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Hypothesis of path integral duality: Applications to QED

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We use the modified propagator for quantum field based on a "principle of path integral duality" proposed earlier in a paper by Padmanabhan to investigate several results in QED. This procedure modifies the Feynman propagator by the introduction of a fundamental length scale. We use this modified propagator for the Dirac particles to evaluate the first order radiative corrections in QED. We find that the extra factor of the modified propagator acts like a regulator at the Planck scales thereby removing the divergences that otherwise appear in the conventional radiative correction calculations of QED. We find that:(i) all the three renormalization factors Z_1, Z_2, and Z_3 pick up finite corrections and (ii) the modified propagator breaks the gauge invariance at a very small level of $\mathcal{O}(10^{-45})$. The implications of this result to generation of the primordial seed magnetic fields are discussed.

1. Introduction

The space-time structure at Planck scales $L_P \equiv (G\hbar/c^3)^{1/2}$ will be drastically affected by the quantum gravitational effects and it is generally believed that L_P acts as a physical cutoff for space-time intervals. The two main approaches to quantum gravity, Superstring theory and Loop quantum gravity, incorporates the fundamental length scale by considering extended structures, rather than point particles, as fundamental blocks.

The existence of a fundamental length implies that processes involving energies higher than Planck energies will be suppressed, and the ultraviolet behavior of the theory will be improved. This could arise naturally in String theories; several other models also incorporate a Planck length cut-off in a suitable manner to improve the ultra violet behaviour of the theory [1]. One direct consequence of such improved behavior will be that the Feynman propagator(in momentum space) will acquire damping factor for energies larger than Planck energy. How-

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ever, this propagator – which arises in the standard formulation of quantum field theory – does not take into account of the existence of any fundamental length in the space-time. On the other hand, such a fundamental length scale was introduced into the Feynman propagator in a Lorentz invariant manner by invoking the "principle of path-integral duality" [2]. According to this postulate, the weightage given for a path in the path integral should be invariant under the transformation $\mathcal{R} \to L_P^2/\mathcal{R}$, where $\mathcal{R}$ is the length of the path and the fundamental length scale L_P is assumed to be of the order of the Planck length $(G\hbar/c^3)^{1/2}$.[In this paper, when we say that the fundamental length is L_P, we actually mean that it is of $\mathcal{O}(L_P)$.] Padmanabhan [2] has shown by rigorous evaluation of the path integral by lattice techniques that the effect of the duality principle is to modify the weightage given to a path of proper time s from $\exp(-im^2s)$ to $\exp[-i(m^2s - L_P^2/s)]$, where m is associated with the mass of the particle. For example, the Feynman propagator for a free scalar field of mass m, propagating in the flat $(3+1)$ space-time, in Schwinger's proper time formalism, is described by the integral

$$G_F(x,x') = \frac{1}{(4\pi i)^2}\int_0^\infty \frac{ds}{s^2}\exp(-im^2s)\exp[i(x-x')^2/4s]. \tag{1}$$

The duality principle modifies the Feynman propagator to the form

$$\begin{aligned} G_F^P(x,x') &= \frac{1}{(4\pi i)^2}\int_0^\infty \frac{ds}{s^2}\exp(-im^2s)\exp[i((x-x')^2+L_P^2)/4s] \\ &= \frac{m}{4\pi^2}\frac{K_1(im\sqrt{x^2-L_P^2-i\epsilon})}{\sqrt{x^2-L_P^2-i\epsilon}}, \end{aligned} \tag{2}$$

where $K_1(z)$ is the modified Bessel function of order 1. [The metric signature we follow is $(+,-,-,-)$.] In momentum space, the modified propagator is

$$G_F^P(p) = -i\int_0^\infty dz\,\exp(iL_P^2/4z + i(p^2-m^2+i\epsilon)z). \tag{3}$$

The presence of a fundamental length scale is a feature that is expected to arise in a quantum theory of gravity. Hence, the modification of the weightage factor as mentioned above can be interpreted as being equivalent to introducing quantum gravitational corrections into standard field theory. The ultraviolet divergences in quantum field theory arise from the singularities of the propagator functions on the light cone, and a smearing out of the light cone due to the quantum gravitational corrections, using the principle of "path integral duality", will lead to the suppression of these divergences.

In an earlier work [3], the implications of the modified propagator to certain conventional non-perturbative quantum field theoretic results were discussed in detail. It was found that the essential feature of this prescription of path integral duality is to provide an ultra-violet cutoff at the Planck energy scales, thereby obtaining a Lorentz invariant finite results. Encouraged by this fact, we would like to estimate the renormalization factors in QED, and other radiative correction terms which cannot be obtained in the conventional QED calculations.

The standard definition of Feynman propagator for Dirac particles is

$$S(x) = -(-i\gamma^{\mu}\partial_{\mu} + m)G_F(x), \tag{4}$$

where, $G_F(x)$ is the usual Feynman propagator for the scalar particles. In our analysis of evaluating quantum gravitational corrections to the standard QED calculations, we assume that the effect of "principle of path integral duality" on the Dirac propagator is defined as

$$S^P(x) = -(-i\gamma^{\mu}\partial_{\mu} + m)G_F^P(x). \tag{5}$$

Hence, the effect of summing over the quantum fluctuations of the space-time structure, in the low energy scales, can be realized in any field theoretic calculations where the propagator of the scalar particles appear explicitly.

