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ZEL'DOVICH APPROXIMATION AND THE PROBABILITY DISTRIBUTION FOR THE SMOOTHED DENSITY FIELD IN THE NONLINEAR REGIME

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ABSTRACT

The study of large-scale structures in the universe is often based on the observed density distribution of matter smoothed by a suitable filter function. The probability distribution for this smoothed density field in the nonlinear regime is studied using the Zel'dovich approximation. When the shear term of the velocity field is not too large, one can obtain a reasonably good analytic approximation to this probability distribution. The properties of this distribution are discussed and compared with other attempts along similar lines.

Subject headings: cosmology: theory — large-scale structure of universe

1. INTRODUCTION

It is generally believed that large-scale structures in the universe, like galaxies, clusters, etc., formed via gravitational instability (see, e.g., Peebles 1980). In this picture we start with a small density inhomogeneity $\delta_i \equiv \delta(\boldsymbol{x}, t_i) = [\rho_i(\boldsymbol{x}, t_i) - \rho_b(t_i)]/\rho_b(t_i)$ with $\delta_i \ll 1$ at some time t_i and evolve it using standard dynamical equations. In principle, one can predict the exact density distribution in the universe today if the initial conditions are known precisely. In practice, of course, this is neither possible nor necessary. Our information about the present-day universe is largely statistical (like, e.g., the mean number of galaxies with certain properties), and we only require statistical predictions from the theory. It is therefore usual to consider the initial density contrast to be one particular realization from an ensemble governed by some probability distribution. One of the most popular assumptions is to take this probability distribution to be a Gaussian.

It is easy to show that linear evolution of the density perturbations (valid when $\delta \ll 1$) preserves the original probability distribution. But when $\delta \simeq 1$, different Fourier components couple strongly, and the correlations develop. The probability distribution will no longer be Gaussian.

Further, in most situations of interest, one deals with a smoothed version of the final density field. For example, in calculating the rms fluctuations in the mass contrast or counts of galaxies in cells, one smoothens the density field over a region of size L. It is therefore essential that we compute the probability density function (PDF) of the final, smoothed density field in order to compare theoretical predictions with observations.

In this paper we shall attempt to derive this probability distribution $P[\delta, t; L]$ even when the mean density contrast is of order unity. To do this, we shall use the Zel'dovich approximation (Zel'dovich 1970) which allows us to handle density contrasts which are mildly nonlinear. What is more, we shall invoke a further approximation which is simple enough to be handled analytically but is nontrivial enough to give some insight into the dynamics.

2. PROBABILITY DISTRIBUTION IN THE ZEL'DOVICH APPROXIMATION

The Zel'dovich approximation provides a relation between the Eulerian and Lagrangian coordinates of a particle in the form

$$\boldsymbol{r}(t) = a(t)[\boldsymbol{q} + \boldsymbol{f}(\boldsymbol{q}, t)] \equiv a(t)\boldsymbol{x} . \quad (1)$$

It is possible to show that the function $\boldsymbol{f}(\boldsymbol{q}, t)$ can be expressed in the form $b(t)\boldsymbol{p}(\boldsymbol{q})$ to a good degree of accuracy. Here $b(t)$ is the growing solution to the linear perturbation equation, and $\boldsymbol{p}(\boldsymbol{q})$ is expressible as a gradient ($\boldsymbol{p} = \nabla\psi$) where ψ is proportional to the initial gravitational potential.

In this approximation, one can write the matter density $\rho(\boldsymbol{r}, t)$ at any time t as

$$\rho(\boldsymbol{r}, t) = \rho_0(\boldsymbol{q}, t_i)\left(\frac{a_i}{a}\right)^3 \frac{1}{(\det J)} \simeq \rho_0(t_i)\left(\frac{a_i}{a}\right)^3 \frac{1}{(\det J)} , \quad (2)$$

where the determinant of the Jacobian is given by

$$\det J = \det\left(\delta_{ij}^{\mathrm{K}} + b\,\frac{\partial^2\psi}{\partial q_i\,\partial q_j}\right) , \quad (3)$$

where δ_{ij}^{K} is the Kronecker delta function. In arriving at equation (2), we have also assumed that $\rho(\boldsymbol{q}, t_i) \simeq \rho_0(t_i)$ at sufficiently early t_i. By diagonalizing the matrix J, we can write this density as

$$\rho(\boldsymbol{r}, t) = \frac{\rho_0(t_i)(a_i^3/a^3)}{[1 - b\lambda_1(\boldsymbol{q})][1 - b\lambda_2(\boldsymbol{q})][1 - b\lambda_3(\boldsymbol{q})]} , \quad (4)$$

where $[-\lambda_i(\boldsymbol{q})]$ are the eigenvalues of a matrix

$$M_{ij} \equiv \frac{\partial p_i}{\partial q^j} = \frac{\partial^2 \psi}{\partial q_i \partial q_j}\,; \quad \boldsymbol{p} \equiv \nabla\psi\ . \tag{5}$$

We shall assume that $\lambda_1 > \lambda_2 > \lambda_3$.

