

Tolman-Bayin type static charged fluid spheres in general relativity

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ABSTRACT

In a static spherically symmetric Einstein-Maxwell spacetime the class of astrophysical solution found out by Ray and Das (2002) and Pant and Sah (1979) are revisited here in connection to the phenomenological relationship between the gravitational and electromagnetic fields. It is qualitatively shown that the charged relativistic stars of Tolman (1939) and Bayin (1978) type are of purely electromagnetic origin. The existence of this type of astrophysical solutions is a probable extension of Lorentz's conjecture that electron-like extended charged particle possesses only 'electromagnetic mass' and no 'material mass'.

Key words: gravitation – relativity – stars : general – stars : interior.

1 INTRODUCTION

The study of the interior of stars is always fascinating to the astrophysicists, specially in connection to general theory of relativity. This is obvious because of the fact that towards the late stages of stellar evolution, general relativistic effects become much important. One of the remarkable works in this direction was that of the Tolman (1939) solutions. Tolman extensively studied the stellar interior and provided a class of explicit solution in terms of known analytic functions for the static, spherically symmetric equilibrium fluid distribution. Subsequently Wyman (1949), Leibovitz (1969) and Whitman (1977) generalized some of Tolman's solutions. Bayin (1978) also obtained some more new analytic solutions related to static fluid spheres using the method of quadratures.

Recently we (Ray & Das, 2002) have obtained the charged generalization of Bayin's work (1978) motivated by the idea that in stellar astrophysics the coupled Einstein-Maxwell field equations may have some physical implications. In connection to singularity problem it is observed that in the presence of charge, the gravitational collapse of a spherically symmetric distribution of matter to a point singularity may be avoided. The mechanism is such that the gravitational attraction is counterbalanced by the repulsive Coulombian force in addition to the thermal pressure gradient due to fluid. Also, it is seen that the presence of the charge function serves as a safety valve, which absorbs much of the fine-tuning, necessary in the uncharged case (Ivanov, 2002). Thus, the problem of coupled charge-matter distributions in general relativity has received considerable attention.

The present paper is based on the simple investigation of the solutions already obtained by us (Ray & Das, 2002) and Pant & Sah (1979) in connection to the electromagnetic origin of the gravitational mass. It is worthwhile to mention here that there is a fairly long history of investigations about the nature of the mass of electron. Einstein (1919) himself believed that "... of the energy consti-

tuting matter three-quarters is to be ascribed to the electromagnetic field, and one-quarter to the gravitational field" whereas Lorentz's (1904) conjecture of extended electron was that "there is no other, no 'true' or 'material' mass," and thus provides only 'electromagnetic masses of the electron'. Wheeler (1962) also believed that electron has a 'mass without mass'. Feynman (1964) termed this type of models as 'electromagnetic mass models' in his classic volume. Starting from 60's in the last century several authors (e.g., Florides, 1962; Cooperstock & De La Cruz, 1978; Tiwari et al., 1984; Gautreau, 1985; Grøn, 1986; Ponce de Leon, 1987; and the references therein) took up the problem again and studied electromagnetic mass models for the static spherically symmetric charged perfect fluid distribution in the framework of general relativity. Very recently the idea is extended to the Einstein-Cartan theory and Kaluza-Klein theory by adding torsion and higher dimension respectively (Tiwari & Ray, 1997; Ponce de Leon, 2003). Most of these workers exploit an equation of state $\rho + p = 0$ where, in general, the matter density $\rho > 0$ and pressure $p < 0$. This type of equation of state implies that the matter distribution under consideration is in tension and hence the matter is known in the literature as a 'false vacuum' or 'degenerate vacuum' or ' ρ -vacuum' (Davies, 1984; Blome & Priester, 1984; Hogan, 1984; Kaiser & Stebbins, 1984).

