template family must be chosen so as to render acceptable loss of SNR.

The study of templates has been made by many research workers in time domain [Schutz (1991), Królak (1997), Brady et. al. (1998), Brady and Creighton (2000), Jaranowski et al. (1998) and Jaranowski and Królak (2000)]. However, the analysis in the frequency domain has the advantage of incorporating interferometer’s spectral noise density. In order to determine the number of templates required for matched filtering analysis, we make use of the formula for FF as given by Apostolatos (1995) i.e.

$$FF = \max_{\theta, \phi} \frac{\langle h(f) | h_T(f; \theta_T, \phi_T) \rangle}{\sqrt{\langle h_T(f; \theta_T, \phi_T) | h_T(f; \theta_T, \phi_T) \rangle \langle h(f) | h(f) \rangle}} \quad (1)$$

where $h(f)$ and $h_T(f; \theta_T, \phi_T)$ represent respectively the FTs of the actual signal wave form and the templates.

In earlier papers we have made an analysis of continuous gravitational wave output through Fourier transform (FT) (Srivastava and Sahay, 2002 a, b). These papers are referred in the sequel respectively as I and II. It has been established there that the AM of CGW data output results into redistribution of power at four additional frequencies $f \pm 2f_{rot}, f \pm f_{rot}$ in accordance with the frequency modulation (FM). Hence it is sufficient for the analysis of FF to consider only the frequency modulated FT. The results obtained in paper-II regarding FT of frequency modulated data output i.e. (II-27, 28) may be arranged, making use of the symmetry property of the Bessel functions, as follows.

$$\tilde{h}(f) \simeq \frac{\nu}{w_{rot}} \left[\frac{J_o(\mathcal{Z})J_o(\mathcal{N})}{2\nu^2} [\{\sin(\mathcal{R} + \mathcal{Q}) - \sin(\mathcal{R} + \mathcal{Q} - \nu\xi_o)\} + \right.$$

$$\begin{aligned}
& i \{ \cos(\mathcal{R} + \mathcal{Q}) - \cos(\mathcal{R} + \mathcal{Q} - \nu\xi_o) \} + \\
& J_o(\mathcal{Z}) \sum_{m=1}^{m=\infty} \frac{J_m(\mathcal{N})}{\nu^2 - m^2} [(\mathcal{Y}\mathcal{U} - \mathcal{X}\mathcal{V}) - i(\mathcal{X}\mathcal{U} + \mathcal{Y}\mathcal{V})] + \\
& \left. \sum_{k=1}^{k=\infty} \sum_{m=-\infty}^{m=\infty} e^{iA\mathcal{B}} (\tilde{\mathcal{C}} - i\tilde{\mathcal{D}}) \right] ; \tag{2}
\end{aligned}$$

$$\left. \begin{aligned}
\mathcal{X} &= \sin(\mathcal{R} + \mathcal{Q} - m\pi/2) \\
\mathcal{Y} &= \cos(\mathcal{R} + \mathcal{Q} - m\pi/2) \\
\mathcal{U} &= \sin \nu\xi_o \cos m(\xi_o - \delta) - \frac{m}{\nu} \{ \cos \nu\xi_o \sin m(\xi_o - \delta) - \sin m\delta \} \\
\mathcal{V} &= \cos \nu\xi_o \cos m(\xi_o - \delta) + \frac{m}{\nu} \sin \nu\xi_o \sin m(\xi_o - \delta) - \cos m\delta
\end{aligned} \right\} \tag{3}$$

Now it is straight-forward to compute FF . To understand the procedure let us consider a source at location $(\theta, \phi) = (25^\circ, 30^\circ)$ emitting frequency $f_o = 0.5$ Hz. In order to evaluate the summation given in Eq. (2) let us note that the value of Bessel function decreases rapidly as its order exceed the argument. Accordingly, for the present case it is sufficient to take the ranges of k and m as 1 to 1610 and -4 to 4 respectively. We wish to analyze the data set for $T_o =$ one sidereal day. To maximize FF over θ and ϕ , we first maximize over ϕ by fixing ϕ_T to some arbitrarily selected value say, $\phi_T = \phi = 30^\circ$. Having done this we maximize over θ by varying θ_T in discrete steps over its entire range i.e. 0° to 180° . The results obtained are plotted in Fig. (1). It is remarked that in order to compute the inner product of two waveform h_1 and h_2 which is defined as