There is a similar approach in the literature by Ohanian [4], who had used a smeared propagator which is Poincare invariant, to calculate the radiative corrections in QED. The modified Feynman propagator in our case is obtained by rigorous evaluation and the duality principle introduces the fundamental length scale in a Lorentz invariant manner. We also comment on the differences between our approaches and results at the appropriate sections below.

Before proceeding to the technical aspects, it is necessary to outline certain conceptual issues related to this approach. In the conventional approaches to quantizing a field based on some classical Lagrangian, one will invariably obtain quantum corrections to the classical Lagrangian. When these corrections are local in the configuration space and contain terms which are of the same form as those in the classical Lagrangian, it is necessary to absorb them into the parameters of the original Lagrangian. This process of renormalization, has — *a priori* — nothing to do with divergences. Usually, however, the quantum corrections to the theory — calculated by perturbative methods — lead to divergent expressions. If the new (divergent) terms are not of the form of the original terms, then the theory cannot be interpreted perturbatively. However, if the divergent terms have the same structure as the terms in the original Lagrangian, it is possible to use the procedure of renormalization (which, *a priori*, has nothing to do with divergences) to give meaning to the theory. To do this properly, it is necessary to first evolve a procedure (called regularization) which allows the divergent expressions to be recast as the limit of some finite quantities. After performing the renormalization subtractions, one is free to take the required limit, leading to finite corrections.

The above approach gets modified in two essential aspects when the modified propagator is used. Firstly, when quantum gravitational corrections lead to a cutoff, the quantum corrections will not have any divergent terms; that is, the regularization is now built into the theory with Planck length acting as regulator. But renormalization of the theory is still needed and the physical and bare coupling constants will differ by a finite amount. Secondly, *the regularization procedure is now fixed by our ansatz*. This is important because, it is well known from standard work in quantum field theory that different regularization procedures are not equivalent. For example, dimensional regularization and momentum space techniques are not completely equivalent as regards their treatment of the symmetries of the system. Since we have no freedom in choosing a regularization, it is necessary to accept and investigate the final results arising from the ansatz. The use of modified propagator to the field theoretic calculation is a more realistic approach towards the removal of the divergences in field theory based on the relevant physics, rather than a formalistic approach based on improper mathematical manipulations. In an earlier work [3], it was shown that the physical parameters in this system like mass and coupling constant have additive finite terms, that are proportional to the powers of L_P by calculating the effective potential for a self-interacting scalar field theory using the modified propagator.

In this paper, we evaluate the second order radiative corrections in QED using the modified propagator. The three corrections are to the vacuum polarization, electron self-energy, and vertex function. In the conventional QED calculations the divergent terms are absorbed into the physical parameters like mass, charge and spin. The modified propagator here again acts as a regulator to the ultra-violet divergences in the theory. The modified propagator introduces two kinds of regulators in the radiative correction calculations, which are logarithmic and power law. In the three radiative corrections the leading order power law regulator is $\mathcal{O}(L_P^2 m^2)$ and this is very small when compared to the other non-divergent terms in the conventional QED results. The three renormalization factors has the logarithmic corrections which is of the $\mathcal{O}(\ln(L_P m))$.

The following point needs to be stressed regarding the actual values of the corrections. In pure QED, the perturbative expansion is in a series in α and corrections are of the order of $\alpha \simeq 10^{-2}$, $\alpha^2 \simeq 10^{-4}$ etc in successive orders. The lowest order quantum gravitational corrections are, by and large, of order $L_P^2 m^2 \simeq 10^{-45}$. Since $\alpha^{22} \approx (L_P m)^2$, the first 22 order corrections of QED will dominate over the quantum gravitational corrections computed here! Much before this, electroweak corrections will start modifying the results. Thus the finite corrections computed here are only of conceptual significance — providing the lowest order corrections *from quantum gravity* — rather than of any operational significance. The only exception to this general situation is when quantum gravitational effects break a symmetry originally present in the theory, which — as we shall see — does happen.

In sections II and III, we evaluate the second order radiative corrections in QED, and we discuss the results and present other applications in Sec. IV.

2. Vacuum Polarization and Electron Self Energy

2.1. *Vacuum Polarization*

The interaction of the photon field with electron field modifies the free photon propagator. The photon propagator of momentum q, with the one loop radiation correction included, is given by

$$iD'^{F}_{\mu\nu}(q) = iD^{F}_{\mu\nu} + iD^{F}_{\mu\rho}\frac{i\Pi^{\rho\sigma}(q)}{4\pi}iD^{F}_{\sigma\nu},$$

where $\Pi^{\mu\nu}$ is the vacuum polarization tensor, and $D^{F}_{\mu\nu}$ is the free photon propagator. The free photon propagator, $D^{F}_{\mu\nu}$, in the above relation is the conventional QED propagator. We concentrate here on the corrections to $D_{\mu\nu}$ arising from the modification of $\Pi_{\mu\nu}$ rather than from direct modification of $D_{\mu\nu}$ due to our ansatz. Using the Feynman rules of QED in the momentum space, we can write the vacuum polarization tensor as

$$\begin{aligned}\Pi_{\mu\nu}(q) &= 16\pi i e^2\int\frac{d^4k}{(2\pi)^3}\left(k_\mu(k-q)_\nu+(k-q)_\mu k_\nu-g_{\mu\nu}(k^2-qk-m^2)\right)\\ &\times\ G^P_F(k)G^P_F(k-q).\end{aligned} \tag{6}$$