It is interesting to note that in the present study, even though the solutions related to pressure and density in general follow the ordinary equation of state, viz., $\rho + p \neq 0$ but ultimately in connection to electromagnetic mass models it turns out to be the exotic kind of equation of state (Davies, 1984; Blome & Priester, 1984; Hogan, 1984; Kaiser & Stebbins, 1984) in both the cases of Bayin and Tolman solutions. We have investigated here that related to this type of vacuum- or imperfect-fluid equation of state the charged analogue of Bayin (1978) and Tolman (1939) type astrophysical class of solution show the electromagnetic field dependency of gravitational mass. Therefore, the existence of this type

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of solutions, in our opinion, is a probable extension of Lorentz's conjecture in connection to astrophysical models.

2 EINSTEIN-MAXWELL FIELD EQUATIONS

We write the line element for static spherically symmetric spacetimes in the form

$$ds^2 = A^2 dt^2 - B^2 dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

in the standard coordinates $x^i = (t, r, \theta, \phi)$, where the quantities $A(r)$ and $B(r)$ are the metric potentials.

The Einstein-Maxwell field equations, for the metric (1) in the co-moving coordinates read as

$$\frac{1}{B^2} \left(\frac{2B'}{Br} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 8\pi\rho + \frac{q^2(r)}{r^4}, \quad (2)$$

$$\frac{1}{B^2} \left(\frac{2A'}{Ar} + \frac{1}{r^2} \right) - \frac{1}{r^2} = 8\pi p - \frac{q^2(r)}{r^4}, \quad (3)$$

$$\frac{1}{B^2} \left[\frac{A''}{A} - \frac{A'B'}{AB} + \frac{1}{r} \left(\frac{A'}{A} - \frac{B'}{B} \right) \right] = 8\pi p + \frac{q^2(r)}{r^4}, \quad (4)$$

where the prime denotes differentiation with respect to radial coordinate r only. In the equation (2) - (4), the quantities ρ , p and q represent the energy density, isotropic pressure and total electric charge respectively. The total charge within a sphere of radius r is defined as

$$q(r) = 4\pi \int_0^r J^0 r^2 AB dr, \quad (5)$$

J^i being the 4-current takes here the form, via the electromagnetic field F^{01} , as

$$F^{01} = \frac{q(r)}{ABr^2}. \quad (6)$$

Now, eliminating p from equations (3) and (4) and assuming $A'/Ar = C(r)$ one can get

$$\left(\frac{1}{B^3 r^2} + \frac{C}{B^3} \right) dB - \left(\frac{1}{B^2} \right) dC - \left(\frac{B^2 - 1}{B^2 r^3} + \frac{C^2 r}{B^2} - \frac{2q^2}{r^5} \right) dr = 0. \quad (7)$$

which is a Pfaffian differential equation in three dimensions having the general form as

$$f_1(B, C, r)dB + f_2(B, C, r)dC + f_3(B, C, r)dr = 0. \quad (8)$$

3 ELECTROMAGNETIC MASS MODELS FOR STATIC CHARGED FLUID SPHERES

3.1 Bayin's class of solution

The Pfaffian differential equation (7) can be solved in different ways as shown by us (Ray & Das, 2002) in details. It is shown that in terms of $B(r)$ when $C(r)$ is known, the Pfaffian differential equation (7) becomes

$$\frac{dB}{dr} = \left[\frac{1 - \frac{2q^2}{r^2}}{\left(C + \frac{1}{r^2}\right)r^3} \right] B^3 + \left[\frac{C^2 r - \frac{1}{r^3} + \frac{dC}{dr}}{C + \frac{1}{r^2}} \right] B. \quad (9)$$

Also, in terms of $C(r)$ when $B(r)$ is given, the Pfaffian differential equation (7) modifies to

$$\frac{dC}{dr} = \left(\frac{1}{Br^2} \frac{dB}{dr} - \frac{B^2 - 1}{r^3} \right) + \left(\frac{1}{B} \frac{dB}{dr} \right) C - C^2 r + \frac{2B^2 q^2}{r^5}, \quad (10)$$

which is a Riccati equation for $C(r)$ with known value of charge q .

