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Reviewed work(s):
Source: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 270, No. 1343 (Dec. 11, 1962), pp. 553-561
Published by: The Royal Society
Stable URL: http://www.jstor.org/stable/2414566
Accessed: 30/11/2011 06:17

Neutrinos and the arrow of time in cosmology

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(*Communicated by F. Hoyle, F.R.S.—Received* 5 *June* 1962)

Hogarth's approach to the time-symmetric electrodynamics of Wheeler & Feynman is extended to apply to the neutrinos. It is shown that in some cosmological models neutrinos travelling both into the past and the future exist whereas in others only those going into the the future can exist. The Einstein–de Sitter and the steady-state models are the respective examples of the two cases. An interesting way of testing the various cosmologies by means of neutrino emitters and receivers is suggested.

1. Introduction

It is well known that the fundamental equations of electro-magnetism—Maxwell's equations—are time-symmetric. Yet in order to explain the natural phenomena it is necessary to discard certain solutions of these equations for purely empirical reasons. For example, the fields present in the neighbourhood of an oscillating charge are described by means of the retarded potentials. A second solution involving advanced potentials is also possible because of the time symmetry of the equations; but this is rejected on the grounds that it contradicts causality. This apparently arbitrary rejection of a whole class of solutions leads to the phenomena such as radiation that specify a definite arrow of time.

This state of affairs was regarded as unsatisfactory by Wheeler & Feynman (1945, 1949) who developed electromagnetism from a single-action principle based on direct interparticle action. Two particles contribute to the action if and only if they lie on each other's null cone. From this it is possible to derive formally all Maxwell's equations. The theory has a further attractive feature that it rules out self-action and the infinities associated with it.

However, as the action is defined impartially along both the branches of the null cone there is no distinction between past and future—or between advanced and retarded potentials. The usual retarded fields $F_{\text{ret.}}$ of a source O are replaced by a time-symmetric field

$$F_0 = \tfrac{1}{2}(F_{\text{ret.}} + F_{\text{adv.}}). \tag{1}$$

The objection that (1) contradicts experience is met by saying that (1) refers to an isolated source in an otherwise empty space whereas experience refers to the universe which is far from being empty. In fact, Wheeler & Feynman were able to show that in a static Euclidean universe the net contribution of the rest of the particles (collectively known as the 'absorber' in the theory) at O is a field given by

$$R = \tfrac{1}{2}(F_{\text{ret.}} - F_{\text{adv.}}). \tag{2}$$

This when added to F_0 gives the familiar solution $F_{\text{ret.}}$.

It is somewhat puzzling at first to find that a time-symmetric theory in a time-symmetric universe can lead to a solution that is not time-symmetric. Indeed, a little consideration shows that the above solution is not the only one. For example, by reversing the arrow of time throughout the above argument one can obtain another possible solution $F_{\text{adv.}}$. Any normalized linear combination of the two solutions is also a solution. This was recognized by Wheeler & Feynman and a way out of this difficulty was also suggested by them. They postulated unsymmetrical initial conditions and argued on statistical grounds that this asymmetry would persist and favour $F_{\text{ret.}}$ to $F_{\text{adv.}}$.

2. Cosmological considerations

This extra postulate of unsymmetrical initial conditions was criticized by Hogarth (1962) who proposed an alternative approach. This approach is based on the generally accepted evidence that the universe is expanding. If one defines an arrow of time with respect to which the universe expands then one expects the state of affairs on the 'past' and the 'future' parts of the null cone of O to be in general different. Hogarth accordingly makes a distinction between the past and the future absorbers with respect to O. The behaviour of these two absorbers will in general depend on the type of cosmological model chosen to describe the universe. Hogarth's subsequent argument is given below in a somewhat different form. Only models that continually expand will be considered throughout.

Suppose the total field near the source O is given by

$$F_{\text{tot.}} = AF_{\text{ret.}} + BF_{\text{adv.}}. \tag{3}$$

Of this $F_{\text{ret.}}$ travels into the future and interacts with the future absorber. The advanced effects of this interaction will reach O instantaneously after the propagation of $F_{\text{ret.}}$. These effects add up to R in a static universe, as was shown by Wheeler & Feynman. If the fields are Fourier analyzed and a single monochromatic component is considered, the contribution of the advanced effects from the future absorber can be written as fR, where f is a complex number depending on the frequency of the field and the model of the universe. Similarly the response of the past absorber to $F_{\text{adv.}}$ can be written as $-pR$. For the static Euclidean universe $f = p = 1$.

