

On a Peculiar Family of Static, Axisymmetric, Vacuum Solutions of the Einstein Equations.

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Abstract

The Zipoy-Voorhees family of static, axisymmetric vacuum solutions forms an interesting family in that it contains the Schwarzschild black hole excepting which all other members have naked singularity. We analyze some properties of the region near singularity by studying a natural family of 2-surfaces. We establish that these have the topology of the 2-sphere by an application of the Gauss-Bonnet theorem. By computing the area, we establish that the singular region is ‘point-like’. Isometric embedding of these surfaces in the three dimensional Euclidean space is used to distinguish the two types of deviations from spherical symmetry.

PACS numbers: 04.20.-q, 04.20.jb

Typeset using REVTeX

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I. INTRODUCTION

The Weyl class of static, axisymmetric vacuum solutions of Einstein equation is very well studied and a systematic procedure for constructing solutions is available [1]. This class of solutions can be looked upon as describing finite size static bodies with axial symmetry. These admit a regular horizon only when the symmetry turns spherical and the solution is the Schwarzschild black hole. Among these is a class of so called ‘prolate spheroidal’ solutions found by Zipoy [2]. A special family of this class is the subject of this note.

Solutions of this family are parameterized by two parameters, m and A . All are asymptotically flat and all except $A = 1$, which is the Schwarzschild solution, have ‘naked singularity’. It is interesting to note that arbitrarily small deviations from the Schwarzschild black hole value of $A = 1$, which also introduces deviations from sphericity, convert the event horizon into an eternal naked (curvature) singularity. This makes the study of this family interesting and this is undertaken here.

In section II, we sketch an alternative derivation of the solution. Traditional approaches to construction of Weyl solutions have involved use of ‘Newtonian potentials’ corresponding to various source configurations eg [1,3]. We discovered this solution while studying the Kerr metric derivation given by Chandrasekhar [4] in some other context. The solution is arrived at essentially by specializing Chandrasekhar’s equations [4] leading to the Kerr solution, by switching off rotation ($\omega = 0$). This is the static limit of the Kerr geometry which would reduce to the Schwarzschild black hole by the demand of the existence of a regular horizon.

In section III, we analyze some properties of the solution. After stating some elementary properties we focus on its $t = \text{constant}$, $r = \text{constant}$ surfaces, Σ_2 . We analyze

its topology and intrinsic geometry and examine its ‘shape’ as viewed in a three dimensional Euclidean space. It turns out that for the solutions for $A > 1$ and r very close to $2m$, these surfaces are *not* completely embeddable in the Euclidean three dimensional space. This is a rather unusual and interesting feature of this family of solutions. What *physical* implication this feature has is at present an open question.

In the final section, we comment on possible physical implications of the results and open issues. The notation and conventions used are those of Chandrasekhar [4] with the metric signature $(+ - - -)$.

II. DERIVATION OF THE SOLUTION

The solution is fairly straightforward to derive. We begin by the general equations given in [4] for stationary, axisymmetric space-times and specialize to the static form of the ansatz by setting $\omega = 0$. The organization of equations and subsequent simplifications is identical to that given in [4]. Hence we give only the key steps.

The metric is given by,

$$ds^2 = e^{2\nu} dt^2 - e^{2\psi} d\phi^2 - e^{2\mu_2} dr^2 - e^{2\mu_3} d\theta^2. \quad (1)$$

The metric coefficient functions depend only on r, θ and $\Delta \equiv \exp[2(\mu_3 - \mu_2)]$ is freely specifiable. Defining $\beta \equiv \psi + \nu$, the vacuum equations become,

$$\partial_r(\sqrt{\Delta}\partial_r e^\beta) + \partial_\theta\left(\frac{1}{\sqrt{\Delta}}\partial_\theta e^\beta\right) \quad (2)$$

$$\partial_r\{e^\beta\sqrt{\Delta}\partial_r(\psi - \nu)\} + \partial_\theta\left\{\frac{e^\beta}{\sqrt{\Delta}}\partial_\theta(\psi - \nu)\right\} \quad (3)$$

$$\partial_r\partial_\theta\beta - \partial_r\beta\partial_\theta\mu_2 - \partial_\theta\beta\partial_r\mu_3 + \partial_r\nu\partial_\theta\nu + \partial_r\psi\partial_\theta\psi \quad (4)$$

$$\begin{aligned}
& 4\sqrt{\Delta}\{\partial_r\beta\partial_r\mu_3 + \partial_r\psi\partial_r\nu\} - \frac{4}{\sqrt{\Delta}}\{\partial_\theta\beta\partial_\theta\mu_2 + \partial_\theta\psi\partial_\theta\nu\} \\
& = 2e^{-\beta}\{\partial_r(\sqrt{\Delta}\partial_re^\beta) - \partial_\theta(\frac{1}{\sqrt{\Delta}}\partial_\theta e^\beta)\}
\end{aligned} \tag{5}$$