By solving these differential equations (9) and (10), and also some other simple cases we (Ray & Das, 2002) obtained the solutions for Einstein-Maxwell field equations related to Bayin (1978) type astrophysical class of models. The solutions thus obtained for the parameters A , B , ρ , p and q respectively the gravitational potentials, energy density, isotropic pressure and electric charge are involved with several integration constants. Some of these may, in principle, be determined by matching of the interior solution to the exterior Reissner-Nordström metric at the boundary $r = a$ of the spherical matter distribution. The exterior Reissner-Nordström metric is given by

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2} \right) dt^2 - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2} \right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (11)$$

Now, considering the metric components g_{00} , g_{11} and $\frac{\partial g_{00}}{\partial r}$ to be continuous across the boundary $r = a$ of the sphere and assuming for the total charge on the sphere

$$q(a) = Ka^n, \quad (12)$$

one can get the following cases of the gravitational mass (vide equations (53), (56), (61), (65), (69) and (72) of Section 4 in Ray & Das (2002)) in the explicit forms with electric charge.

Case I (i): For $n = 1$

$$m = q^2 + a_0 a_1 \left(\frac{q}{K} \right)^2 + a_1^2 \left(\frac{q}{K} \right)^3, \quad (13)$$

(ii): For $n = 3$

$$m = a_1^2 \left(\frac{q}{K} \right) + 4K^2 \left(\frac{q}{K} \right)^{5/3} + a_0 a_1 \left(\frac{q}{K} \right)^{2/3}, \quad (14)$$

Case II (i): For $n = 1$

$$m = \frac{1}{W_0^2} \left[W_0^2 (1 - 2K^2) + \left(\frac{q}{K} \right)^2 \left(C_1 - \frac{q^2}{K^2} \right) \right]^{1/2} \times \left[\frac{q}{K} \right]^3, \quad (15)$$

(ii): For $n = 3$

$$m = \frac{1}{W_0^2} \left[W_0^2 + C_1 \left(\frac{q}{K} \right)^{2/3} + (W_0^2 K^2 - 1) \left(\frac{q}{K} \right)^{4/3} \right]^{1/2} \times \left[\frac{q}{K} \right] - 2K^{1/3} q^{5/3}, \quad (16)$$

Case III: For $n = 1$

$$m = \frac{1 + K^2}{C_3} = (K^2 - 1) \left[C_3 \left(\frac{q}{K} \right) - 2 \right] \left[\frac{q}{K} \right], \quad (17)$$

Case IV: For $n = 1$

$$m = 3C_6^2 \left[3C_5 + \left(\frac{q}{K} \right)^3 \right] \left[\frac{q}{K} \right]^4, \quad (18)$$

where K , a_0 , a_1 , W_0 , C_1 , C_3 , C_5 and C_6 all are constant quantities. Now, m for the cases I(i), I(ii) and IV are as usual positive

whereas for the rest of the cases II(i), II(ii) and III the conditions for positivity are $(1/2) > K^2 > (q^2/C_1)$, $K > (1/W_0)$ and $1 < K < (C_3q/2)$ respectively.

It is observed from the explicit forms of the above set of expressions that the effective gravitational mass m , along with the central pressures and densities at $r = 0$ (vide equations (22), (27), (31), (40), (45) and (21), (28), (32), (39) respectively in Ray & Das (2002)), is related to the charge q of equation (12) of the spherical system. Therefore, vanishing of the charge makes all the physical quantities including the gravitational mass also to vanish. This means that the gravitational mass originates from the electromagnetic field alone. Thus, the gravitational mass is purely 'electromagnetic mass' (Lorentz, 1904) and this type of model is known as 'electromagnetic mass model' in the literature (Feynman, 1964). It is relevant to note here that this particular important feature of the solution set, viz., the electromagnetic nature of the gravitational mass is obviously not available in the uncharged case of Bayin (1978) and thus indicates that the presence of charge allows for a wider range of behaviour.