Adding the total contribution of the two absorbers to the basic field (1) at O gives the total field near O

$$F_{\text{tot.}} = F_0 + AfR - BpR. \tag{4}$$

Comparing (4) with (3) and equating coefficients of $F_{\text{ret.}}$ and $F_{\text{adv.}}$ gives the following equations for A, B

$$\tfrac{1}{2}Af - \tfrac{1}{2}Bp + \tfrac{1}{2} = A, \tag{5}$$

$$-\tfrac{1}{2}Af + \tfrac{1}{2}Bp + \tfrac{1}{2} = B. \tag{6}$$

Adding (5) and (6) gives

$$A + B = 1 \tag{7}$$

which is the normalizing condition. Except when $f = p = 1$ the solution is

$$A = \frac{1-p}{2-f-p}, \quad B = \frac{1-f}{2-f-p}. \tag{8}$$

In the exceptional case (5) and (6) reduce to (7). This is the situation encountered by Wheeler & Feynman. It is clear that an extra postulate is necessary to determine A and B.

However, in the general case there is a unique solution. It is interesting to note that cosmological models with $f = 1$ give only retarded solutions *whatever the value of p may be*, provided $p \neq 1$. Similarly models with $p = 1$ admit only advanced solutions. Hogarth examined various cosmological models with a view to determining whether they satisfy the required criterion $f = 1 \neq p$. He found that general models which maintain a constant density in spite of the expansion (and therefore involve continuous creation of matter) are more favourably placed than models based on conservation of matter. He called the above two types of models class I and class II universes respectively.

3. Application to neutrinos

An interesting picture emerges when one considers the above description in terms of particles instead of fields. Then one has a source emitting photons into past and future. Those that travel into the future are scattered by the future absorber. The scattering at each point is again into past as well as future. The net effect of both the absorbers is to produce photons going only into the future in the actual universe. In the class I universes there is in general a sufficient number of scatterers per unit proper volume to ensure the eventual scattering of every photon emitted by O. The class II universes do not have this advantage and therefore fail to satisfy the condition $f = 1$ in general.

When looked at from this point of view the situation presents new possibilities. One can ask: what will happen in the case of the neutrinos because of their very low reaction cross-section? It is proposed to consider this problem in the rest of the paper.

Though there is no classical theory of the neutrino (analogous to the electromagnetic theory for the photon) the three essential things to describe the above picture are fortunately available. These are: (*a*) the mode of transmission of a free neutrino in the expanding universe; (*b*) the cross-section of elastic scattering of neutrinos by the particles in the absorber and (*c*) the refractive index of the medium for neutrino propagation. These will be considered in this section.

(*a*) *The mode of transmission of a free neutrino in an expanding universe*

Because of the simplicity of geometrical description it is proposed to work in a conformally flat space with metric given by

$$ds^2 = e^{2\zeta(\tau)}[c^2 d\tau^2 - dx^2 - dy^2 - dz^2]. \tag{9}$$

It will be assumed that at the time of emission $\tau = \tau_0$ and $\zeta(\tau_0) = 0$. The transformation

$$t - t_0 = \int_{\tau_0}^{\tau} \zeta(\tau')\,d\tau', \quad R(t) = \exp[\zeta(\tau)] \tag{10}$$

reduces (9) to the familiar Robertson–Walker form

$$ds^2 = c^2 dt^2 - R^2(t)[dx^2 + dy^2 + dz^2], \tag{11}$$

where t is the cosmic time.