$$\begin{aligned}
\langle h_1|h_2 \rangle &= 2 \int_0^\infty \frac{\tilde{h}_1^*(f)\tilde{h}_2(f) + \tilde{h}_1(f)\tilde{h}_2^*(f)}{S_n(f)} df \\
&= 4 \int_0^\infty \frac{\tilde{h}_1^*(f)\tilde{h}_2(f)}{S_n(f)} df \tag{4}
\end{aligned}$$

where $*$ denotes complex conjugation, $\tilde{}$ denotes the Fourier transform of the quantity underneath ($\tilde{a}(f) = \int_{-\infty}^\infty a(t) \exp(-2\pi ift) dt$) and $S_n(f)$ is the spectral density of the detector's noise. One would require to integrate the expression over the band width of Doppler modulated signal. The band width may be determined by computing the maximum value of the Doppler shift. In accordance with Eq. (I-23) the Doppler shift is $\sim 10^{-4} f_o$. However, for the present case we consider the band width equal to 0.002 Hz. In a similar manner one may fix the θ_T and obtain the variation of FF with template parameter ϕ_T . Figures (2) and (3) represent such a plot for a signal of $f_o = 25$ Hz emitted by a source located at $\theta = 1^\circ$ and $\phi = 35^\circ$ and 220° , respectively.

The following points in reference to these plots may be noted.

- (i) The FF is unity for $\theta_T = 25^\circ, 155^\circ$ [Fig (1)] for $\phi_T = 35^\circ, 145^\circ$ [Fig (2)] and for $\phi_T = 220^\circ, 320^\circ$ [Fig (3)].

(ii) It is found that the dependence of FF on the template variables θ_T and ϕ_T may be expressed via

$$FF = e^{-0.00788(\theta - \theta_T)^2} \quad (5)$$

$$FF = e^{-0.01778(\phi - \phi_T)^2} \quad (6)$$

(iii) The oscillatory behaviour is more or less typical in waveform that match well or bad depending on their parameters. However, we are content with the technique we have employed as the region of such artificial facets fall in the region of the $FF < 0.25$.

Finally, we conclude this section by noting the symmetry property of the FF with template parameters. A closer look of the graphs and the remark (i) above reveal the following symmetry property. The FF is symmetrical under following transformations.

$$\theta_T \longrightarrow \pi - \theta_T \quad 0 \leq \theta_T \leq \pi \quad (7)$$

$$\phi_T \longrightarrow \pi - \phi_T \quad 0 \leq \phi_T \leq \pi \quad (8)$$

$$\phi_T \longrightarrow 3\pi - \phi_T \quad \pi \leq \phi_T \leq 2\pi \quad (9)$$

Let us note that these symmetry properties are based on our results for one day observation time. The generic nature of the symmetries may be established only after studying the variation of FF with T_o . Unfortunately, we could not make this analysis because of our limitations on the computational facilities.

3 Number of Templates

It is important to study the problem of number of templates for all sky search in the light of FF . The results of the previous section reveal that the grid spacing $\Delta\theta$ in the θ -parameter of templates may be expressed symbolically as a function of FF , f_o , T_o , θ and ϕ as

$$\Delta\theta = \mathcal{F}(FF, f_o, \theta, \phi, T_o) \quad (10)$$

Similarly, we have

$$\Delta\phi = \mathcal{G}(FF, f_o, \theta, \phi, T_o) \quad (11)$$

In the literature it is reported that the choice of grid spacing $\Delta\phi$ depends insignificantly on ϕ , whereas $\Delta\theta$ depends as $\sin 2\theta$ (Brady and Creighton, 2000). However, in order to arrive at such conclusions one may have to make analysis for different values of θ and ϕ . Unfortunately, due to the limited memory and the efficiency of the computer we could not make a study of this aspect.