In order to admit possibility of a horizon, following Chandrasekhar, we set $\exp\{\beta(r, \theta)\} \equiv \sqrt{\Delta(r)}f(\theta)$. Equation (2) then leads to $\Delta = r^2 + d_1r + d_2, f(\theta) = \sin(\theta)$. In anticipation of spherical topology we have required $f(\theta)$ to vanish at the ‘poles’ at $\theta = 0, \pi$. We will assume that d_i constants are such that $\Delta = 0$ could have two real roots and we may write $\Delta(r) = (r - r_+)(r - r_-)$. The remaining equations now become equations for ν and μ_2 (say) by eliminating ψ in favour of ν, β and by eliminating μ_3 in favour of μ_2, Δ . The class of solutions we are interested in is obtained by assuming that $\nu(r, \theta)$ is independent of θ . The equations simplify further and lead to the following solution:

$$\begin{aligned}
\nu &= \frac{C}{r_+ - r_-} \ln\left\{\frac{r - r_+}{r - r_-}\right\} + D, & \bar{C} &\equiv C^2 - \frac{(r_+ - r_-)^2}{4}; \\
\mu_2 &= -\frac{C}{r_+ - r_-} \ln\left\{\frac{r - r_+}{r - r_-}\right\} - \frac{2\bar{C}}{(r_+ - r_-)^2} \ln\left\{\frac{\sin^2\theta}{\Delta} + \frac{4}{(r_+ - r_-)^2}\right\} + \Phi_0 \\
\mu_3 &= \mu_2 + \frac{1}{2}\ln\Delta, & \Delta &= (r - r_+)(r - r_-) \\
\beta &= \ln(\sin(\theta)) + \frac{1}{2}\ln(\Delta), & \psi &= \beta - \nu
\end{aligned} \tag{6}$$

Here C, D and Φ_0 (and of course $r_\pm$) are constants of integration. By redefining $r \rightarrow r + r_-$ one sees that r_- now appears only in the combination $r_+ - r_- \equiv 2m \geq 0$. For vacuum solution, the metric is always defined up to an over all multiplicative constant. We also have the freedom to redefine t and ϕ . Using these one can effectively absorb the constants D, Φ_0 . Trading C for mA , the final form of the metric can be written as:

$$\begin{aligned}
ds^2 &= F^A dt^2 - F^{-A} G^{-B} dr^2 - \Delta F^{-A} G^{-B} d\theta^2 - \Delta F^{-A} \sin^2(\theta) d\phi^2 \\
\text{where, } \Delta &= r^2 - 2mr, \quad F = \frac{\Delta}{r^2}, \quad G = 1 + \frac{m^2}{\Delta} \sin^2(\theta) \quad \text{and} \quad B \equiv A^2 - 1
\end{aligned} \tag{7}$$

Let us quickly remark that the solution can be expressed in a standard form [1] by the following identifications:

$$\rho = \sqrt{\Delta} \sin \theta \quad , \quad z = \frac{1}{2} \frac{d\Delta}{dr} \cos \theta \quad , \quad e^{2U} = F^A \quad \text{and} \quad e^{2k} = G^{-A^2}. \quad (8)$$

So that the metric is given by,

$$ds^2 = e^{2U} dt^2 - \rho^2 e^{-2U} d\phi^2 - e^{-2U+2k} (d\rho^2 + dz^2) \quad (9)$$

We also note that this solution corresponds to the ‘prolate’ family of solutions found by Zipoy [2]. In equations (43-45) of [2], put $\gamma = 0$. A trivial shift of his r gives the above solution with his β same as our A . Voorhees [3] also obtained the solution as an illustration of his method of constructing Weyl solutions. He also elaborated some of its properties.

In the next section we recall some of the basic properties and then focus on the $t = \text{constant}$ and $r = \text{constant}$ surfaces Σ_2 .