The neutrino propagation in curved space time is given by the Dirac equation

$$\gamma^i \nabla_i \psi = 0, \tag*{(12}$$

where γ^i are the Dirac matrices satisfying the commutation relations

$$\gamma^i\gamma^k + \gamma^k\gamma^i = 2g^{ik} \quad (i, k = 1, 2, 3, 4). \tag{13}$$

$\nabla_i \psi$ is the covariant derivative of the 4-spinor ψ,

$$\nabla_i \psi = \partial\psi/\partial x^i - \Gamma_i \psi, \tag{14}$$

where the matrices Γ_i satisfy the relations

$$\partial\gamma_i/\partial x^k - \Gamma^l_{ik}\gamma_l + \gamma_i\Gamma_k - \Gamma_k\gamma_i = 0. \tag{15}$$

These formulae have been given by Wheeler & Brill (1957) and many other authors.

With $x' = x$, $x^2 = y$, $x^3 = z$, $x^4 = \tau$ and with g_{ik} given by (9) the relations (13) and (15) are satisfied if

$$\gamma^i = e^{-\zeta}\hat{\gamma}^i, \quad \Gamma_\mu = -\tfrac{1}{2}\dot{\zeta}\hat{\gamma}_\mu\hat{\gamma}_4, \quad \Gamma_4 = 0 \quad (i = 1, 2, 3, 4, \mu = 1, 2, 3), \tag{16}$$

where a hat specifies the corresponding quantity for the Minkowski space. The equation (12) then reduces to

$$\hat{\gamma}^i \partial\hat{\psi}/\partial x^i = 0 \tag{17}$$

where

$$\hat{\psi} = \psi e^{\frac{3}{2}\zeta}. \tag{18}$$

Thus $\hat{\psi}$ satisfies the wave equation in the Minkowski space. The solution may be further restricted by the requirement that the particle is left or right handed, i.e.

$$i\gamma_5\psi = \pm\psi. \tag{19}$$

Equations (17) and (19) can be solved explicitly by using a specific representation. In the Dirac representation

$$\hat{\gamma}_1 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}, \quad \hat{\gamma}_2 = \begin{bmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{bmatrix}, \quad \hat{\gamma}_3 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix},$$

$$\hat{\gamma}_4 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad i\hat{\gamma}_5 = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}.$$

The condition (19) then requires that, with $i\gamma_5\psi = \psi$,

$$\hat{\psi} = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_1 \\ \chi_2 \end{bmatrix}, \tag{20}$$

where χ_1, χ_2 satisfy the equations

$$\frac{\partial\chi_2}{\partial x} - i\frac{\partial\chi_2}{\partial y} + \frac{\partial\chi_1}{\partial z} + \frac{1}{c}\frac{\partial\chi_1}{\partial\tau} = 0, \tag{21}$$

$$\frac{\partial\chi_1}{\partial x} + i\frac{\partial\chi_1}{\partial y} - \frac{\partial\chi_2}{\partial z} + \frac{1}{c}\frac{\partial\chi_2}{\partial\tau} = 0. \tag{22}$$

Thus it is necessary to consider only two components of ψ.

Waves representing radially outgoing neutrinos from O behave at large distance r from O as a linear combination of

$$\frac{1}{r}\exp\left[-\mathrm{i}\omega_0(\tau - r/c)\right]\begin{bmatrix}1+\cos\theta\\ \sin\theta\,\mathrm{e}^{\mathrm{i}\phi}\end{bmatrix}, \quad \frac{1}{r}\exp\left[-\mathrm{i}\omega_0(\tau - r/c)\right]\begin{bmatrix}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}\\ 1-\cos\theta\end{bmatrix}, \tag{23}$$

where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

Such a solution will be denoted by $\psi_{\text{ret.}}$. Similarly $\psi_{\text{adv.}}$ at large r is a linear combination of

$$\frac{1}{r}\exp\left[-\mathrm{i}\omega_0(\tau + r/c)\right]\begin{bmatrix}1-\cos\theta\\ -\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}\end{bmatrix}, \quad \frac{1}{r}\exp\left[-\mathrm{i}\omega_0(\tau + r/c)\right]\begin{bmatrix}-\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}\\ 1+\cos\theta\end{bmatrix}. \tag{24}$$

The frequency of the wave on the τ scale is ω_0. On the t-scale it is

$$\omega = \omega_0(\mathrm{d}\tau/\mathrm{d}t) = \omega_0\mathrm{e}^{-\zeta}. \tag{25}$$

Since it is the t-scale that characterizes the atomic processes the energy of the neutrino is not $\hbar\omega_0$ but

$$E = \hbar\omega = \hbar\omega_0\mathrm{e}^{-\zeta} = \hbar\omega_0 R(t_0)/R(t). \tag{26}$$

This is the well-known red-shift formula.