Given the form of $\psi(\boldsymbol{q})$, the above equation determines the density at any event $(t, \boldsymbol{r})$, provided $(1 - b\lambda_1) > 0$. (When this condition is violated, caustics form in the density field, and the entire Zel'dovich approximation breaks down.) It follows that the statistical properties of ρ is determined completely by the statistical properties of λ_i which, in turn, are decided by the statistical properties of ψ. Given the statistical properties of ψ, one can in principle determine the probability $P(\lambda_i)d^3\lambda_i$ for the occurrence of eigenvalues in the range $(\lambda_i, \lambda_i + d\lambda_i)$. Given $P(\lambda_i)$ we can determine all the statistical properties of ρ.

In particular, the probability that the ratio (ρ/ρ_b) between density ρ and the background density (ρ_b) has a value between η and $\eta + d\eta$ is

$$\mathscr{P}(\eta)d\eta \propto d\eta \int P(\lambda_1, \lambda_2, \lambda_3)\delta_{\rm D}[z(\lambda_i)]d^3\lambda_i\ , \tag{6}$$

where $\delta_{\rm D}(z)$ is the Dirac delta function with the argument:

$$z = \prod_{i=1}^{3} (1 - b\lambda_i)^{-1} - \eta\ . \tag{7}$$

Of special importance is the case in which the initial perturbations form a realization of a Gaussian random field with the variance σ^2. In that case the probability distribution $P(\lambda_i)$ depends on the initial distribution through σ^2: $P(\lambda_i) = P(\lambda_i;\, \sigma^2)$. Using the above formula we can determine $\mathscr{P} = \mathscr{P}(\eta;\, \sigma^2)$. All the moments of the density contrast $(\eta - 1)$ can be expressed in terms of σ^2:

$$F_n(\sigma^2) \equiv \langle(\eta - 1)^n\rangle = \int_{-\infty}^{\infty} d\eta\, \mathscr{P}(\eta;\, \sigma^2)(\eta - 1)^n\ . \tag{8}$$

Given the form of $P(\lambda_i)$ all the moments can be computed.

Though the above analysis might seem straightforward, there arises an interesting subtlety in the computation outlined above. We shall now discuss this issue.

Since most of the cosmological observations deal with a smoothed density field, what is actually relevant in our study is the probability distribution for the density contrast in the *present-day* universe *filtered* over some length scale L

$$\delta(\boldsymbol{x};\, L) = \int_{-\infty}^{\infty} d^3y\,\theta\left(\frac{|\boldsymbol{y} - \boldsymbol{x}|}{L}\right)\left[\frac{\rho(\boldsymbol{y}) - \rho_b}{\rho_b}\right], \tag{9}$$

where $\theta(z) = 1$ for $z < 1$ and zero otherwise. The measure of fluctuation generally used is in $\sigma^2(L) = \langle\delta^2(\boldsymbol{x};\, L)\rangle$ which is defined as the average of δ^2

$$\sigma^2(L) = \int_V \frac{d^3\boldsymbol{x}}{V}\,\delta^2(\boldsymbol{x};\, L) \tag{10}$$

over a large volume V. Hence, to be of any practical use, we should express the *filtered final* density in terms of the *filtered initial* density.

Note that once we fix the filtering region for the final density field, the dynamical evolution determines a corresponding filtering region for the initial density field. In fact, such initial smoothening may be required for another reason as well: The probability distribution $P(\lambda_i)$ and $\mathscr{P}(\eta)$ are well defined only if the integral

$$\sigma^2 = \int \frac{d^3k}{(2\pi)^3}\,|\delta_k|^2 \tag{11}$$

calculated from the original power spectrum is finite. For some power spectra which are of interest in cosmology, like the CDM spectra, this integral for σ^2 diverges due to small-scale fluctuations, and a smoothening of the density distribution by a suitable window function is essential. The smoothening of the final density field automatically takes care of this requirement.

The fact that one is smoothening the *final* density field, however, makes the problem of calculating the PDF much more difficult, since the shape of the filtering region gets distorted in going from the $\boldsymbol{r}$-coordinates to $\boldsymbol{q}$-coordinates. Nevertheless, this is a "secondary" effect and hence can be handled by some approximation.

The simplest—but yet, nontrivial—approximation is to assume that we can replace the matrix $M_{ij}(\boldsymbol{q})$ by some smoothed-out matrix $\bar{M}_{ij}$ in describing the distortion of the shape. This assumption, it should be stressed, is always implicit when the Zel'dovich approximation is used with a *filtered field.* Consider the mean density inside a sphere of radius L centered at the comoving location $\boldsymbol{x}$ today. If we evolve back this spherical region to the past, it will become an ellipsoid. (Since the smoothening scale itself is L, we can consider $\bar{M}_{ij}$ to be a constant in determining the distortion in the shape of the smoothening region; this fact makes the transformation $(\boldsymbol{x} \leftrightarrow \boldsymbol{q})$ linear.) The semimajor axes of this ellipsoid will be

$$(\xi_1, \xi_2, \xi_3) = \left[\frac{L}{(1 - b\bar{\lambda}_1)}, \frac{L}{(1 - b\bar{\lambda}_2)}, \frac{L}{(1 - b\bar{\lambda}_3)}\right]. \tag{12}$$