In view of this, Eqs. (5) and (6) may be equivalently expressed as

$$\mathcal{F}(FF, 0.5, 25^\circ, 30^\circ, T) = \left[-(0.00788)^{-1} \ln(FF) \right]^{1/2} \quad (12)$$

$$\mathcal{G}(FF, 25, 1^\circ, 35^\circ, T) = \left[-(0.01778)^{-1} \ln(FF) \right]^{1/2} \quad (13)$$

For any chosen value of FF one can determine $\Delta\theta$ and $\Delta\phi$. However, there is no unique choice for it. Our interest would be in the assignment of $\Delta\theta$ and $\Delta\phi$ such that the spacing is maximum resulting into the least number of templates. As we have mentioned earlier, there is stringent requirement on reducing computer time. Accordingly, there is serious need of adopting some procedure/formalism to achieve this. For example, one may adopt the method of hierarchical search given by Mohanty and Dhurandhar (1996) and Mohanty (1998). This search is carried out in two steps. At the first level one would start with template bank with a coarse spacing in the parameter space but with a lower threshold. At the next level a more finely spaced set of templates and a higher threshold would be used but only around those templates of the previous level which exceeded the previous threshold.

However, an important issue related to the problem of number of templates is regarding the study of the behaviour of number of templates with FF for different f_o and T_o . We have made an investigation of this aspect. We assume a source location $(\theta, \phi) = (1^\circ, 30^\circ)$. We choose some value of FF , say 0.995. Now we maximize FF over ϕ by taking $\phi = \phi_T = 30^\circ$ and choose the spacing $\Delta\theta$ so as to yield the selected FF . In the case under investigation $\Delta\theta$ is found to equal 4.5×10^{-5} . Thereafter we maximize over ϕ by introducing spacing $\Delta\phi$ in the so obtained bank of templates and determine the resulting FF . The results obtained may be expressed in the form of a graph such as shown in Figs. (4) and (5). Interestingly, the nature of these curves are similar. We have obtained a best fit to the graphs as

$$N_{Templates} = \exp[a - bx + cx^2 - dx^3 + ex^4]; \quad 0.85 \leq x \leq 0.99 \quad (14)$$

where a, b, c, d and e are constants as given in Table (1).

Let us note from the graphs, for sake of comparison, that the number of templates required for FF equal to 0.97 are respectively 1.44×10^{10} , 3.5×10^{10} and 5.5×10^{10} for observation data sets of 30, 120 and 365 days and $f_o = 50$ Hz. Similarly to analyse the observation data set of 120 days of GW frequencies 20, 50 and 100 Hz the number of templates required are 1.22×10^{10} , 2.16×10^{10} and 5×10^{10} . It is observed that higher FF requires exponentially increasing large number of templates.

4 Discussion

In view of the complexity of the FT, which contains trigonometric as well as Bessel functions; one has to be careful in computing FF . We have found useful to employ the

f_o (Hz)	T_o (days)	a $\times 10^{-2}$	b $\times 10^{-4}$	c $\times 10^{-5}$	d $\times 10^{-6}$	e $\times 10^{-7}$
50	30	2138.05	2071.43	7225.73	6239.43	2036.14
	180	2317.05	-71.3155	1746.61	2146.55	944.931
	365	2382.96	216.917	2464.42	2464.42	1031.54
20	120	2047.55	-794.473	3564.56	4650.68	1945.87
50		2266.59	4269.44	15655.0	17509.5	6484.51
100		2360.23	-206.906	1158.27	1491.01	733.520

Table 1: Coefficients of the best fit graphs obtained for the number of templates

Romberg integration using Padé approximation. We have used (i) QROMO of numerical recipes instead of QROMB as the former takes care of singularities, and (ii) RATINT routine for Padé approximation.

We have noticed marked symmetries in all sky search in both θ and ϕ space for one day observation time. It has been found that the two different template values in source location, each in θ_T and ϕ_T , have same FF . Accordingly, computation burden will be reduced by a factor of four. However, it is not clear whether this symmetry property can be established analytically as well. The source location, because of these symmetries is uncertain and some other analysis is to be adopted for getting the exact location. We have computed the number of templates assuming the noise power spectral density $S_n(f)$ to be flat which is justified as the bandwidth is extremely narrow.