(*b*) *Neutrino scattering cross-sections*

As the neutrinos going into the future are red-shifted their scattering at very low energies is of interest. Observation suggests that the most important scatterers are likely to be electrons and protons. For such a scattering Feynman & Gellmann (1958) proposed the interaction

$$G\sqrt{8}[\bar{\psi}_A\gamma^i(1\pm\mathrm{i}\gamma_5)\psi_\nu][\bar{\psi}_\nu\gamma_i(1\pm\mathrm{i}\gamma_5)\psi_A], \tag{27}$$

where G is a coupling constant and A stands for the scatterer. This leads to differential scattering cross-sections

$$\mathrm{d}\sigma = \frac{G^2m_A^2c^4E^2\,\mathrm{d}\Omega}{2\pi^2[m_Ac^2+E(1-\cos\theta)]^2}, \quad \mathrm{d}\sigma = \frac{G^2m_A^4c^8E^2\,\mathrm{d}\Omega}{2\pi^2[m_Ac^2+E(1-\cos\theta)]^2} \tag{28}$$

in the two cases of (19). m_A is the mass of A.

At low energies the scattering is spherically symmetric and the cross-section is proportional to the square of incident energy. The energy of the scattered neutrino is

$$E' = \frac{m_Ac^2}{m_Ac^2+E(1-\cos\theta)}E \tag{29}$$

so that as $E\to 0$, $E'/E\to 1$.

For neutrinos going into the past the energy increases owing to red shift. The formulae (28) obtained by using perturbation theory are not applicable. It is generally believed that the cross-sections tend to saturate at high energies. Even so, (29) shows that as $E\to\infty$, $E'/E\to 0$ if $\theta \neq 0$. This, as will be seen later, greatly simplifies the discussion of the past absorber.

(c) *Refractive index for neutrino propagation*

The presence of scatterers gives the medium a refractive index different from unity. It can be shown that if a plane wave $\exp[i\omega x/c]$ gives rise to a scattered wave $(a/r)\exp[i\omega r/c]$ from the scatterer the refractive index is given by

$$n = 1 + (2\pi c^2/\omega^2) Na, \tag{30}$$

where N is the number of scatterers per unit proper volume of the medium (Hamilton 1959). The scattering length a is related to the cross-section σ by

$$\sigma = 4\pi |a|^2. \tag{31}$$

Since $\sigma \propto E \propto e^{-\zeta}$ for the neutrinos going into the future, one can write

$$a = a_0 e^{-\zeta} = (\alpha_0 + i\beta_0) e^{-\zeta} \tag{32}$$

where a_0 is the value of a at $\tau = \tau_0$. As absorption is also present

$$\beta_0 > 0. \tag{33}$$

For reasons mentioned in (*b*) no such simple expressions are available for the neutrinos going into the past.

4. Evaluation of f and p

It is now possible to consider the interaction of the past and the future absorbers with the source O. It is convenient to work with an explicit example. Accordingly, consider $\psi_{\text{ret.}}$ to be the first of the two solutions (23).

At a point $P_1(r_1, \theta_1, \phi_1)$ of the future absorber, with $r_1 \gg (c/\omega_0)$, $\psi_{\text{ret.}}$ behaves as a plane wave

$$\psi_{\text{ret.}} \sim \frac{1}{r_1} \exp[-\tfrac{3}{2}\zeta_1] \exp[-i\omega_0 \tau + (i\omega_0/c) \times (x \sin\theta_1 \cos\phi_1 + y \sin\theta_1 \sin\phi_1 + z\cos\theta_1)] \begin{bmatrix} 1+\cos\theta_1 \\ \sin\theta_1 e^{i\phi_1} \end{bmatrix}, \tag{34}$$

where $\zeta_1 = \zeta(\tau_0 + r_1/c)$. In order to evaluate fR it is necessary to consider the advanced scattered wave from P_1 at a general point P in the neighbourhood of O. The approximation implied by $r_1 \gg r \gg c/\omega_0$, where (r, θ, ϕ) are the co-ordinates of P, will be used in what follows.