III. PROPERTIES OF THE SOLUTION

We have a two parameter family of static, axisymmetric solutions potentially containing a horizon. For $A = 1$ we obtain the Schwarzschild solution with mass m , r being the usual areal radial coordinate and $r = 2m$ being the horizon. The spherical symmetry is restored for this value. For large r , the behaviour of the solution shows that it is asymptotically flat with mA being the ADM mass. We are primarily interested in parameter values closer to the Schwarzschild case. Thus we take $m > 0$ and $A > 0$ so that the mass is also positive ($m = 0$ gives Minkowski space-time). We now concentrate on $A \neq 1$.

To locate possible singular regions, we compute curvature components. For example (in Chandrasekhar’s notation),

$$R_{0101} = -AG^B \Delta^{A-2} r^{-2A} m r \left(1 - \frac{(A+1)m}{r}\right) \quad (10)$$

As $r \rightarrow 2m$, $\Delta \rightarrow 0$. For $A \neq 1$ ($B \neq 0$), the net power of Δ becomes $A - A^2 - 1$ which is always negative for all *real* A . Thus we get a curvature singularity as $\Delta \rightarrow 0$. Other curvature components and Kretschmann scalar show similar behaviour. For $A = 1$, there is no contribution from G and the curvature component reduces to $-\frac{m}{r^3}$. Since for $r \rightarrow 2m$ itself we encounter curvature singularity, there is no point in exploring $r < 2m$ region. Since there is no other $r > 2m$ where the norm of the stationary Killing vector (g_{tt}) can vanish, the singularity must be naked. Thus except for $A = 1$ where one has a non singular Killing horizon, for all other values of A the space-time has naked singularity (diverging curvatures). This is an eternal naked singularity in an asymptotically flat space-time with *positive* ADM mass. From now on we limit ourselves to the non-singular coordinate range $r > 2m$. In order to gain some intuitive understanding of the region near the singularity as well as to see how the $A \neq 0$ ‘effects’ are seen in large r regions, we focus now on the $t = \text{constant}$ and $r = \text{constant}$ surfaces, generically denoted by Σ_2 . This naturally leads us to study the intrinsic geometry of Σ_2 .

The metric as given has all the coordinates as local coordinates. We have implicitly assumed the θ, ϕ to be the standard spherical polar angles on S^2 . As such the non-singular form of the metric is not directly in conflict with such an interpretation of the coordinates. (While ϕ can be taken as azimuthal angle referring to the axisymmetry, interpretation of θ as the polar angle is not automatic.) One way to check that Σ_2 indeed can have spherical topology is to verify the Gauss-Bonnet theorem. This is precisely what is done below.

It is straight forward to compute the Ricci scalar for the intrinsic metric on a Σ_2 . Since r is a constant, defining $\alpha \equiv \Delta F^{-A}$ and $\beta \equiv \frac{m^2}{\Delta}$ one obtains the Ricci scalar as,

$$R(g) = -\frac{1}{2} \left[\partial_\theta \left(\frac{\partial_\theta g_{\phi\phi}}{\det(g)} \right) + \frac{1}{\det(g)} \partial_\theta^2 g_{\phi\phi} \right] \quad (11)$$

$$= \frac{2}{\alpha} \{1 + \beta \sin^2 \theta\}^{(B-1)} \{(\beta B - 1) - \beta(1 + B) \sin^2 \theta\} \quad (12)$$

If topology of Σ_2 is indeed S^2 , then integral of $\sqrt{\det(g)}R(g)$ over the full range of the coordinates should equal 8π . In terms of $x = \cos\theta$, the integral can be expressed as,

$$\frac{1}{8\pi} \int_{\Sigma_2} \sqrt{\det(g)}R(g) = \beta^{B/2} \int_0^1 dx \frac{\frac{1+\beta}{\beta} - (B+1)x^2}{\left(\frac{1+\beta}{\beta} - x^2\right)^{1-B/2}} \quad (13)$$

Clearly for $B = 0$ (Schwarzschild), the right hand side is 1 as expected. Defining $\xi \equiv (1 + \beta)/\beta$, we get,

$$\frac{1}{8\pi} \int_{\Sigma_2} \sqrt{\det(g)}R(g) = (\xi - 1)^{-B/2} [(B+1) - 2\xi\partial_\xi] g(\xi, B) \quad (14)$$

where, $g(\xi, B) \equiv \int_0^1 dx (\xi - x^2)^{B/2}$

Note that $\xi > 1$ and hence the function $g(\xi, B)$ is well defined and is in fact differentiable. One can simplify the ξ dependence by defining $x = \sqrt{\xi}y$ in the integral. It is now easy to evaluate the ξ -derivative and check explicitly that the right hand side is indeed 1 for all values of B and β . *Thus topology of Σ_2 surfaces is indeed that of S^2 .*