Thus, the filtering of the density field over a sphere of radius L (today) is equivalent to filtering the original field over an ellipsoid. If the original power spectrum is $S(k) = |\delta_k|^2$, the variance σ^2 calculated within an ellipsoidal region will be

$$\begin{aligned}\sigma^2_{\rm elli} &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, W_{\rm sph}(k^i\xi_i)S(k)\ ,\\ &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, S(k)\,\frac{9}{(k^i\xi_i)^6}\,[\sin(k^i\xi_i) - (k^i\xi_i)\cos(k^i\xi_i)]^2\ ,\end{aligned} \tag{13}$$

where $W_{\rm sph}(\boldsymbol{k})$ is the Fourier transform of the spherical window function with radius L. Note that $\sigma^2_{\rm elli}$ now depends on $\bar{\lambda}_i$ through the combination $\xi_i = L(1 - b\bar{\lambda}_i)^{-1}$. The probability distribution for the (average values) $\bar{\lambda}_i$ will be a function

$$\bar{P}(\bar{\lambda}_i) \propto P[\bar{\lambda}_i;\, \sigma^2_{\rm elli}(\bar{\lambda}_i)] \equiv NF(\bar{\lambda}_i)\ . \tag{14}$$

The following point should be stressed. The dependence of F on $\bar{\lambda}_i$ is determined by the initial potential field, which is taken to be Gaussian random variable. Further dependence on $\bar{\lambda}_i$ arises due to the fact that the filtering region gets distorted in a λ-dependent way. We need to take both the effects into account to get the correct result.

The proportionality constant N can be fixed by normalizing F. Since the density contrast at any point in the Zel'dovich approximation,

$$\delta_{\rm zel} = \prod_i \frac{1}{(1 - b\bar{\lambda}_i)} - 1 \equiv \delta_{\rm zel}(\bar{\lambda}_i)\ , \tag{15}$$

is a well-defined function of $\bar{\lambda}_i$ the computation of the moments $\langle \delta^n_{\rm zel} \rangle$ is straightforward. They can be expressed as

$$\langle \delta^n_{\rm zel} \rangle = N \int d^3\bar{\lambda}_i \, F(\bar{\lambda}_i) \delta^n_{\rm zel}(\bar{\lambda}_i) \prod_i (1 - b\bar{\lambda}_i) \, . \quad (16)$$

The last term is the Jacobian which arises in transforming from $\boldsymbol{q}$ to $\boldsymbol{x}$. This is needed because the original probability distribution was in $\boldsymbol{q}$-space while the final one is in the $\boldsymbol{x}$-space. It is easily verified that

$$\langle \rho(\boldsymbol{x}, t)/\rho_b \rangle = \left\langle \prod_i (1 - b\lambda_i)^{-1} \right\rangle = 1 \quad (17)$$

independent of the form[1] of F. We shall now compute $F(\lambda_i)$, $\mathscr{P}(\eta)$, etc., explicitly.

3. APPROXIMATE EVALUATION OF THE PROBABILITY

The probability distribution of the eigenvalues $\bar{\lambda}_i$ was derived by Doroshkevich (1970) and is given by

$$P(\bar{\lambda}_i; \sigma^2) = C\left(\frac{1}{\sigma^6}\right)(\bar{\lambda}_1 - \bar{\lambda}_2)(\bar{\lambda}_2 - \bar{\lambda}_3)(\bar{\lambda}_1 - \bar{\lambda}_3) \exp(-Q) \, , \quad (18)$$

$$Q = \frac{3}{5}\left(\frac{\sum_i \bar{\lambda}_i}{\sigma}\right)^2 - \frac{3}{2}\left(\frac{\bar{\lambda}_1\bar{\lambda}_2 + \bar{\lambda}_2\bar{\lambda}_3 + \bar{\lambda}_3\bar{\lambda}_1}{\sigma^2}\right) . \quad (19)$$

Here $\sigma^2 = (\sigma^2_{\rm elli}/5)$ and C is a normalization constant that can be absorbed into N. It is also assumed that $\bar{\lambda}_1 > \bar{\lambda}_2 > \bar{\lambda}_3$. Given the original power spectrum one can compute from equation (13) the filtered dispersion $\sigma^2 = \sigma^2(L, b\bar{\lambda})$ (note that σ^2 depends on b and $\bar{\lambda}_i$ through the combination $b\bar{\lambda}_i$). Combining equations (14) and (18), we can determine $F(\bar{\lambda}_i)$. The constant N is fixed by integrating over all λ and setting the result to unity.

While the above calculation can be performed numerically to determine the moments $\langle \delta^n \rangle$, such a procedure gives no insight into the physics of the problem. We shall, therefore, follow a more approximate procedure leading to a closed analytic expression for $\mathscr{P}(\eta)$.