The issues of optimum template parameterization and placement, and the related computational burden have been discussed in the literature by several authors notably by Sathyaprakash and Dhurandhar (1991), Dhurandhar and Sathyaprakash (1994), Owen (1996), Apostolatos (1995, 1996), Mohanty and Dhurandhar (1996), Mohanty (1998), Owen and Sathyaprakash (1999). The question of possible efficient interpolated representation of the correlators is a problem of current interest and remains still unsolved.

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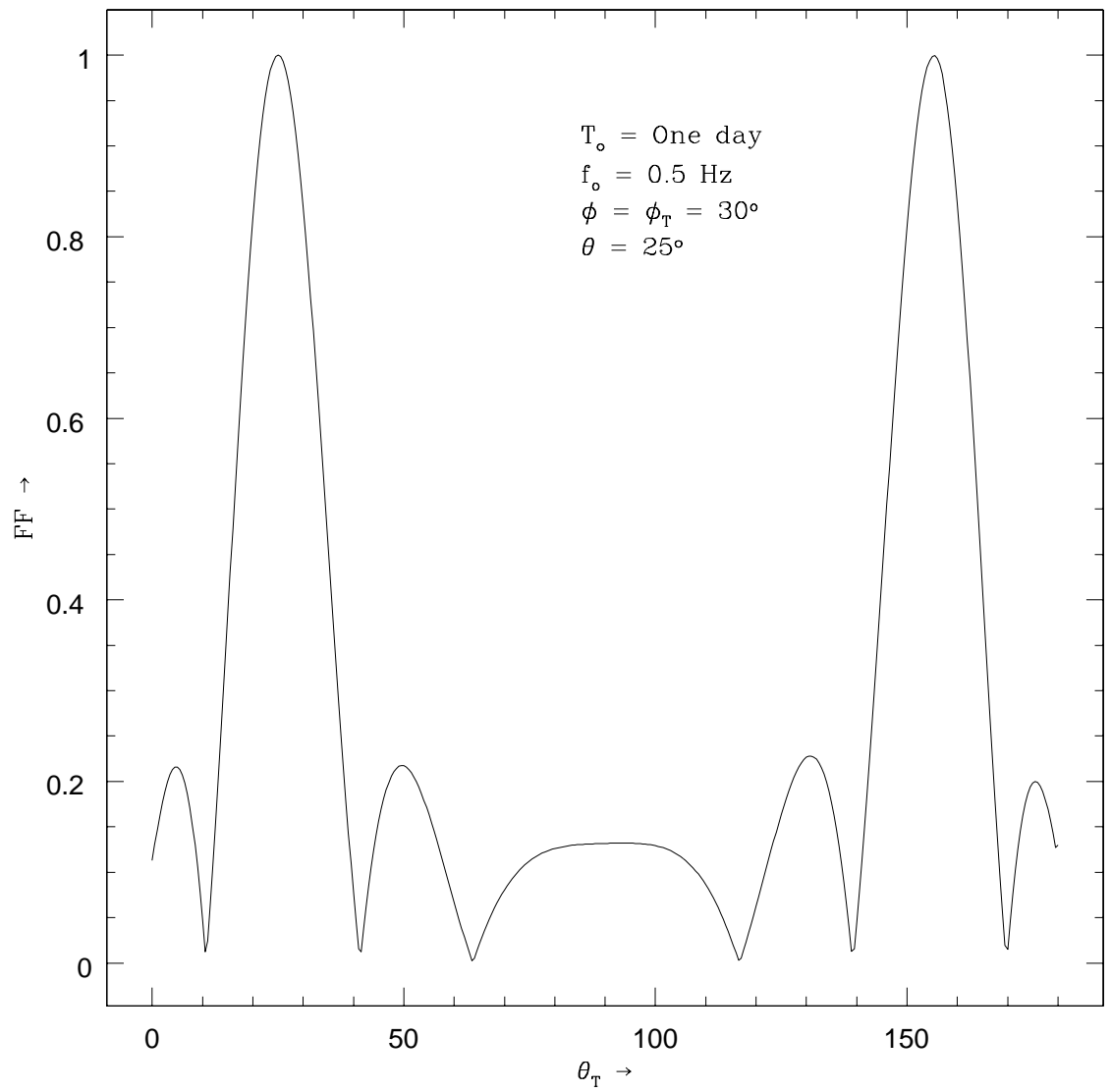


Figure 1: Variation of FF with θ_T .