3.2 Tolman's solution VI

In the introduction we have mentioned that motivated by the work of Tolman (1939), a similar kind of new class of solution was found by Bayin (1978) and hence in view of the results of sub-section 3.1 related to Bayin's work it will be interesting to examine the solutions of Tolman whether they are also a member of electromagnetic mass models. As a ready-made example we would like to present here the solution obtained by Pant & Sah (1979) to meet our, at least partial, requirement. For a static spherically symmetric distribution of charged fluid the solution set (vide equations (10a), (10b), (10c) and (11) in Pant & Sah (1979)) is as follows.

$$A^2 = e^\nu = br^{2n}, \quad (19)$$

$$B^{-2} = e^{-\lambda} = c, \quad (20)$$

$$\rho = \frac{1}{16\pi r^2} [1 - c(n-1)^2], \quad (21)$$

$$p = \frac{1}{16\pi r^2} [c(n+1)^2 - 1], \quad (22)$$

$$\sigma = \pm \frac{1}{4\pi r^2} \left[\frac{c}{2} \{1 - c(1 + 2n - n^2)\} \right]^{1/2}, \quad (23)$$

$$E^2 = \frac{1}{2r^2} [1 - c(1 + 2n - n^2)], \quad (24)$$

where

$$b = a^{-2n} \left[1 - \frac{2m}{a} + \frac{q^2}{a^2} \right], \quad (25)$$

$$c = \left[1 - \frac{2m}{a} + \frac{q^2}{a^2} \right] = \left[1 - \frac{2q^2}{a^2} \right] (1 + 2n - n^2)^{-1}. \quad (26)$$

The above set of solutions, in view of c , with $\Lambda = 0$ and $B = 0$ represents the charged analogue of Tolman's (1939) solution VI and thus in the absence of the total charge q reduces to the neutral one (the sub-case C of uncharged fluid sphere in the Pant & Sah (1979)). Now, the equation (2) can be expressed in the form

$$B^{-2} = 1 - \frac{2m(r)}{r}, \quad (27)$$

where the gravitational mass $m(r) = M(r) + \mu(r)$ being defined as

$$M(r) = 4\pi \int_0^r \rho r^2 dr \quad (28)$$

and

$$\mu(r) = \int_0^r (q^2/2r^2) dr, \quad (29)$$

respectively the Schwarzschild mass and the mass equivalence of electromagnetic field. Hence, the total gravitational mass, $m(r = a)$, can be calculated as

$$m = \frac{na^2(2-n) + 2q^2}{2(1+2n-n^2)a}. \quad (30)$$

If we now make the specific choice $n = 0$ for the parameter n appearing in the above solution set then one get the following expressions.

$$A^2 = \left[1 - \frac{2m}{a} + \frac{q^2}{a^2} \right], \quad (31)$$

$$B^{-2} = \left[1 - \frac{2m}{a} + \frac{q^2}{a^2} \right] = \left[1 - \frac{2q^2}{a^2} \right], \quad (32)$$

$$\rho(r) = \frac{1}{8\pi r^2} \left[\frac{q^2}{a^2} \right], \quad (33)$$

$$p(r) = -\frac{1}{8\pi r^2} \left[\frac{q^2}{a^2} \right], \quad (34)$$

$$\sigma(r) = \pm \frac{1}{4\pi r^2} \left[\frac{q}{a} \right] \left[1 - \frac{2q^2}{a^2} \right]^{1/2}, \quad (35)$$

$$E(r) = \frac{q}{ar}, \quad (36)$$

and

$$m = \frac{q^2}{a}. \quad (37)$$

Thus, for vanishing electric charge all the physical quantities including gravitational mass vanish and the spacetime becomes flat. It is interesting to note that, in the present situation, the equations (33) and (34) related to the isotropic pressure and matter density provide an equation of state $\rho + p = 0$, which is known as the vacuum- or imperfect-fluid equation of state. As is evident, from the equations (21) and (22), this is not true for the general case when $n \neq 0$ and can be read as

$$\rho + p = \frac{1}{4\pi r^2} \left[\frac{n(a^2 - 2q^2)}{a^2(1 + 2n - n^2)} \right]. \quad (38)$$