To motivate the nature of the approximation, let us consider the matrix M_{ij} more closely. We can decompose M_{ij} into the irreducible parts in the form

$$M_{ij} = \frac{\partial p_i}{\partial q^j} = \frac{1}{3}(\nabla_q \cdot \boldsymbol{p})\delta^{\rm K}_{ij} + Q_{ij} = \left[-\frac{1}{3}\left(\frac{\dot{b}}{b}\right)_i \delta_i\right]\delta^{\rm K}_{ij} + Q_{ij} \, , \quad (20)$$

where $\delta^{\rm K}_{ij}$ is the Kronecker delta function and δ_i is the initial density contrast. The diagonal term represents the effect due to the density contrast and the matrix Q_{ij} represents the traceless shear tensor of the velocity field

$$Q_{ij} = \frac{1}{2}\left(\frac{\partial p_i}{\partial q_j} + \frac{\partial p_i}{\partial q_i}\right) - \frac{1}{3}(\nabla \cdot \boldsymbol{p})\delta^{\rm K}_{ij} \, . \quad (21)$$

From the eigenvalues we may separate out the trace of the matrix $T = (\lambda_1 + \lambda_2 + \lambda_3)$. If the original density field follows Gaussian statistics, then the quantities (δ, Q_{ij}) will also be distributed in the same way. What is more, the density δ will be uncorrelated with the shear field Q_{ij} because of the isotropy of the background. That is, $\langle Q_{ij}\delta \rangle = 0$. This implies that the probability $P(\bar{\lambda}_i)$ *must* be expressible in the form $P(\lambda_1, \lambda_2, \lambda_3) = P_1(T)P_2(\alpha, \beta)$ where α, β are the two eigenvalues of Q_{ij}. This result is easily verified by writing

$$\lambda_i = \tfrac{1}{3}T - x_i ; \quad \sum x_i = 0 ; \quad x_2 > x_1 \, , \quad x_3 > x_1 \quad (22)$$

and introducing the coordinates $u = x_1/x_2$; $v = x_1 + x_2$. Straightforward algebra will allow us to express $P(\bar{\lambda}_i)$ in the form

$$P = NP_1(T)P_2(u, v) \quad (23)$$

with

$$P_1(T) = \exp\left[-\frac{1}{10}\left(\frac{T}{\sigma}\right)^2\right] ,$$
$$P_2(u, v) = uv^2(u^2 - v^2) \exp -\frac{3}{8\sigma^2}(u^2 + 3v^2) \, . \quad (24)$$

The conditions $(x_2 > x_1, \; -\infty < x_1 < \infty)$ become $(u < 0, -\infty < v < \infty)$. It is now clear that the most relevant quantity characterizing the density distribution is the trace $T = (\lambda_1 + \lambda_2 + \lambda_3)$. One can easily determine the marginal probability distribution for T by integrating out (u, v). Since the integral will be a constant we obtain

$$P(T) = N \exp\left[-\frac{1}{10}\left(\frac{T}{\sigma}\right)^2\right]$$
$$= \left(\frac{1}{2\pi\sigma^2_{\rm elli}}\right)^{1/2} \exp\left[-\frac{1}{2}\frac{(\sum \lambda_i)^2}{\sigma^2_{\rm elli}}\right] . \quad (25)$$

This result shows that the quantity $T = \sum \lambda_i$ is distributed like the density contrast of the linear theory *even in* the nonlinear regime. This suggests the approximation in which each of the variables $(1 - b\lambda_i)$ is replaced by a quantity similar to their geometric mean. That is, we take

$$\prod_i \frac{1}{(1 - b\lambda_i)} \simeq \frac{1}{(1 - bT/3)^3} \, . \quad (26)$$

Given this approximation it is fairly straightforward to calculate the PDF. However, before we do so, we shall discuss the validity and possible limitations of this approximation. It may be noted that the above approximation is effectively the same as the spherical model (e.g., Peebles 1980) for the nonlinear evolution of an overdense region. Such an approximation is extensively used in the literature to study the nonlinear evolution in the context of CDM-like models. (The main difference between the conventional spherical model and the approximation we have used above is that we have invoked a Zel'dovich-type analysis of the spherical model. Such an analysis is discussed in detail by the authors elsewhere [Padmanabhan & Subramanian 1992], where it has been shown that the Zel'dovich version of the spherical model tracks the exact spherical model quite well for density contrasts up to about 3 or so.)

At first sight it may seem that a spherical approximation may not do justice to the asymmetries in the density field. This

[1] With this normalization $\langle 1 + \delta \rangle = 1$ but $\langle 1 \rangle \neq 1$; we can also arrange a normalization with $\langle 1 \rangle = 1$ but $\langle 1 + \delta \rangle \neq 1$. These two choices will differ by a constant for any smoothing scale.

fear is unfounded, however, because of the following reason. Consider the density field sometime in the past when the density contrasts are small compared to unity. At this epoch the peaks of the density field coincide with the peaks of T. Now suppose we take a fiducial sphere around one such peak, defined by a set of particles and follow its evolution into the future. This spherical region will distort with time; but as long as caustics do not form over the length scale of the sphere, the particles originally inside the sphere will stay contained by the distorted sphere. Also for moderate density contrasts, say those which obtain before turn around, the distortion of the sphere will also be moderate. Due to these facts it is justifiable to smoothen the density field inside the sphere and use the spherical model. And indeed this is the standard practice in the literature. Therefore our approximation is both useful and valid as long as the density field is smoothed and the average density contrasts do not exceed ~ 3.