Substituting for the propagator from Eqn.(3), and following similar calculations as in conventional QED, the vacuum polarization tensor can be separated into gauge invariant and gauge non-invariant part [5]. The resultant gauge invariant part is given by

$$\Pi^1_{\mu\nu}(q^2) = -\frac{4e^2}{\pi}\left(q_\nu q_\mu - g_{\mu\nu}q^2\right)\int_0^1 dz z(1-z)K_0(\xi), \tag{7}$$

where

$$\xi^2 = L_P^2\frac{m^2-q^2z(1-z)}{z(1-z)}.$$

The series expansion of $K_0(z)$, about the origin, is given by

$$K_0(z) = -\gamma - \ln(z/2) - \frac{z^2}{4}\left[1-\gamma-\ln(z/2)\right] + \cdots. \tag{8}$$

Hence, Eqn.(7) takes the form

$$\begin{aligned}\Pi^1_{\mu\nu}(q^2) &= \left(q_\nu q_\mu - g_{\mu\nu}q^2\right)\Bigg[(Z_3-1) - \frac{2e^2}{\pi}A_1 - \frac{e^2L_P^2}{\pi}\Bigg(\frac{q^2}{2}A_1\\ &+ \frac{1}{2}\left(1-\gamma-\ln(L_Pm/2)\right)\left(m^2-\frac{q^2}{6}\right)\Bigg)\Bigg],\end{aligned} \tag{9}$$

where

$$A_1 = \int_0^1 dz\ z(1-z)\ln\left(\frac{m^2}{m^2-q^2z(1-z)}\right) \tag{10}$$

to the lowest order of $K_0(\xi)$. The term A_1 is the familiar conventional QED non-divergent term and the remaining terms (of the order $\mathcal{O}(L_P^2)$) are the leading order quantum gravitational power law corrections/regulators to the conventional QED terms. The contribution of the quantum gravitational corrections to the vacuum polarization is extremely small as compared to the conventional non-divergent QED term, i.e. they are of the order 10^{-45} which is much smaller compared to the next order radiative corrections of QED. But as we can notice the quantum gravitational corrections to the conventional QED non-divergent terms become important when $L_P q \simeq 1$. In this high momentum transfer limit, the propagation effects of the virtual photons will probe the small scale quantum gravitational effects and the corrections will be of the same order as the conventional QED non-divergent terms. The charge renormalization factor, Z_3, which is divergent in the usual QED calculations is now finite and is given by

$$Z_3 - 1 = \frac{2e^2 \ln(L_P m/2)}{\pi}\left[\frac{1}{3} + \frac{6\gamma - 5}{18\ln(L_P m/2)} + \mathcal{O}(L_P^2 m^2)\right]. \qquad (11)$$

The estimate of this factor comes out to be

$$Z_3 - 1 = \frac{2e^2}{3\pi}\ln(L_P m) \simeq -0.1. \qquad (12)$$

In the earlier work [3], authors have calculated charge renormalization factor using Effective action approach, which is a non-perturbative technique. In this approach, they calculated the effect of the classical electro-magnetic background on the quantum charged scalar fields, propagating in the flat space-time, using the above modified propagator. The charge renormalization factor they obtain using the effective action approach is of the form $Z^P = \frac{q^2}{6\pi}K_0(2L_P m)$. On expanding $K_0(z)$ using Eqn.(8), their estimate of the charge renormalization factor was also of the order of 0.1.

In the standard QED calculations, there does arise a gauge non-invariant part of the vacuum polarization tensor which is divergent and is renormalized to zero(for instance, see Hatfield [6]). Even though, this is a standard result and can be found in textbooks we have given the relevant steps in the Appendix for the sake of completeness. By retracing the steps given in Appendix, we now obtain a finite quantity to the gauge non-invariant part of the vacuum polarization tensor using the modified propagator. The gauge non-invariant part comes out to be

$$\begin{aligned}\Pi^2_{\mu\nu}(q) &= \frac{iL_P^2 e_0^2}{(4\pi)^2} g_{\mu\nu}\int_0^\infty \frac{dz}{z}\int_0^\infty \frac{dz'}{z'}\frac{1}{(z+z')^2}\exp\left(\frac{iL_P^2}{4}(z^{-1}+z'^{-1})\right) \\ &\times \exp i\left(\frac{q^2 z z'}{z+z'} - m_0(z+z')\right). \qquad (13)\end{aligned}$$

In Eqn.(13), transforming the variables z and z' to a new set of variables by the relation $z = 1/t$ and $z' = 1/t'$ leads to

$$\Pi^2_{\mu\nu}(q) = \frac{iL_P^2 e_0^2}{(4\pi)^2} g_{\mu\nu} \int_0^\infty t\ dt \int_0^\infty t'\ dt' \frac{1}{(t+t')^2} \exp\left(\frac{iL_P^2}{4}(t+t')\right) \times \exp i \left(\frac{q^2}{t+t'} - m_0(t^{-1} + t'^{-1})\right). \tag{14}$$