Hence starting from a perfect fluid type equation of state via $n = 0$ we are arriving at the imperfect-fluid type equation of state and thus n here is taking a definite and peculiar role for deciding the form of the equation of state. This particular aspect is also true via the equations (2) and (3) for the equations (19) and (20) which reduce to the equations (31) and (32) respectively, with $n = 0$ when we get $\nu + \lambda = 0$. This again, for the Reissner-Nordström metric (equation (11)) related to the spherically symmetric static charged fluid distribution, can be expressed in the form $g_{00}g_{11} = -1$. Thus, in view of equations (2) and (3), we see that for the

boundary condition $\nu + \lambda = 0$ one can get $\rho + p = 0$ and vice versa, so that $\lambda = -\nu \leftrightarrow p = -\rho$.¹ This result means that if $\rho > 0$ then must be $p < 0$ for the inside of the fluid sphere though, in general, ρ and p are positive for the condition $0 \leq n \leq 1$. This partially admits the comment by Ivanov that "... electromagnetic mass models all seem to have negative pressure". Partially because, in our opinion, there are some examples of electromagnetic mass models where positive pressures are also available will be shown elsewhere.

4 CONCLUSIONS

We have revisited in the present paper the work already done by us (Ray & Das, 2002) and that one of Pant & Sah (1979) motivated by the fact that the gravitational mass m for a charged matter distributions always can be seen to be divided into two parts, viz., (i) the Schwarzschild mass $M(r)$ and (ii) the mass equivalence of electromagnetic field $\mu(r)$ as is evident from the equation (27). Thus, the total mass is increasing due to electromagnetic energy (Florides, 1964; Mehra, 1980) which is obviously an extra feature in comparison to the neutral case. Keeping this aspect in mind we wanted to examine whether the gravitational mass obtained by us (Ray & Das, 2002) in one of our previous papers is of purely electromagnetic origin or not. We have, in the present simple investigation, shown that the charged generalized solutions of Bayin (1978) is purely of electromagnetic origin. In this connection, we have also shown, by citing the solution of Pant & Sah (1979), that the charged generalization of Tolman's solution VI (1939) yields electromagnetic mass models which, of course, needs further investigations with a direct study of the Tolman's whole set of solutions by inclusion of charge.

We would also like to mention here that the works done by different investigators on electromagnetic mass models so far are mainly concerned with the structure of the classical electron (special references are Gautreau, 1985 and Tiwari et al, 1986). Even though Tiwari et al. (1986) find astrophysically interesting Lane-Emden equations in connection to electromagnetic mass models but at the same time, instead of studying the stellar structures, they apply the radii of some of the models for the comparison with the classical electron radius. This particular aspect of electromagnetic mass models related to Lane-Emden equations in the astrophysical context needs further investigations.

As is mentioned in Ray & Das (2002), to justify the present work with a charged fluid distribution, that even though the astrophysical systems are by and large electrically neutral, recent studies do not rule out the possibility of the existence of massive astrophysical systems that are not electrically neutral (Treves & Turella, 1999). The mechanism, though not completely understood, is mainly related to the acquiring a net charge by accretion from the surrounding medium. On the other hand, there are some other views of acquiring charge by a compact star during its collapse from the supernova stage. In this regard it will be worth mentioning that to study the effect of electric charge in compact stars Ray et al. (2003) assume an *ansatz* such that $\sigma = \alpha\rho$ where α is related to the charge fraction f as $\alpha = 869.24f$ and show by numerical calculation that in order to see any appreciable effect on the phenomenology of the compact stars, the total electric charge is to be $\sim 10^{20}$ Coulomb. Therefore, in our opinion, even such a remote possibility

gives enough scope to theoretical speculations and hence the corresponding modeling and investigations become as much pertinent for these cases as for the established neutral systems.

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¹ A coordinate-independent statement of the relation $g_{00}g_{11} = -1$ and hence $\nu + \lambda = 0$ is given by Tiwari et al. (1984).