We shall proceed with the calculation of the PDF for the final smoothed density field based on the above approximation. Using the same order of approximation, we can set $\xi_i \simeq L(1 - bT/3)^{-1}$ and

$$\sigma^2_{\rm elli} = \sigma^2_{\rm elli}(R)\Big|_{R=L(1-bT/3)^{-1}} \equiv \sigma^2_L(T)$$
$$\prod_i (1 - b\lambda_i)\delta^n_{\rm zel} = \left\{\frac{1}{[1-(bT/3)]^3} - 1\right\}^n \left(1 - \frac{bT}{3}\right)^3 . \qquad (27)$$

We have denoted by $\sigma^2_L(T)$ the function obtained by substituting in the original variance $\sigma^2(R)$ the value $R = L(1 - bT/3)^{-1}$. Then

$$\langle \delta^n_{\rm zel} \rangle = N \int_{-\infty}^{3/b} dT \left(1 - \frac{bT}{3}\right)^3 \left[\left(1 - \frac{bT}{3}\right)^{-3} - 1\right]^n \times \frac{1}{\sigma_L(T)} \exp\left[-\frac{1}{2}\frac{T^2}{\sigma^2_L(T)}\right] \qquad (28)$$

with N fixed by the condition

$$N \int_{-\infty}^{3/b} \frac{dT}{\sigma_L(T)} \left(1 - \frac{bT}{3}\right)^3 \exp\left[-\frac{1}{2}\frac{T^2}{\sigma^2_L(T)}\right] = 1 . \qquad (29)$$

The upper limit to the integration is taken to be $(3/b)$ since our approximation is valid only prior to the formation of caustics. The behavior of this function depends on the form of $\sigma_{\rm elli}(x)$. In the standard CDM model, $\sigma(x) \propto x^{-2}$ for large x and $\sigma \simeq q \simeq$ constant for small x. As $bT \to 3$, $\sigma_L(T) \equiv \sigma[L(1 - bT/3)^{-1}]$ will vanish as $(1 - bT/3)^2$. The factor in front of the exponent will behave as $(1 - bT/3)^{-3n+1}$; however, the exponential will behave as $\exp[-L^2T^2/(1 - bT/3)^2]$. The vanishing of the exponential will dominate and render the expression finite. Similar cutoff occurs at the lower limit, $T \to -\infty$, as well. In this limit, σ_L is effectively a constant but the exponent behaves as $\exp(-T^2/2q^2)$ thereby cutting off the integral. This implies all the moments of the distribution are finite.

The expression above allows us to extract the probability distribution for the nonlinear density contrast $\mathscr{P}[\delta]$ by inspection. Transforming from the variable T to δ by the relation

$$\delta = \left(1 - \frac{bT}{3}\right)^{-3} - 1 ; \quad T = \frac{3}{b}\left[1 - \frac{1}{(1+\delta)^{1/3}}\right]$$

in the range $-1 \le \delta \le \infty$, we find that

$$\langle \delta^n \rangle = N \int_{-1}^{\infty} \left(\frac{dT}{d\delta}\right) \frac{1}{(1+\delta)} \delta^n \frac{1}{\sigma_L(\delta)} \times \exp\left\{-\frac{9}{2b^2\sigma^2_L(\delta)}\left[1 - \frac{1}{(1+\delta)^{1/3}}\right]^2\right\} d\delta ,$$
$$= N \int_{-1}^{\infty} d\delta \frac{\delta^n}{(1+\delta)^{7/3}} \frac{1}{b\sigma_L(\delta)} \times \exp\left\{-\frac{9}{2b^2\sigma^2_L(\delta)}\left[1 - \frac{1}{(1+\delta)^{1/3}}\right]^2\right\} , \qquad (30)$$

where $\sigma^2_L(\delta)$ stands for the standard $\sigma^2(R)$ linear theory evaluated at $R = L(1 - \delta)^{1/3}$. Since the variance in linear theory grows as $b(t)$, the combination $b\sigma_L$ denotes the variance today (at $z = 0$) calculated by linear theory. We will call this quantity $\sigma_0(\delta)$. It is now clear that the exact probability distribution for δ is given by

$$\mathscr{P}[\delta] = \frac{N}{\sqrt{2\pi}\,\sigma_0(\delta)} \frac{1}{(1+\delta)^{7/3}} \times \exp\left\{-\frac{9}{2\sigma^2_0(\delta)}\left[1 - \frac{1}{(1+\delta)^{1/3}}\right]^2\right\} \qquad (31)$$

for $\delta > -1$ and zero otherwise. We have redefined N by factoring out $[1/(2\pi)^{1/2}]$ for later convenience. To be precise, $\mathscr{P}$ also depends on the filtering scale L, since σ_0 depends on it:

$$\sigma_0(\delta) = \sigma_{\rm lin}[R = L(1 + \delta)^{1/3}] . \qquad (32)$$

We will examine the nature of $\mathscr{P}(\delta)$ in detail in the next section. Before doing this, we discuss another approach to defining the probability distribution function which came to our notice after the completion of this work. (Kofman 1991a, b).