Using the standard identity,

$$1 = \int_0^\infty \frac{d\beta}{\beta} \delta\left(1 - \beta(z+z')\right)$$

and scaling $t_i \to t_i/\beta$, as in the conventional QED calculations, we get

$$\Pi^2_{\mu\nu}(q) = -\frac{2ie_0^2}{(4\pi)^2} g_{\mu\nu} L_P^6 \int_0^1 t(1-t)\frac{K_2(\xi)}{\xi^2}\ dt, \tag{15}$$

where

$$\xi^2 = L_P^2 \left(\frac{m_0^2}{t(1-t)} - q^2\right)$$

The expansion of $K_2(z)$ near the origin is given by

$$K_2(z) = \frac{2}{z^2} - \frac{1}{2} + \left(\frac{3-4\gamma}{32} + \frac{\ln(2) - \ln(z)}{8}\right) z^2 + \cdots.$$

Substituting the series expansion of $K_2(\xi)$ in Eqn.(15), we obtain

$$\Pi^2_{\mu\nu}(q) = g_{\mu\nu} \frac{-2ie_0^2}{(4\pi)^2} \left[2L_P^2 \int_0^1 dt \frac{(t-t^2)^3}{(m_0^2 - q^2 t(1-t))^2} - \frac{L_P^4}{2} \int_0^1 dt \frac{(t-t^2)^2}{(m_0^2 - q^2 t(1-t))} + \cdots \right]. \tag{16}$$

This clearly shows that the the gauge non-invariant part is nonzero for $L_P \neq 0$ and vanishes as $L_P \to 0$. While the term which breaks the gauge invariance is small to be of operational significance, it does have certain conceptual importance. The following points need to be noted regarding this result:

(i) Mathematically speaking, this result arises from the fact that our ansatz is equivalent to a momentum space regularization procedure in conventional QED. It is known that, momentum space regularization, in contrast to dimensional regularization, can lead to gauge breaking terms. Usually, this is considered as an argument in favor of dimensional regularization. In our approach, of course, we have no choice and the result arises automatically. In fact, it is very likely that any quantum gravitational cutoff will appear like a momentum space regulator and will break the gauge symmetry.

(ii) Previously, Ohanian [4] had obtained a gauge breaking term using a gravitationally smeared propagator. There is however one vital difference between our result and the one obtained by him. Note that, in our approach, the propagator reduces to that of conventional field theory when $L_P \to 0$. If the procedure is to be consistent, the gauge breaking term should vanish when the limit of $L_P \to 0$ is taken. This is true as regards our result in Eqn.(16) showing that this is indeed a quantum gravitational effect. However, Ohanian [4] obtains a gauge breaking term which does not vanish in the corresponding limit. This suggests that, our approach does allow a consistent interpretation of the results.

(iii) It is the extra term $L_P^2 (z+z')^{-2}/(4zz')$ which breaks the gauge invariance of the electro-magnetic field. This extra factor can be associated to the current in the charge conservation relation and hence, implying that the charge conservation is no more valid. There have been – more drastic! – attempts in the literature to break even the Lorentz invariance by introducing a coupling of the photon to charged scalar field, through gravitational couplings to the photon, etc [7]. The basic aim in the process of breaking the conformal symmetry of the electro-magnetic field is allowing for the possibility of generating large scale magnetic fields within inflationary scenarios. Most of these studies have used ad-hoc interaction potential to break the conformal invariance. The breaking of gauge invariance of the electro-magnetic field in our case is from a much more deeper "principle of path integral duality". The connection to the cosmological seed magnetic field is still under investigation.

2.2. *Electron Self Energy*

The interaction of the electron field with photon field modifies the free electron propagator. The electron propagator of momentum p, with the one-loop correction included, is given by

$$iS'_F(p) = iS_F(p) + iS_F(p)(-i\Sigma(p))iS_F(p),$$

where $\Sigma(p)$ (a 4-spinor) is the self-energy function, and $S_F(p)$ is the free electron propagator. Using the Feynman rules in the momentum space, we get

$$\Sigma(p) = -4\pi i e^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu (\gamma^\nu p_\nu - \gamma^\nu k_\nu + m) \gamma_\mu G_F^P(k) G_F^P(p-k). \tag{17}$$

Substituting the propagator from the Eqn.(3), the above equation reduces to

$$\Sigma(p) = \frac{e^2}{4\pi^2} \int_0^1 dz\, (2m_0 - \gamma^\mu p_\mu z)\, K_0(\xi), \tag{18}$$

where

$$\xi^2 = \frac{L_p^2}{z(1-z)} \left[m_0^2 z - p^2 z(1-z) \right].$$

On expanding $K_0(\xi)$ using Eqn.(8) we obtain, the lower order terms corresponding to the conventional QED results and the higher order terms as contributions of the quantum fluctuations of the space-time to the self energy of electron. The electron wave function renormalization, and the shift in the mass are obtained by recasting the interacting field propagator to look like the free field propagator i.e. we set

$$Z_2(\gamma^\mu p_\mu - m_0 - \Sigma(p)) = \gamma^\mu p_\mu - m + \text{finite terms.}$$