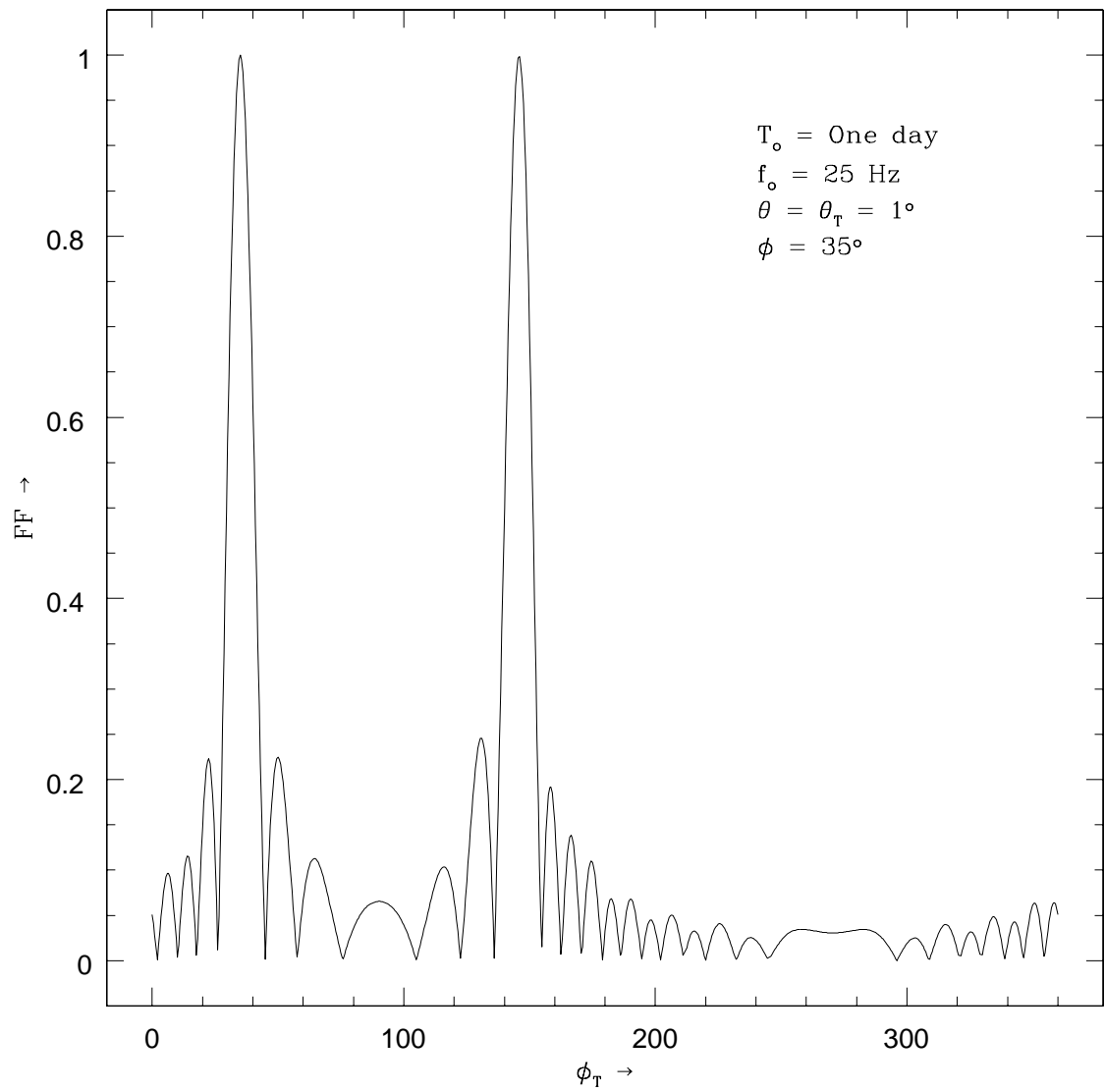


Figure 2: Variation of FF with ϕ_T .

