# Quantum Structure of Spacetime and Entropy of Schwarzschild Black Holes

T. Padmanabhan*
*IUCAA, Post Bag 4, Ganeshkhind, Pune 411 007, India*
(Received 5 January 1998)

The gap between a microscopic theory for quantum spacetime and the semiclassical physics of Schwarzschild black holes is bridged by treating the black hole spacetimes as highly excited states of a class of *nonlocal* field theories. All of the black hole thermodynamics are shown to arise from an asymptotic form of the dispersion relation satisfied by the elementary excitations of these field theories. These models involve, quite generically, fields which are (i) smeared over regions of the order of Planck length and (ii) possess correlation functions which have universal short distance behavior. [S0031-9007(98)07638-8]

PACS numbers: 04.70.Dy, 04.20.Gz, 04.62.+v

Simple thought experiments suggest that there exists an operational limitation in measuring lengths (and times) smaller than the Planck length $L_P \equiv (G\hbar/c^3)^{1/2}$ (see, for example, Ref. [1]). The correct theory of quantum gravity should incorporate this limitation in a natural manner just as the correct quantum mechanical theory (based on noncommuting operators) incorporates the uncertainty principle. String theories as well as the spin network formalism based on Ashtekar variables do seem to have ingredients to describe the quantum microstructure of spacetime in such a manner that $L_P$ arises as a natural lower bound to length scales [2,3]. In these approaches, the spacetime continuum arises as a nonperturbative quantum condensate of the basic variables. The description of spacetime geometry as a solution to Einstein equation is analogous to the description of, say, a gaseous system by macroscopic parameters through an equation of state. Given the microscopic theory, there should exist certain well-defined procedures for obtaining the continuum description.

Unfortunately, we do not yet have a clearly spelled out quantum theory of gravity. It is therefore important to ask whether one could obtain some general features of such a theory based on our knowledge of the macroscopic description. It is obvious that such an "inverse" process—which is similar to an attempt in understanding the quantum nature of radiation field, given the macroscopic description of the blackbody radiation—will not possess a unique solution; however, some key features, which must be respected by any underlying quantum description of spacetime, should arise from such an analysis.

Since quantum structure of spacetime is likely to reveal itself only at energies close to Planck energy, $E_P = L_P^{-1}$, its effects are most likely to influence the description of virtual processes taking place at Planck energies and above. In general, such virtual processes cannot be described without knowing the details of the theory. One fortunate exception occurs in classical Schwarzschild spacetime containing a compact infinite redshift surface. I give detailed arguments elsewhere [4] to suggest that black hole evaporation can be thought of as the deexcitation of degrees of freedom located within a few Planck lengths from the event horizon. The existence of infinite redshift allows virtual processes at super-Planckian energies to manifest as phenomena at sub-Planckian energies which are describable in terms of semiclassical continuum physics. It follows that the existence of black hole evaporation will put certain constraints on the correct description of quantum microstructure though it is still likely to leave a fair amount of liberty as regards the details [5].

I show below that it is possible to construct toy effective field theories which correctly describe the semiclassical thermodynamics of black holes. This analysis helps to delineate those features of the effective field theories which are essential to reproduce the correct behavior and reveals the tremendous amount of freedom which still exists as regards the microscopic description.

Let the quantum microstructure of spacetime be described by certain degrees of freedom $q_A$ and energy levels $\epsilon_j$. (The precise definition of these quantities, including that of energy levels $\epsilon_j$, needs to await the full quantum theory of the microscopic structure.) Classical, asymptotically flat, spacetimes with mass-energy $M \gg E_P$ will be made of a large number of degrees of freedom combining together in a coherent manner. Normally, these elementary degrees of freedom will remain in their ground state. An exception occurs in the description of Schwarzschild spacetimes possessing a compact, infinite redshift surface. In such a case, highly excited states of the basic degrees of freedom can be populated, at least around the event horizon, due to the presence of virtual excitations of arbitrarily high energies. (This paper concentrates on Schwarzschild spacetime in which the infinite redshift surface coincides with the event horizon. There are conceptual issues involved in extending these results to spacetimes in which these two surfaces do not coincide. This will be addressed in a future paper.) A black hole spacetime of mass $M$ will correspond to a *highly excited* quantum state of $q_A$'s such that the mean energy of the state $\bar{E} = M$ with $M \gg E_P$.