In the present approach we have related the *filtered* final density field to the *filtered* initial density field. This resulted in σ^2 being not only a function of L but also of λ_i. Kofman (1991a, b), on the other hand, asks a different question for calculating the probability distribution function of δ in the nonlinear regime. He starts with the initial filtered δ and evolves it according to Zel'dovich approximation, and defines the probability $P_{\rm K}(\eta)$ for the density contrast (ρ/ρ_b) to lie in the range $(\eta, \eta + d\eta)$ to be proportional to the volume in $\boldsymbol{r}$ space where

$$\eta < \frac{\rho}{\rho_b} = \prod_i (1 - b\lambda_i)^{-1} < \eta + d\eta . \qquad (33)$$

Note that in Kofman's case the final density field is not smoothed. This probability $P_{\rm K}(\eta)$ can be written down by using the relation:

$$\int_{\eta}^{\infty} P_{\rm K}(\eta) d\eta = \int d^3\lambda_i\, G(\bar{\lambda}_i) \prod_i (1 - b\bar{\lambda}_i)\theta[\eta^{-1} - \prod (1 - b\bar{\lambda}_i)] . \qquad (34)$$

Here $\prod_i (1 - b\bar{\lambda}_i)$ is once again the Jacobian which arises in going from $\boldsymbol{x}$ space to $\boldsymbol{q}$ space, and the theta function picks out the regions where $(\rho/\rho_b) > \eta$; $G(\bar{\lambda}_i)$ is the probability distribution of eigenvalues $\bar{\lambda}_i$ given earlier, with $\sigma^2(L)$, now being a function of initial filtering scale L but *not* $\bar{\lambda}_i$. Differentiating equation (34) with respect to η we have

$$P_K(\eta)d\eta = \frac{d\eta}{\eta^2}\int d^3\bar{\lambda}_i\, G(\bar{\lambda}_i)\prod_i (1 - b\lambda_i)\delta_D\left[\eta^{-1} - \prod_i (1 - b\bar{\lambda}_i)\right]. \tag{35}$$

Changing to variables $p = 1 - b\lambda_1$, $q = 1 - b\lambda_2$, $r = 1 - b\lambda_3$, and integrating over p we get

$$P_K(\eta)d\eta = \frac{d\eta}{\eta^3}\iint_{\mathscr{V}} \frac{dq\,dr}{(\sigma b)^6|qr|}(r - q) \times \left(q - \frac{1}{\eta qr}\right)\left(r - \frac{1}{\eta qr}\right)\exp\left(-\frac{\bar{Q}}{\sigma^2 b^2}\right), \tag{36}$$

where

$$\bar{Q} = \frac{3}{5}\left[3 - \left(\frac{1}{\eta qr} + q + r\right)\right]^2 - \frac{3}{2}\left[3 - 2\left(\frac{1}{\eta qr} + q + r\right) + qr + \frac{1}{\eta qr}(q + r)\right]. \tag{37}$$

Here the range of integration $\mathscr{V}$ is such that $p = 1/\eta qr < q < r$. This integral can be evaluated numerically for an arbitrary fixed η, to derive $P_K(\eta)$. However, the asymptotic form for $P_K(\eta)$ for large η can be seen from equation (36) without doing the integral explicitly. In the limit $\eta \to \infty$, the integral in equation (36) becomes independent of η and $P_K(\eta)d\eta \propto d\eta/\eta^3$. This asymptotic form has also been pointed out by Kofman (1991a, b). The asymptotic form of $P_K(\eta)$ can be derived more simply as follows. Note that the density near a pancake caustic, scales with the distance l from the caustic as $\rho \propto l^{-1/2}$. So the probability to have a density contrast $(\rho/\rho_b) > \eta$, will be

$$\int_\eta^\infty P_K(\eta')d\eta' \propto Al \propto A\rho^{-2} \propto A\eta^{-2}, \tag{38}$$

where A is the area of the surface, which is at a distance l from the pancake. Differentiating equation (38) we then get $P_K(\eta) \propto (1/\eta^3)$, for large η as given above.

Quite clearly, the probability $P_K(\eta)$ and $\mathscr{P}[\eta]$ are answers to two *different* questions. One has to consider a specific physical context to decide which result is applicable. If the observations are related to a *filtered* final density, then $\mathscr{P}[\eta]$ discussed in this paper is more relevant. Also notice that all moments of the distribution $\langle\eta^k\rangle$ with $k \geq 2$ diverge for Kofman's distribution due to the η^{-3} tail. We saw that these moments are finite for $\mathscr{P}[\eta]$. On the other hand $P_K(\eta)$ explicitly takes into account the fact that universe is dominated by caustics at late stages while $\mathscr{P}[\eta]$ is inapplicable unless the filtering scale is large enough to exclude caustics.