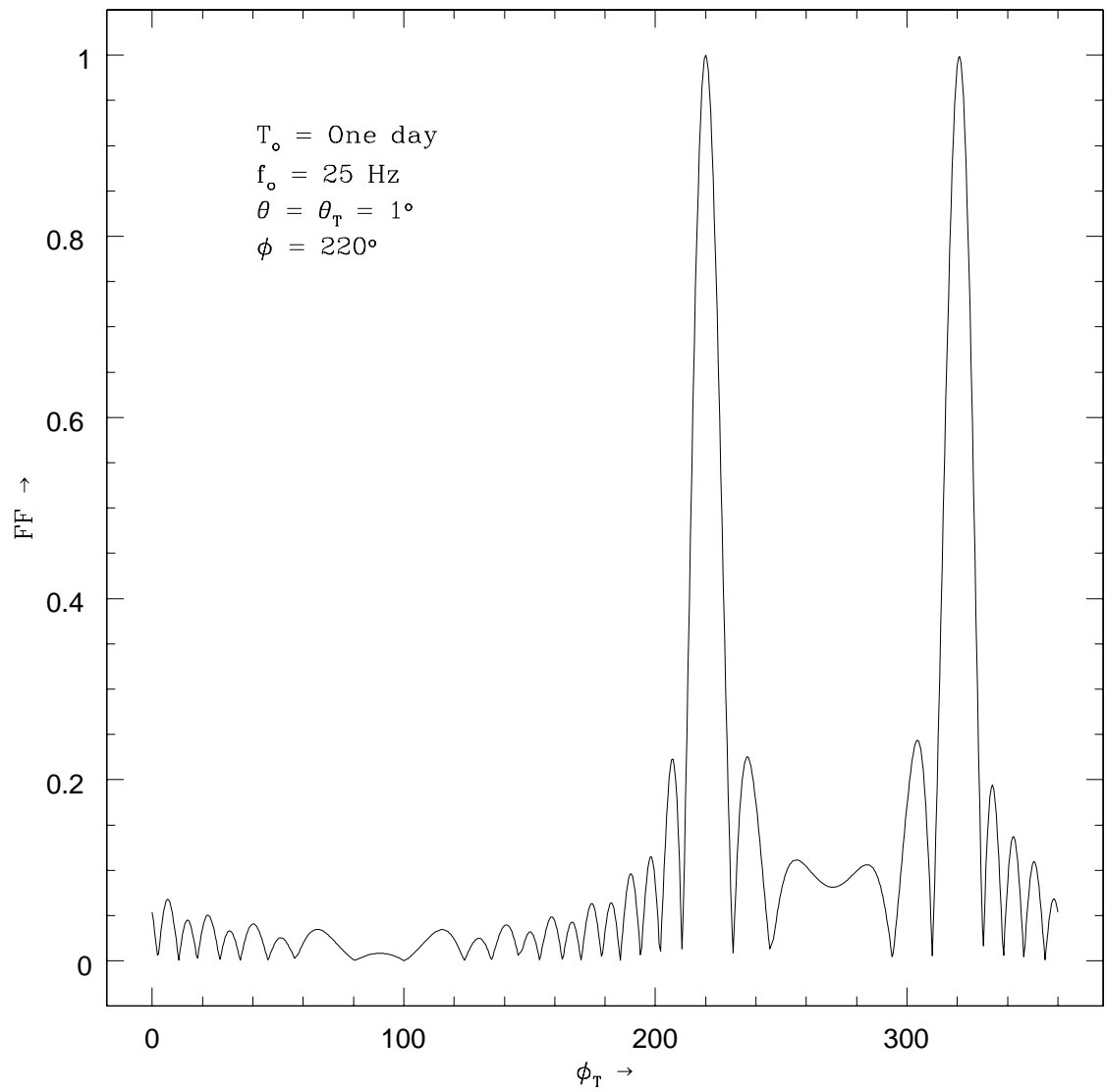


Figure 3: Variation of FF with ϕ_T .