The advanced wave from P will again be a plane wave like (34) near P and will therefore have the same spin parts. With the use of the scattering length a the advanced wave contribution from points near P can be written as

$$\frac{1}{2} \frac{a}{|\mathbf{r}-\mathbf{r}_1| e^{\zeta_1}} \exp\left[-i\omega_0\left(\tau + \frac{|\mathbf{r}-\mathbf{r}_1|}{c}\right)\right] \frac{1}{r_1} \exp\left[-\tfrac{3}{2}\zeta\left(\tau_0 + \frac{r_1}{c} - \frac{|\mathbf{r}-\mathbf{r}_1|}{c}\right)\right] \begin{bmatrix} 1+\cos\theta_1 \\ \sin\theta_1 e^{i\phi_1} \end{bmatrix}. \tag{35}$$

The factor e^{ζ_1} appears in the denominator as the ratio of proper to co-ordinate distance at time $\tau_0 + (r_1/c)$. The above expression needs correction for phase factors. First to account for the change of phase from O to P_1 it is necessary to multiply (35) by $\exp[\mathrm{i}\omega_0 r/c)]$. Over and above this there is a phase change by an angle χ owing to the refractive index of the medium being different from unity. χ is given by

$$\chi = \int_0^{r_1} \omega(n-1)\exp\zeta'\,\mathrm{d}r', \tag{36}$$

where $\zeta' = \zeta(\tau_0 + (r'/c)\}$. The corrected form of (35) after making approximations for $r \ll r_1$ is

$$\frac{a\mathrm{e}^{-\zeta_1}}{2r_1^2}\exp\left[-\mathrm{i}\omega_0\{\tau + (r/c)(\cos\theta_1\cos\theta + \sin\theta_1\sin\theta\cos(\phi_1-\phi))\}\right]\exp(\mathrm{i}\chi)\begin{bmatrix}1+\cos\theta_1\\ \sin\theta_1\,\mathrm{e}^{\mathrm{i}\phi_1}\end{bmatrix}. \tag{35}$$

The total contribution from the future absorber is therefore

$$\begin{aligned} fR = \int_0^{R_1} \frac{a_0}{2r_1^2}\times N_0\exp(M\zeta_1 + \mathrm{i}\chi - \mathrm{i}\omega_0\tau)\,r_1^2\,\mathrm{d}r_1 \\ \times\iint \exp\left[(\mathrm{i}\omega_0 r/c)\{\cos\theta_1\cos\theta + \sin\theta_1\sin\theta\cos(\phi_1-\phi)\}\right] \\ \times\begin{bmatrix}1+\cos\theta_1\\ \sin\theta_1\,\mathrm{e}^{\mathrm{i}\phi_1}\end{bmatrix}\sin\theta_1\,\mathrm{d}\theta_1\,\mathrm{d}\phi_1, \end{aligned} \tag{37}$$

where
$$M = \begin{cases} +1 & \text{for class I universes,} \\ -2 & \text{for class II universes.}\end{cases}$$

Using the result proved in the appendix we can write (37) in the form

$$\begin{aligned} fR = \left\{\frac{\exp[-\mathrm{i}\omega_0(\tau - r/c)]}{2r}\begin{bmatrix}1+\cos\theta\\ \sin\theta\,\mathrm{e}^{\mathrm{i}\phi}\end{bmatrix} - \frac{\exp[-\mathrm{i}\omega_0(\tau + r/c)]}{2r}\begin{bmatrix}1-\cos\theta\\ -\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}\end{bmatrix}\right\} \\ \times\int_0^{R_1}\frac{2\pi a_0 N_0 c}{\mathrm{i}\omega_0}\exp[M\zeta_1 + \mathrm{i}\chi_1]\,\mathrm{d}r_1 = f[\tfrac{1}{2}\psi_{\text{ret.}} - \tfrac{1}{2}\psi_{\text{adv.}}], \end{aligned} \tag{38}$$

where
$$f = \int_0^{R_1}\frac{2\pi a_0 N_0 c}{\mathrm{i}\omega_0}\exp[M\zeta_1 + \mathrm{i}\chi_1]\,\mathrm{d}r_1, \tag{39}$$