0031-9007/98/81(20)/4297(4)$15.00  

For a description of such a state, with the least amount of additional assumptions, I rely on pure combinotrics used in conventional statistical mechanics. In such a description the mean energy can be written in the form

$$\bar{E}(\beta) = -\frac{\partial}{\partial \beta} \ln Z(\beta);$$
$$Z(\beta) = \sum_j g_j e^{-\beta \epsilon_j} \cong \int_C \rho(z) e^{-\beta z}\, dz, \tag{1}$$

where $Z(\beta)$ is the partition function and $\rho(z)$ is the density of states of the system. The second equality for $Z(\beta)$ assumes the validity of continuum description but keeps the contour of integration $C$ in the complex plane unspecified; this is convenient since I have to deal with density of states which are unbounded along the real axis. Given the form of $\bar{E}(\beta)$ one can determine $Z(\beta)$ and, by inverting the relation between $Z$ and $\rho$, obtain $\rho(z)$. If the quantum state describes a semiclassical black hole, then we must have $M = \bar{E}(\beta) = (E_P^2 \beta / 8\pi)$; this gives $Z(\beta) = \exp(-\beta^2 E_P^2 / 16\pi)$. Choosing the contour of integration $C$ along the imaginary axis, we get $\rho(iy/E_P) = \exp[-4\pi(y/E_P)^2]$. This corresponds to the density of states for real energies given by $\rho(\bar{E}) = \exp[4\pi(\bar{E}/E_P)^2]$ for $\bar{E} \gg E_P$. The corresponding entropy is $S(\bar{E}) \equiv \ln \rho(\bar{E}) = 4\pi(\bar{E}/E_P)^2 = \mathcal{A}/4L_P^2$, where $\mathcal{A}$ is the area of the event horizon. Given a system with this form of density of states and a mechanism to populate the excited states as required in the statistical description, one can reproduce the standard thermodynamics of black holes. Hence, any description of quantum microstructure which, in the continuum limit, leads to a density of states of the form $\rho(E) \approx \exp[4\pi(\bar{E}/E_P)^2]$ will correctly reproduce the results of black hole thermodynamics. (Populating the excited states is possible in black hole spacetimes due to the existence of the infinite redshift surface.) In fact, this density of state can be trusted only for $\bar{E} \gg E_P$. In general, the density of states can have a form

$$\rho(\bar{E}) \approx \exp[4\pi(\bar{E}/E_P)^2 + \mathcal{O}(\ln(\bar{E}/E_P)\ldots], \tag{2}$$

where the leading log corrections are unimportant for $\bar{E} \gg E_P$.

I will now show how several toy field theoretical models can be constructed, which have such a density of state. While this shows that black hole entropy, by itself, cannot provide more information regarding the quantum microstructure, it will reveal one essential feature: *All of these effective field theory models constructed here are nonlocal and involve smearing of the fields over a distance of the order of Planck length.* This appears to be the key ingredient which is needed to produce the correct continuum limit.

Let us assume that the transition to continuum limit can be described in terms of certain fields $\phi(t, \mathbf{x})$ which are to be constructed in some suitable manner from the fundamental variables $q_A$. (This process is analogous to obtaining the fluid description of a gaseous system made of discrete molecules; in addition, even $t$ and $\mathbf{x}$ have to arise in terms of the microscopic variables in some, as yet unknown, fashion.) I take the Lagrangian describing the effective field $\phi(t, \mathbf{x})$ to be

$$L = \frac{1}{2} \int d^D \mathbf{x} \dot{\phi}^2 - \frac{1}{2} \int d^D \mathbf{x} d^D \mathbf{y} \phi(\mathbf{x}) F(\mathbf{x} - \mathbf{y}) \phi(\mathbf{y})$$
$$= \int \frac{d^D \mathbf{k}}{(2\pi)^D} \left(\frac{1}{2}\right) [|\dot{Q}_\mathbf{k}|^2 - \omega_\mathbf{k}^2 |Q_\mathbf{k}|^2]. \tag{3}$$

The Lagrangian is nonlocal in the space coordinates $\mathbf{x}$ which is taken to be $D$ dimensional; the corresponding Fourier space coordinates are labeled $\mathbf{k}$. The quadratic nonlocality allows us to describe the system in terms of free harmonic oscillators with a dispersion relation $\omega(\mathbf{k})$ related to $F(\mathbf{r})$ by

$$\omega^2(\mathbf{k}) = \int d^D \mathbf{r} F(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{4}$$