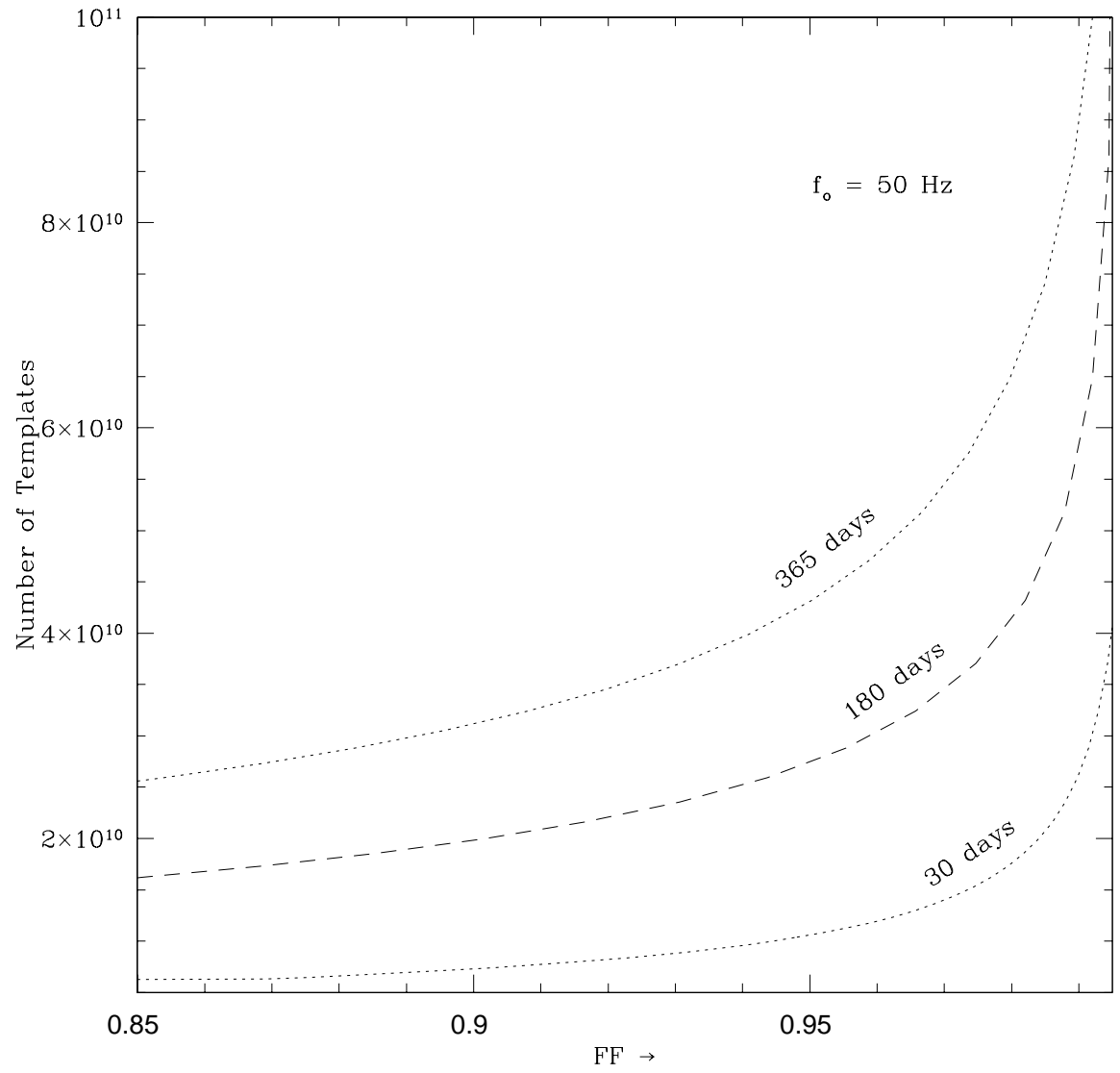


Figure 4: Variation of number of templates with FF for fixed f_o at different T_o .