The electron wave function renormalization factor is obtained by equating terms proportional to p in the above equation. This results in

$$Z_2^{-1} - 1 = \frac{e^2}{8\pi^2}\left[\gamma + \ln 2 - \ln(L_p m_0) + \mathcal{O}(1)\right]. \tag{19}$$

[Note that we have neglected the higher order terms of $K_0(\xi)$ as these have the dependence as $L_P^2 m_0^2$ whose contributions are negligible]. The last term in the above expression is a finite quantity and is of the order one and it has the dependence of the electron momentum. The shift in the mass is given by

$$\frac{\delta m}{m_0} = -\frac{e^2}{8\pi^2}\left[3\ln(L_P m_0/2) + 3\gamma + \mathcal{O}(1)\right] \tag{20}$$

The estimate of the scale factor Z_2 comes out to be

$$Z_2^{-1} - 1 = -\frac{\alpha}{\pi}\ln m_0 L_P \simeq 0.1, \tag{21}$$

and the fractional shift in the mass is

$$\frac{\delta m}{m} \simeq -\frac{\alpha}{\pi}\ln m_0 L_P \simeq 0.1. \tag{22}$$

The usual infrared divergences is ignored in the calculation of the renormalization factor Z_2. Here again, we see that both the mass renormalization factor and the mass shift are also of the same order as the charge renormalization factor i.e. 0.1.

3. Vertex Correction and Anomalous magnetic moment

3.1. *Vertex Correction*

For a free Dirac field, the current density is defined as

$$J^\mu = \bar{\psi}\gamma^0\gamma^\mu\psi.$$

Thus, in the low energy QED processes the current transfer is related by γ^μ, which is the low energy vertex function. The radiative corrections will modify the vertex, to the one-loop correction, as

$$-ie\Lambda_\mu = -ie\gamma_\mu - ie\Gamma_\mu, \tag{23}$$

where, Γ_μ is the vertex function. Using the Feynman rules, we obtain

$$\begin{aligned}\Gamma_\mu(p', p) &= (-ie_0)^2 \int \frac{d^4k}{(2\pi)^4} \left[\gamma_\mu(\gamma_\rho p'^\rho - \gamma_\rho k^\rho + m_0)\gamma_\mu(\gamma_\sigma p'^\sigma - \gamma_\sigma k^\sigma + m_0)\right] \\ &\times \; G_F^P(k) G_F^P(p' - k) G_F^P(p - k). \end{aligned} \tag{24}$$

Substituting for the propagator from the Eqn. (3), the above equation reduces to

$$\begin{aligned}\Gamma_\mu(p', p) &= (Z_1^{-1} - 1)\gamma_\mu + \frac{(-ie_0)^2}{(4\pi)^2} \int_0^1 dz \int_0^{1-z} dz' \frac{\xi}{T_1} \\ &\times \; \left[2\gamma_\mu (p' - p)^2 (1 - z')(1 - z) - 4imz(1 - z - z')(p' - p)^\mu \sigma_{\mu\nu}\right], \end{aligned} \tag{25}$$

where

$$\xi^2 = L_P^2 T_1 T_2, \quad T_1 = (zp' + z'p)^2 - zp'^2 - z'p^2 + (z + z')m_0^2,$$

and

$$T_2 = (1 - z - z')^{-1} + z^{-1} + z'^{-1}.$$

The vertex renormalization factor Z_1 is given by

$$\begin{aligned}Z_1^{-1} - 1 &= i\frac{(-ie_0)^2}{(4\pi)^2} \int_0^1 dz \int_0^{1-z} dz' \Big[4K_0(\xi) + 2m_0^2 \frac{\xi}{T_1} K_1(\xi) \\ &\times \; (-2 + 2(z + z') + (z + z')^2)\Big]. \end{aligned} \tag{26}$$

Substituting for $K_0(\xi)$ using Eqn.(8), and using the series expansion of $K_1(\xi)$ as

$$K_1(z) = \frac{1}{z} + \frac{z}{2}\left(\ln(z/2) + \frac{2\gamma - 1}{2}\right) + \frac{z^3}{16}\left(\ln(z/2) - \frac{5 - 4\gamma}{4}\right), \tag{27}$$

in the Eqn. (26), the vertex normalization factor comes out to be

$$\begin{aligned}Z_1^{-1} - 1 &= i\frac{(-ie_0)^2}{(4\pi)^2} \int_0^1 dz \int_0^{1-z} dz' \Big[-4\ln(\xi/2) - 4\gamma + \frac{2m_0^2}{T_1} \\ &\times \; (-2 + 2(z + z') + (z + z')^2)\Big] \end{aligned} \tag{28}$$

$$\cong -\frac{\alpha}{\pi} \ln(m_0 L_P), \tag{29}$$

which is roughly of the order of 0.1. The estimated renormalization factors Z_1 and Z_3 using the modified propagator are not equal unlike in conventional QED.