Next we compute the areas of these spheres. This will allow us to estimate the ‘size’ of the region of high curvature. The area is given by,

$$\text{Area}(r) = 4\pi r^{2A} \Delta^{(1-A)^2/2} \left[\int_0^1 dx \{ \Delta + m^2(1 - x^2) \}^{(1-A^2)/2} \right] \quad (15)$$

Putting $r = m(\mu + 1)$, $x = \mu y$ and $\gamma = (1 - A^2)/2$, the area can be expressed as,

$$\text{Area}(\mu, A) = 4\pi m^2 \mu^{2-A^2} (\mu + 1)^{(1+A)^2/2} (\mu - 1)^{(1-A)^2/2} \int_0^{1/\mu} dy (1 - y^2)^\gamma \quad (16)$$

Since $r > 2m$ we have $\mu > 1$ and manifestly $\gamma < 1/2$. The area is clearly well defined and is a positive, finite quantity for all finite $\mu > 1$. Our interest is to estimate the behaviour of the area as $\mu \rightarrow 1$. Once again for $A = 1$, we recover the expected answer.

It is easy to see that for $0 < A < 2$, $A \neq 1$, the area vanishes as $r \rightarrow 2m$ while for larger values of A the area blows up. The vanishing area indicates that the singular region is ‘point-like’. For diverging area we can not say so. Since we are mainly interested

in values of A near the Schwarzschild value, we see that for these values the area vanishes and hence the curvature singularity is ‘point-like’. The coordinate value $r = 2m$ really corresponds to areal radial coordinate vanishing. For the subsequent analysis we will limit ourselves to solutions with ‘point-like’ singular region, i.e. to $0 < A < 2$.

Note: If we assume that the Ricci scalar satisfies $a \leq R \leq b$ everywhere on a spherical surface for some constants a, b depending on the surface, then the Gauss-Bonnet integral for spherical topology gives,

$$a \text{ (Area)} \leq 8\pi \leq b \text{ (Area)} \quad (17)$$

Thus if the area vanishes then the Ricci scalar must be unbounded above while if the area diverges then either the Ricci scalar must vanish in a precise manner (eg usual S^2 metric for large radius) or a must be negative or a can go to $-\infty$. One can check from our expressions that this is indeed the case. Observe that, the $\Delta \rightarrow 0$ behaviour of the Ricci scalar is different along the poles and along generic θ directions.

Having estimated the ‘size’ of the singular region, we now look for its ‘shape’. All our intuitive understanding of ‘shape’ of an object is based on its embedding in three dimensional *Euclidean* space. A natural formulation of ‘shape’ of Σ_2 is to look for its embedding in the Euclidean space of dimension 3. The embedding is to be such that the induced metric on the image is the same as the intrinsic metric on it, in short, an isometric embedding.

Using the symmetries of the Euclidean space and choosing the axis of symmetry to be the Z-axis, a natural ansatz for the embedding is:

$$X(\theta, \phi) = x(\theta) \cos\phi \quad , \quad Y(\theta, \phi) = y(\theta) \sin\phi \quad , \quad Z(\theta, \phi) = z(\theta) \quad . \quad (18)$$

The demand that the induced metric be axisymmetric (ϕ -independent) and diagonal,

leads to $y(\theta) = \pm x(\theta)$ and without loss of generality we choose the $+$ sign. Equating with the intrinsic metric then gives (prime denotes derivative with respect θ),

$$x(\theta) = \pm\sqrt{\alpha} \sin\theta \quad , \quad (z') = \pm\sqrt{\alpha} \left[(1 + \beta \sin^2\theta)^{-B} - \cos^2\theta \right]^{1/2} \quad (19)$$

Without loss of generality we can choose positive sign for $x(\theta)$ and negative sign for $z(\theta)$ equations above. The equation for $z(\theta)$ is an ordinary differential equation with one constant of integration which corresponds to the choice of origin. It is invariant under reflection about the equator. We can thus solve the equation for $0 \leq \theta \leq \pi/2$. For convenience, let us denote the expression in the square brackets in the above equation by $f(\theta, \beta, B)$. We are interested in the behaviour of this function for $0 \leq \theta \leq \pi/2$, for $0 < \beta$ and for $-1 < B < 3$ ($0 < A < 2$).