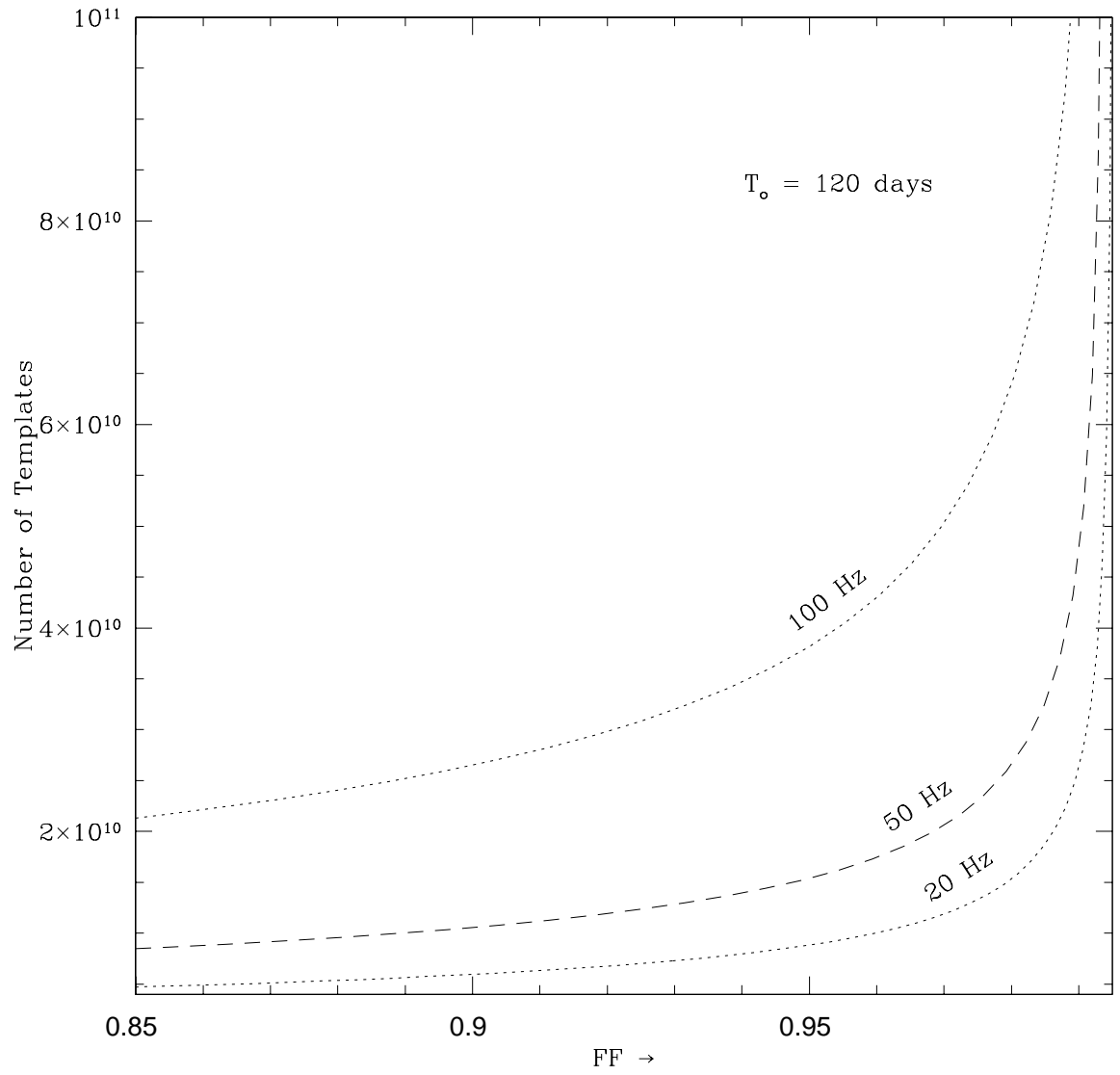


Figure 5: Variation of number of templates with FF for fixed T_o and of frequencies.