The energy levels of the system are built out of elementary excitations with energy $\hbar\omega(\mathbf{k})$, and the partition function for a one-particle excited state is given by

$$Z(\beta) \cong \int \frac{d^D \mathbf{k}}{(2\pi)^D} \exp[-\beta \omega(\mathbf{k})] = \int dE\, \rho(E) e^{-\beta E}, \tag{5}$$

where $\rho(E)$ is the Jacobian $\rho(E) = |d^D \mathbf{k}/dE|$. It is straightforward to see that if the dispersion relation $\omega(\mathbf{k})$ has the asymptotic form

$$\omega^2(\mathbf{k}) \to \frac{E_P^2 D}{8\pi} \ln k^2 \quad (\text{for } k^2 \gg E_P^2) \tag{6}$$

then the density of states has the asymptotic form as given in (2). Thus a wide class of toy effective field theories of a very simple nature can reproduce the black hole thermodynamics. The only nontrivial ingredient, compared to conventional field theories, is the nonlocality in the $\mathbf{x}$ space. Note that any field theoretic model with a dispersion relation $\omega^2 \propto \ln k^2$ for $k^2 \gg E_p^2$ will necessarily be nonlocal in $\mathbf{x}$ space. Further aspects related to the uniqueness of such a choice are elaborated in Ref. [4].

I will now illustrate the above phenomena using the simplest possible choice, corresponding to $D = 1$ and a dispersion relation

$$\omega^2(k) = \frac{E_P^2}{8\pi} \ln\left(1 + \frac{8\pi k^2}{E_P^2}\right). \tag{7}$$

The form of $\omega(k)$ for $k \ll E_P$ is somewhat arbitrary. I have chosen it so as to reduce the model to a local, massless, free field theory (with $\omega^2 = k^2$) in this limit. Since quantum gravitational effects should be negligible for $E_P \to \infty$, this appears reasonable. (In addition to simplicity, this form can be motivated by other considerations

[4].) The density of states corresponding to this dispersion relation is given by

$$\rho(\bar{E}) \cong \exp\left[4\pi \frac{\bar{E}^2}{E_P^2} + \mathcal{O}\left(\ln \frac{\bar{E}}{E_P}\right)\right] \equiv \exp S(\bar{E}) . \quad (8)$$

The corresponding black hole temperature is

$$T(\bar{E}) = \left(\frac{\partial S}{\partial \bar{E}}\right)^{-1} = \frac{E_P^2}{8\pi \bar{E}}\left[1 + \mathcal{O}\left(\frac{E_P^2}{\bar{E}^2}\right)\right] \cong \frac{E_P^2}{8\pi M} \quad (9)$$

for $\bar{E} = M \gg E_P$. The function $F(r)$ corresponding to the $\omega^2(k)$ in Eq. (7) is

$$F(x) = \frac{E_P^2}{8\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ikx} \ln\left(1 + \frac{8\pi k^2}{E_P^2}\right)$$
$$= -\frac{E_P^3}{8\pi}\left(\frac{L_P}{|x|}\right) \exp\left(-\frac{|x|}{\sqrt{8\pi}\, L_P}\right) \quad (10)$$

for finite, nonzero, $x$. (The logarithmic singularity in the $k$ integration can be handled by standard regularization techniques; e.g., by using the integral representation of the ln function.) When $L_P \to 0$, the function $F(x)$ is proportional to the second derivative of Dirac delta function as can be seen from the fact that, as $L_P \to 0$, $\omega^2(k) \to k^2$. In this limit, we recover the standard local field theory.

The functional form of $F(x)$ clearly illustrates the smearing of the fields over a region with correlation length $\sqrt{8\pi}\, L_P$. It can be shown, with more tedious algebra, that this is a generic feature of the dispersion relation (7) in any dimension $D$. The asymptotic structure in (6) governs the short distance behavior of $F(\mathbf{x})$ and in $D$ dimension $F(\mathbf{x}) \propto (|\mathbf{x}|/L_P)^{-D}$ as $|\mathbf{x}| \to 0$. In fact, this is the only feature of $F(\mathbf{x})$ which is needed to reproduce the density of states leading to the correct theory of black hole thermodynamics.

The physical content of the above analysis can be viewed as follows. I start with certain loosely defined dynamical variables $q_A$ describing the quantum microstructure of the spacetime. The dynamical theory describing $q_A$'s must lead, in suitable limit, to a continuum spacetime with quantum states having mean energies much larger than $E_P$. Among them are the classical spacetimes with compact, infinite redshift surfaces like that of a black hole forming out of a stellar collapse. I describe these black hole spacetimes in terms of an intermediate effective field theory in $(D + 1)$ dimension. The existence of an infinite redshift surface allows the elementary excitations of this field with arbitrarily high energies to occur in such spacetimes. (In the absence of infinite redshift surface, one cannot populate high energy states of the toy field so as to obtain a thermodynamic description.) Such a theory is nonlocal in space and is based on smearing of fields over a correlation length of the order of $L_P$.