We would like to make the following cautionary remark at this stage. We came to know recently (L. Kofman 1992, private communication) that Kofman and his collaborators have compared $P_K(\eta)$ with N-body simulations, after the final density field has been smoothed. It appears that the P_K obtained after an *initial* smoothening is in reasonable agreement with the PDF obtained from the simulations after *final* smoothening. We find this result somewhat surprising. Unfortunately the details of the numerical simulations are not yet available, and hence we cannot provide a detailed comparison between our approach and that of Kofman (1991a, b). This issue is under investigation.

4. PROPERTIES OF THE PROBABILITY DISTRIBUTION

To understand the behavior of $\mathscr{P}[\delta]$ we can consider two limiting cases of large and small L. When the field is smoothened over a large scale (i.e., in the limit of $L \to \infty$), σ_0 tends to zero as L^{-2} at finite δ. We see from the exponential that most of the contribution comes from the region with $(1 + \delta)^{-1/3} \simeq 1$, that is, from near $\delta \simeq 0$. In this case we can approximate $\mathscr{P}[\delta]$ by the Gaussian

$$\mathscr{P}[\delta] \cong \frac{1}{\sqrt{2\pi}\,\sigma_0(L)}\exp\left[-\frac{1}{2}\frac{\delta^2}{\sigma_0^2(L)}\right] \tag{39}$$

which is precisely the result from the linear theory. [In this limit $N \cong 1$]. It shows that the original statistical distribution is recovered if L is sufficiently large so that the small-scale irregularities are filtered out. It should, however, be noted that this equivalence exists only for a small range of δ around zero. Outside this range, the probability distribution is more sharply peaked compared to the Gaussian. This result can be seen as follows: For large values of the argument we can approximate δ_0 as $\delta_0(x) \simeq (R_0/x)^2$ where $R_0 \simeq 24\ h^{-1}$ Mpc is the normalization scale fixed from *COBE* data (e.g., Padmanabhan & Narasimha 1992). Therefore

$$\sigma_0[L(1 + \delta)^{1/3}] = \left(\frac{R_0}{L}\right)^2(1 + \delta)^{-2/3} \cong \left(\frac{R_0}{L}\right)^2\left(1 - \frac{2}{3}\delta\right) \tag{40}$$

for small δ. Hence

$$\mathscr{P}[\delta] \cong \frac{N}{\sqrt{2\pi}}\left[\frac{1}{\sigma_0(L)}\right]\left(1 - \frac{5}{3}\delta\right)\exp\left[-\frac{\delta^2}{2\sigma_0^2(L)}\left(1 + \frac{4}{3}\delta\right)\right] \cong P_{\rm lin}[\delta; \sigma_0(L)]\left(1 - \frac{5}{3}\delta\right)\exp\left[-\frac{2}{3}\frac{\delta^3}{\sigma_0^2(L)}\right] \tag{41}$$

where $P_{\rm lin}[\delta; \sigma_0(L)]$ is the linear theory result. The extra exponential factor shows that $\mathscr{P}[\delta] \ll P_{\rm lin}$ when $\delta > \delta_c$ with $\delta_c \simeq [3\sigma_0^2(L)/2]^{1/3}$. For example, if $L = 100\ h^{-1}$ Mpc, $\sigma_0^2(L) \simeq 3.3 \times 10^{-3}$ so that $\delta_c \simeq 0.17$. Thus $\mathscr{P}[\delta]$ is much more sharply peaked compared to $P_{\rm lin}[\delta]$.

The corrections to the linear theory can be worked out by a systematic expansion in δ. For example, using the normalized form of the approximate distribution:

$$\mathscr{P}[\delta] \simeq \frac{1}{\sigma_0\sqrt{2\pi}}\left(1 - \frac{5}{3}\delta\right)\exp\left(-\frac{1}{2}\frac{\delta^2}{\sigma_0^2}\right). \tag{42}$$

It follows that

$$\langle\delta\rangle \simeq -\tfrac{5}{3}\sigma_0^2. \tag{43}$$

Thus to the lowest order both $\langle\delta\rangle$ and $\langle\delta^2\rangle$ are decreased compared to the linear theory result.

The distribution behaves quite differently at small scales. When $L \to 0$, σ_0 becomes effectively a constant, say, q. The probability distribution becomes

$$\mathscr{P} = \frac{N}{q}(1 + \delta)^{-7/3}\exp\left\{-\frac{9}{2q^2}\left[1 - \frac{1}{(1 + \delta)^{1/3}}\right]^2\right\}. \tag{44}$$