3.2. *Anomalous Magnetic Moment*

The triumph of QED has been the precision test of electron anomalous magnetic moment. The experimental value of the anomalous magnetic moment of an electron is in excellent agreement with the predicted perturbative calculations up to the 4^{th} order to the 15^{th} decimal place. It is therefore of interest to compute the quantum gravitational corrections to the magnetic moment of an electron using the modified propagator.

The vertex correction contribution to scattering of an electron in an external field is given by $\bar{u}(p')\Gamma^{\mu}(p',p)u(p)A^{c}_{\mu}(p'-p)$, where $\bar{u}$ and u are the spinor wave-functions. Since, our interest is in calculating the radiative and the quantum gravitational corrections to the gyro-magnetic ratio of an electron, the term involving $\sigma_{\mu\nu}$ in the Eqn. (25) is of our concern. The corresponding $\mathcal{M}$ matrix for this process is given by [5],

$$\begin{aligned} \mathcal{M} &= \bar{u}(p')\gamma^{M}_{\mu}u(p) \qquad (30) \\ &= \bar{u}(p')4m_0\frac{(ie_0)^2}{(4\pi)^2}\int_0^1 dz\int_0^{1-z}dz' z(1-z-z')(p'-p)^{\nu}\frac{\sigma_{\mu\nu}\xi}{T_1}K_1(\xi)u(p). \end{aligned}$$

In the limit of small momentum transfer, $(p'-p)^2 \ll m_0^2$, one obtains by expanding $K_1(\xi)$ from the Eqn. (27), we get

$$\begin{aligned} \mathcal{M} &= -\bar{u}(p')\frac{\alpha}{m\pi}(p'-p)^{\mu}\sigma_{\mu\nu}\int_0^1 dz\int_0^{1-z}dz'\frac{z(1-z-z')}{(z+z')^2}u(p) \\ &- \bar{u}(p')\frac{\alpha}{m\pi}\frac{L_P^2 m_0^2}{24}(p'-p)^{\mu}\sigma_{\mu\nu}\int_0^1 dz\int_0^{1-z}dz' z(1-z-z')T_2 \\ &\times \ln(L_P m_0^2(z+z')^2 T_2/4)u(p). \qquad (31) \end{aligned}$$

The first term in the above equation corresponds to the usual QED radiative vertex correction of the gyro-magnetic ratio (g) to the order e_0^2. The second term in the above equation is the quantum gravitational corrections to the gyro-magnetic ratio of an electron. The integral in the second term of the Eqn. (31) is convergent. The contribution of the quantum gravitational correction to the gyro-magnetic ratio to the first order in α is of the order $L_P^2 m^2 \simeq 10^{-45}$. Obviously, this is not of practical significance.

4. Conclusions and Discussion

In this paper, we evaluated the quantum gravitational corrections(QGC) to three radiative corrections, in the first order of α, in QED using the "principle of path integral duality". The modified propagator is able to remove all the divergences which usually crop up in the conventional QED calculations. The main features of the modified propagator in QED are as follows:

(a) The three renormalization factors(Z_1, Z_2, and Z_3), and the mass shift are all of the same order, $\mathcal{O}(\ln(mL_P)) \simeq 0.1$. The charge renormalization factor Z_3 using the non-perturbative methods in scalar QED is also of order 0.1 [3]. The renormalization factors Z_1 and Z_3 are different in our case as opposed to the conventional QED calculations.

(b) The modified propagator makes the gauge non-invariant part of the vacuum polarization tensor to be non-zero for $L_P \neq 0$ vanishes and in the limit $L_P \to 0$. This breaking of the gauge symmetry is also related to also the difference between the two renormalization factors. We have briefly indicated the possible effect of this breaking of gauge invariance, to the generation of large scale magnetic fields within inflationary scenarios

(c) The contribution of the QGC to the gyro-magnetic ratio is very small i.e. of the order of $L_P^2 m^2 \simeq 10^{-45}$. The contribution of the quantum gravitational correction to the vacuum polarization and the electron self energy is also of the same order.

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Appendix A: Evaluation of Gauge Non-invariant part

For the sake of completeness, we outline the essential steps leading to the gauge non-invariant part of the vacuum polarization tensor in standard QED and how it is regularized to zero. The vacuum polarization tensor in the conventional QED calculations in $2n$ dimensions is given by

$$\begin{aligned}\Pi^{\mu\nu}_{reg}(q) &= (e\mu^{2-n})^2 \int \frac{d^{2n}k}{(2\pi)^{2n}} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k-q)^2 - m^2 + i\epsilon} \\ &\times \; 2^n \left(k^\mu (k-q)^\nu + k^\nu (k-q)^\mu - ((k^2 - m^2) - k\cdot q) g^{\mu\nu}\right),\end{aligned} \tag{A.1}$$

where μ has the dimension of mass. Using the integral representation of the propagator, i.e.