Clearly f is non-negative for all β, θ if $B \leq 0$. *however for $B > 0$ and for some β , it can be negative for θ near the poles, indicating failure of complete embedding.*

Computing the first derivative of f with respect to θ one sees that $\theta = 0, \pi/2$ are two extrema with a possibility of a third extremum at some $\hat{\theta}$. This third extremum, if exists, is always a minimum. For $\beta B < 1$, the third extremum does *not* exist while for $\beta B = 1$, it coincides with $\theta = 0$. Then f is non-negative for all θ . Thus for $\beta B \leq 1$, complete embedding is possible. For $\beta B > 1$, the minimum at $\hat{\theta} > 0$ exists and then f is negative for $0 \leq \theta \leq \theta_{max}$. The θ_{max} is to be obtained by solving the transcendental equation $f = 0$. Thus for $\beta B > 1$, caps nears the poles are not embeddable.

The condition $\beta B > 1$ corresponds to r near $2m$. Since for B close to zero, β is very large and thus corresponds to r very close to $2m$. The farthest r would be for the largest B ($= A^2 - 1$) which for us is 3 ($A < 2$). This translates to the farthest r being $3m$. Thus for $r > 3m$, we will always have complete embedding. Curiously, $r = 3m$ happens to be the radius of the photon circular orbit for the Schwarzschild black hole.

While precise shape for complete embedding can be plotted by numerically integrating the equation, that is however not our main concern. In general the ‘shape’ will be an ellipsoid. For B close to zero, the ellipsoid will shrink as $\Delta \rightarrow 0$.

IV. DISCUSSION

One of the striking features of this family of solutions is that for a slightest deviation from spherical symmetry, the horizon disappears and instead a naked singularity appears which is eternal by virtue of the staticity of the space-time. The departure from sphericity, equivalently from the Schwarzschild geometry, is indicated by the parameter $A \neq 1$. Note that $A = +1$ corresponds to the Schwarzschild black hole while $A = -1$ corresponds to the negative mass Schwarzschild solution. We are assuming of course that the parameter m is positive. In order to include the Schwarzschild black hole, we have chosen $A > 0$. The parameter value $A = 2$, demarcates whether the area of Σ_2 for r near $2m$ vanishes ($A \leq 2$) or diverges ($A > 2$). Likewise, the Schwarzschild value $A = 1$ demarcates whether the surface Σ_2 for r near $2m$ is completely embeddable ($A \leq 1$) or is *partially* embeddable ($A > 1$).

Of course it is more likely that the solutions are valid for r sufficiently large and that a physical body will fill in the interior region. This leads one to the question of a source configuration to which the above family is an exterior space-time [3]. While it would be natural to expect that a two surface across which an interior solution is matched to the above solution is indeed one of the Σ_2 surfaces, it is by no means obvious that it must be so. For instance one could have a 2-surfaces defined by $r = r(\theta)$. The Killing norm however will not be constant over such surfaces. For large r values this will correspond to the matching surface not having constant Newtonian potential. One could of course analyze the shape of any such matching surfaces by similar techniques. One must em-

phasize that the Euclidean space used for the isometric embedding is *not* the physical space whose metric near the vicinity of the surface is not Euclidean. It has been used as a way to discriminate among different members of the family ¹.

These solutions could be viewed as a subset (static) of possible end states of a non-spherical collapse. Black hole uniqueness results would indicate that if censorship holds then either spherical symmetry must be restored (staticity results) or rotation must set in (only stationarity results) during the collapse. For the possibility of static end state, it is only a single special value for which a naked singularity is avoided. In the light of these observations, it would be exceedingly interesting to see what features emerge in an ‘axisymmetric’ collapse with the above family being used for exterior matching just as the Schwarzschild is used in the study of spherical collapse.

Acknowledgments: G. D. would like to acknowledge a helpful e-mail correspondence with Kapil Paranjape of the Institute of Mathematical Sciences regarding the formulation of the notion of ‘shape’. He would also like to thank IUCAA for the warm hospitality and conducive atmosphere provided during his visit during which this work was completed.

¹While the paper was being written, a preprint by N. Pelavas, N. Neary and K. Lake appeared on the e-print archive which applies similar methods to study properties of the instantaneous ergo surface of the Kerr metric, gr-qc/0012052.

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