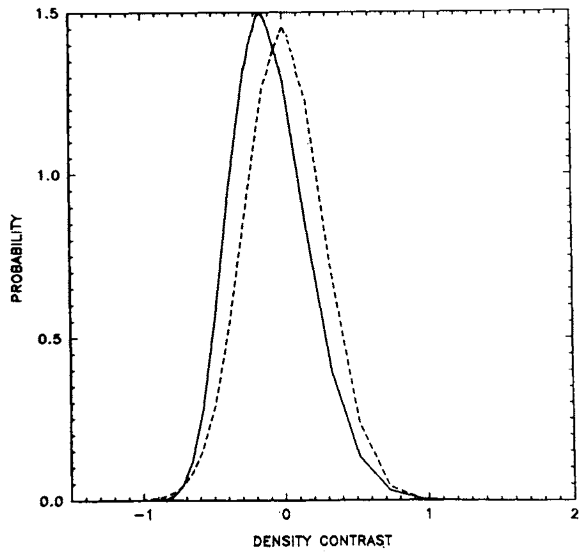

FIG. 1.—Exact probability distribution (*solid line*) and the prediction from linear theory (*dashed line*) for a filtering scale of $L = 50$ Mpc. The latter is based on a CDM model with $\sigma(8\ h^{-1}\ \mathrm{Mpc}) = 1$ and $h = 0.5$.

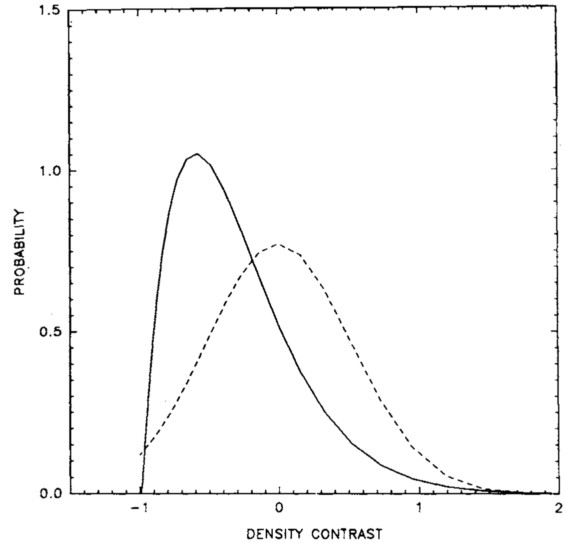

FIG. 2.—Same as Fig. 1 with $L = 30$ Mpc

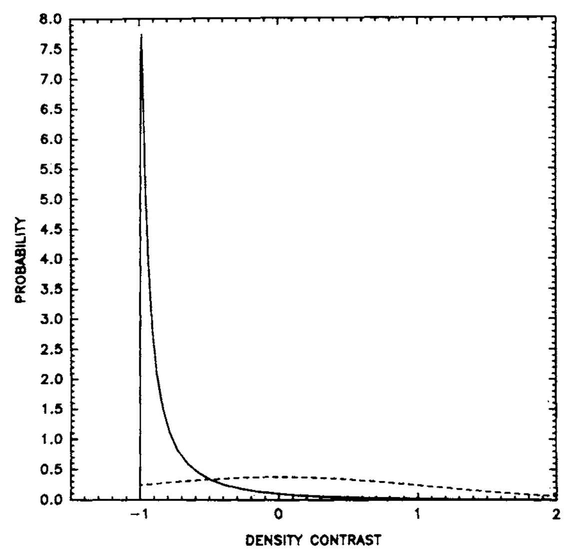

FIG. 3.—Same as Fig. 1 with $L = 15$ Mpc

This distribution is distinctly non-Gaussian, and sharply peaked around $\delta \simeq -1$, showing how nonlinear clustering has affected the statistics of the density field. As the filtering scale varies from a small value to large, the probability distribution changes from equation (44) to equation (39).

These results are shown in Figures 1–3. Figure 1 shows the probability distribution in the nonlinear case when the linear theory is taken to be standard CDM with $\sigma\ (8\ h^{-1}\ \mathrm{Mpc}) = 1$ and $h = 0.5$. In each of the figures, the dashed lines denote the Gaussian probability calculated using linear theory. The filtering scales are 50, 30, and 15 Mpc in the three figures. It is clear that mean value shifts to lower and lower values as the filtering scale is reduced. The distribution also become narrower, which is more clearly seen in Figures 2 and 3. These figures show the probabilities (with proper normalization) for $L = 30$ and 15 Mpc.

The narrowing of the distribution function can be understood on very general terms. Note that (1) The exact probability distribution $P[\delta]$ should vanish for $\delta \leq -1$, and (2) in the limit of $t \to \infty$, the probability distribution should be sharply peaked at $\delta \simeq -1$, since the universe will be dominated by voids. It follows that, at finite times, the peak of the distribution should shift toward (-1), and the width of the distribution should decrease. This implies that the value of $\langle(\delta M/M)^2\rangle$ calculated from linear theory is an overestimate of true $\langle(\delta M/M)^2\rangle$ at small scales. This will have important implications as regards normalization of power spectrum. In particular, it has been noticed that power spectra normalized at large scales tend to overshoot observed values of $\langle(\delta M/M)^2\rangle$ at small scales. The effect under discussion might help to reduce this discrepancy. This issue is under investigation.

Given the probability distribution it is possible to compute other statistical parameters like skewness, kurtosis, etc. We hope to address this and related questions in a future publication.

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