$$\frac{i}{k^2 - m^2 + i\epsilon} = \int_0^\infty dz \exp\left(iz\left(k^2 - m^2 + i\epsilon\right)\right) \tag{A.2}$$

in Eqn.(A.1) and completing the squares in the exponential, we obtain

$$\begin{aligned}\Pi^{\mu\nu}_{reg}(q) &= (e\mu^{2-n})^2 \int_0^\infty dz_1 dz_2 \int \frac{d^{2n}k}{(2\pi)^{2n}} \\ &\times \; 2^n \left(k^\mu (k-q)^\nu + k^\nu (k-q)^\mu - ((k^2 - m^2) - k\cdot q) g^{\mu\nu}\right) \\ &\times \exp\left(i(z_1+z_2)\left(k - \frac{z_2 q}{z_1+z_2}\right)^2 + i\frac{z_1 z_2 q^2}{z_1+z_2} - i(m^2 - i\epsilon)(z_1+z_2)\right).\end{aligned} \tag{A.3}$$

Shifting the variable of integration and using the relations,

$$\int \frac{d^{2n}p}{(2\pi)^{2n}} p^2 \exp(iap^2) = -\frac{n}{(4\pi a)^n}\frac{1}{a}\exp(in\pi/2),$$

$$\int \frac{d^{2n}p}{(2\pi)^{2n}} p^\mu p^\nu \exp(iap^2) = -\frac{g^{\mu\nu}}{(4\pi a)^n}\frac{1}{2a}\exp(in\pi/2),$$

we get

$$\frac{\Pi^{\mu\nu}_{reg}(q)}{(e\mu^{2-n})^2} = \frac{\exp(in\pi/2)}{(4\pi)^n}\left[g_{\mu\nu}Q(q^2,m^2) + (q^\mu q^\nu - g^{\mu\nu}q^2)P(q^2,m^2)\right], \quad \text{(A.4)}$$

where

$$\begin{aligned} Q(q^2,m^2) &= \int_0^\infty dz_1 \int_0^\infty dz_2 \frac{2^n}{(z_1+z_2)^n}\left(\frac{n-1}{(z_1+z_2)} + im^2 - iq^2\frac{z_1 z_2}{(z_1+z_2)^2}\right) \\ &\times \exp\left(iq^2\frac{z_1 z_2}{z_1+z_2} - i(m^2 - i\epsilon)(z_1+z_2)\right), \end{aligned} \quad \text{(A.5)}$$

and

$$\begin{aligned} P(q^2,m^2) &= -i2^{n+1}\int_0^\infty dz_1 \int_0^\infty dz_2 \frac{z_1 z_2}{(z_1+z_2)^{n+2}} \\ &\times \exp\left(iq^2\frac{z_1 z_2}{z_1+z_2} - i(m^2 - i\epsilon)(z_1+z_2)\right). \end{aligned} \quad \text{(A.6)}$$

Global gauge invariance of the action leads to current conservation, $\partial_\mu j^\mu = 0$. This implies $q_\mu \Pi^{\mu\nu}(q) = 0$. The factor proportional to $g^{\mu\nu}$ is the gauge non-invariant part of the vacuum polarization tensor and does not seem to satisfy the gauge invariance condition.

We summarize here the standard argument [see Hatfield [6]] to show that the first term, $Q(q^2,m^2)$, in Eqn.(A.4) can be shown to vanish and hence the regularization preserves the gauge symmetry. Consider only the first term of the gauge non-invariant part and define

$$I(q^2,m^2) \equiv \int_0^\infty dz_1 dz_2 \frac{1}{(z_1+z_2)^{n+1}}\exp\left(iq^2\frac{z_1 z_2}{z_1+z_2} - im^2(z_1+z_2)\right). \quad \text{(A.7)}$$

Rescaling the integration variables, z_1 and z_2, in the above equation by β, i.e. $z_1 \to \beta z_1$ and $z_2 \to \beta z_2$, we obtain

$$\begin{aligned} I(\beta,q^2,m^2) &= \frac{1}{\beta^{n-1}}\int_0^\infty dz_1 dz_2 \frac{1}{(z_1+z_2)^{n+1}} \\ &\times \exp\left(i\beta\left(q^2\frac{z_1 z_2}{z_1+z_2} - m^2(z_1+z_2)\right)\right). \end{aligned} \quad \text{(A.8)}$$

The quantity $I(\beta,q^2,m^2)$ is of course independent of the integration variables z_1 and z_2, and hence is also independent of the parameter(β), i.e. $\partial I(\beta,q^2,m^2)/\partial\beta$ $= 0$. Differentiating the above expression w.r.t β and regrouping terms, we get

$$\begin{aligned} 0 = -\beta\frac{\partial I}{\partial \beta} &= \int_0^\infty dz_1 dz_2 \frac{\beta^{2-n}}{(z_1+z_2)^n}\left(\frac{n-1}{(z_1+z_2)} + im^2 - iq^2\frac{z_1 z_2}{(z_1+z_2)^2}\right) \\ &\times \exp\left(i\beta\left(q^2\frac{z_1 z_2}{z_1+z_2} - m^2(z_1+z_2)\right)\right). \end{aligned} \quad \text{(A.9)}$$

Rescaling the variables z_1 and z_2 by β^{-1}, i.e. $z_i \to z_i/\beta$, in the above expression, we get

$$\begin{aligned} 0 = -\beta\frac{\partial I}{\partial \beta} &= \int_0^\infty dz_1 dz_2 \frac{2^n}{(z_1+z_2)^n}\left(\frac{n-1}{(z_1+z_2)} + im^2 - iq^2\frac{z_1 z_2}{(z_1+z_2)^2}\right) \\ &\times \exp\left(i\left(q^2\frac{z_1 z_2}{z_1+z_2} - m^2(z_1+z_2)\right)\right) = Q(q^2, m^2). \end{aligned} \tag{A.10}$$

Thus, the gauge non-invariant part of the vacuum polarization tensor in the standard QED vanishes. Hence, $q_\mu \Pi^{\mu\nu}(q) = 0$.