In fact, any field theory described by the Lagrangian of the form in (3) can be expressed in terms of a free field $\psi$ such that $\phi$ is obtained by smearing $\psi$ using a window function $W$; in Fourier space, $\psi_{\mathbf{k}} = \phi_{\mathbf{k}} W_{\mathbf{k}}$ with $|W_{\mathbf{k}}|^2 = k^2 F_{\mathbf{k}}^{-1}$. The short distance behavior of such a correlation function is universal and is of the form $(|\mathbf{x}|/L_P)^{-D}$ in $D$ dimension. The black hole spacetimes are interpreted as highly excited states of such a toy field theory which itself is built out of more fundamental, and as yet unknown, variables $q_j$. The semiclassical, thermodynamic behavior of black holes can come from such an effective field theory.

The above analysis shows that there is nothing mysterious in completely different microscopic models (such as those based on strings or Ashtekar variables) leading to similar results regarding black hole entropy [6]. Any theory which has the correct density of states can do this; in fact, the models I have described are only a very specific subset of several such field theories which can be constructed. The situation is reminiscent of one's attempt to understand the quantum nature of light from blackbody radiation. The spectral form of blackbody radiation can be derived from the assumption that $E = \hbar\omega$ and is quite independent of the details of quantum dynamics of the electromagnetic field. Similarly, the black hole thermodynamics can be explained if one treats spacetimes with event horizons as highly excited states of a nonlocal field theory whose elementary excitations obey a dispersion relation with the asymptotic form given by (6). Within the spirit of the current analysis, one need not even identify the $(D + 1)$ dimensional space as a superset of conventional spacetime. The $\mathbf{x}$ and $t$ could represent variables in some abstract space, and the spacetime structure could emerge in a more complicated manner in terms of the fields themselves. Because of this reason, I have not bothered about the Lorentz invariance or other internal symmetries of the toy model.

While it may not be possible to obtain a unique quantum description of spacetime from our knowledge of semiclassical black hole physics, it does give three clear pointers. First is the indirect, but essential, role played by the infinite redshift surface. It is the existence of such a surface which distinguishes the *star* of mass $M$ from a *black hole* of mass $M$. A stellar spactime will not be able to populate high energy states of the toy field as required by the statistical description; in a black hole spacetime, virtual modes of arbitarily high energies near the event horizon will allow this to occur. (It may be possible to model such a process by studying the interaction of this toy field with a more conventional field near the event horizon.) The second one is the asymptotic form of the dispersion relation for the elementary excitations which leads to the correct density of states. The third is the fact that such a dispersion relation almost invariably leads to smearing of local fields over regions of the order of Planck length.

I thank Apoorva Patel and Sayan Kar for several stimulating discussions.

*Electronic address: paddy@iucaa.ernet.in

[1] T. Padmanabhan, Classical Quantum Gravity **4**, L107 (1987).

[2] For a review, see, e.g., A. Sen, Int. J. Mod. Phys. A **9**, 3707 (1994); J. H. Polchinski, Rev. Mod. Phys. **68**, 1245 (1996).

[3] A. Ashtekar, C. Rovelli, and L. Smolin, Phys. Rev. Lett. **69**, 237 (1992); C. Rovelli and L. Smolin, Nucl. Phys. **B442**, 593 (1995).

[4] T. Padmanabhan, e-print hep-th/9801138 (IUCAA-4/98).

[5] Some of the early attempts to understand black hole entropy include J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); **9**, 3292 (1973); S. Hawking, Commun. Math. Phys. **43**, 199 (1975); U. H. Gerlach, Phys. Rev. D **14**, 1479 (1976); J. W. York, Phys. Rev. D **28**, 2929 (1983); W. H. Zurek and K. S. Thorne, Phys. Rev. Lett. **54**, 2171 (1985); G. 't Hooft, Nucl. Phys. **B256**, 727 (1985); L. Bombelli *et al.,* Phys. Rev. D **34**, 373 (1986); V. Frolov and I. Novikov, Phys. Rev. D **48**, 4545 (1993); A. Buonanno *et al.,* Nucl. Phys. **B451**, 677 (1995).

[6] A. Sen, Nucl. Phys. **B440**, 421 (1995); A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996); for a detailed review of attempts based on string theory, see A. W. Peet, e-print hep-th/9712253 (1997).