In the rest of this appendix, we retrace the above steps for the modified propagator. The gauge non-invariant part of the vacuum polarization tensor gets modified to the form,

$$\Pi^{(2)}_{\mu\nu}(q^2, L_P) = (e\mu^{2-n})^2 \frac{\exp(in\pi/2)}{(4\pi)^n} g_{\mu\nu}\Pi^{(2)}_0(q^2, L_P), \tag{A.11}$$

where

$$\begin{aligned} \Pi^{(2)}_0(q^2, L_P) &= \int_0^\infty dz_1 dz_2 \frac{2^n}{(z_1+z_2)^n}\left[\frac{n-1}{(z_1+z_2)} + im^2 - iq^2\frac{z_1 z_2}{(z_1+z_2)^2}\right] \\ &\times \exp\left[iq^2\frac{z_1 z_2}{z_1+z_2} - i(m^2 - i\epsilon)(z_1+z_2) + \frac{iL_P^2}{4}(z_1^{-1} + z_2^{-1})\right], \end{aligned} \tag{A.12}$$

when the modified propagator is used. We now show that the gauge non-invariant part is a finite quantity and is of the order L_P^2. Here again, we consider the first term of the gauge non-invariant part and define

$$\begin{aligned} I(q^2, m^2, L_P) &\equiv \int_0^\infty dz_1 dz_2 \frac{1}{(z_1+z_2)^{n+1}} \exp\left(iq^2\frac{z_1 z_2}{z_1+z_2} - im^2(z_1+z_2)\right) \\ &\times \exp\left(\frac{iL_P^2}{4}(z_1^{-1} + z_2^{-1})\right). \end{aligned} \tag{A.13}$$

Rescaling the variables z_1 and z_2 by β, i.e $z_i \to \beta z_i$, and differentiating the resultant of the above expression w.r.t β and regrouping terms, we obtain

$$-\beta\frac{\partial I(\beta, q^2, m^2, L_P)}{\partial\beta} \equiv Q(\beta, q^2, m^2, L_P) - R(\beta, q^2, m^2, L_P), \tag{A.14}$$

where

$$\begin{aligned} Q(\beta, q^2, m^2, L_P) &= \int_0^\infty dz_1 dz_2 \frac{\beta^{2-n}}{(z_1+z_2)^n} \qquad \text{(A.15)} \\ &\times \left(\frac{n-1}{(z_1+z_2)} + im^2 - iq^2\frac{z_1 z_2}{(z_1+z_2)^2}\right) \\ &\times \exp\left[i\beta\left(q^2\frac{z_1 z_2}{z_1+z_2} - m^2(z_1+z_2)\right) + \frac{iL_P^2}{4\beta}(z_1^{-1} + z_2^{-1})\right], \end{aligned}$$

and

$$R(\beta,q^2,m^2,L_P) \equiv \int_0^\infty dz_1 dz_2\, \frac{iL_P^2}{4z_1z_2}\,\frac{\beta^{-n}}{(z_1+z_2)^n} \times \exp\left(i\beta\left(q^2\frac{z_1z_2}{z_1+z_2} - m^2(z_1+z_2)\right) + \frac{iL_P^2}{4\beta}(z_1^{-1}+z_2^{-1})\right). \tag{A.16}$$

The term $I(\beta,q^2,m^2,L_P)$ is independent of β (by the arguments stated earlier in this appendix) and hence, the partial differential $\partial I/\partial\beta$ vanishes. Hence,

$$Q(\beta,q^2,m^2,L_P) = R(\beta,q^2,m^2,L_P).$$

Rescaling the variables z_1 and z_2 by β^{-1}, i.e. $z_i \to z_i/\beta$, in Eqn.(A.14), we get

$$Q_{rescaled}(q^2,m^2,L_P) = R_{rescaled}(q^2,m^2,L_P) = \Pi_0^{(2)}(q,L_P), \tag{A.17}$$

where

$$R_{rescaled} = \frac{iL_P^2}{4}\int_0^\infty \frac{dz_1}{z_1}\int_0^\infty \frac{dz_2}{z_2}\frac{1}{(z_1+z_2)^n}\exp\left(\frac{iL_P^2}{4}(z_1^{-1}+z_2^{-1})\right) \exp i\left(\frac{q^2z_1z_2}{z_1+z_2} - m_0(z_1+z_2)\right), \tag{A.18}$$

and $\Pi_0^{(2)}(q,L_P)$ is the quantity proportional to the gauge non-invariant part of the vacuum polarization tensor defined in Eqn.(A.11). Using our ansatz, we have shown that the gauge non-invariant part of the vacuum polarization tensor is a finite quantity and vanishes as $L_P \to 0$.

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