R_1 being the radius of the absorber. Now (36) gives

$$\chi = \int_0^{r_1}\frac{2\pi a_0 N_0 c}{\omega_0}\,\mathrm{e}^{M\zeta'}\,\mathrm{d}r'. \tag{40}$$

Writing $U(r_1) = \chi(r_1)$ we can integrate (39) to get

$$f = 1 - \mathrm{e}^{+\mathrm{i}U(R_1)}. \tag{41}$$

Thus the necessary and sufficient condition for f to be unity is

$$\mathscr{I}[U(R_1)] = \infty. \tag{42}$$

Since $\beta_0 > 0$ this condition reduces to

$$\int_0^{R_1}\mathrm{e}^{M\zeta_1}\,\mathrm{d}r_1 = \infty. \tag{43}$$

Turning now to the past absorber it is clear that such simple formulation will not be applicable. On the other hand, as was shown earlier, the scattered neutrinos have negligible energy compared to the incident neutrinos. When these scattered neutrinos travel back to the present $\tau = \tau_0$ they undergo red shift and have negligible energy. In other words, the condition $p = 1$ cannot be satisfied. This situation is also present in the case of the photons, only it is not made clear in a formulation that is entirely classical. For instance, Thomson scattering is not the most important form of scattering. Other forms of scattering such as Compton scattering become relatively more important.

Therefore, unless $f = 1$ the advanced wave functions represented by neutrinos going into the past will also be present. It is therefore interesting to find which cosmological models satisfy this condition. It is easy to verify that the steady state model does satisfy the condition $f = 1$, whereas the Einstein–de Sitter model fails to do so. Although in general the condition is more favourable to the class I models, there are some class II models which do satisfy it; for example, models in which $e^{\zeta(\tau)} \propto \tau^n$, $(0 < n < \frac{1}{2})$. Further, the condition (43) is in general more difficult to satisfy than the corresponding condition for the photon which is similar to (43) but has $M = +2$ for class I and -1 for class II. Thus many cosmologies which satisfy this condition for the photon fail to do so for the neutrino. It is interesting to find that the steady-state model, though it easily satisfies the condition for the photon, only 'just' manages to do so for the neutrino.

5. Conclusions

The above considerations suggest an idealized experiment with a neutrino emitter and detector designed to test the various cosmological models. If there are neutrinos travelling into the past they should be detected *before* their transmission. A negative result of such an experiment will be in agreement with the prediction of the steady-state theory. It will also be in agreement with a unidirectional arrow of time and the experience with causality.

The author is grateful to Professor Hoyle for suggesting the problem and for many helpful discussions. Thanks are also due to Professor W. A. Fowler and Dr J. C. Taylor for discussions on neutrino cross-sections.

6. Appendix

To evaluate the integral

$$I = \iint \exp\left[i\lambda\{\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1)\}\right] f(\theta_1,\phi_1)\sin\theta_1\, d\theta_1\, d\phi_1 \tag{44}$$

in the case $\lambda \gg 1$. The function $f(\theta_1, \phi_1)$ is periodic in θ_1, ϕ_1 with period 2π.

Consider first the integration with respect to ϕ_1. Using the method of stationary phase (Jeffreys & Jeffreys 1956) gives

$$I \simeq \int_0^{2\pi} \{\exp[i\lambda\cos(\theta_1-\theta) - \tfrac{1}{4}i\pi] f(\theta_1,\phi_1) - \exp[i\lambda\cos(\theta_1-\theta) + \tfrac{1}{4}i\pi] f(\theta_1, \pi+\phi)\} \times \sqrt{\left(\frac{2\pi\sin\theta_1}{\lambda\sin\theta}\right)}\, d\theta_1. \tag{45}$$

Using the method of stationary phase once again leads to

$$I \simeq \frac{2\pi}{\lambda \mathrm{i}}[f(\theta,\phi)\,\mathrm{e}^{-\mathrm{i}\lambda} - f(\pi-\theta,\pi+\phi)\,\mathrm{e}^{-\mathrm{i}\lambda}]. \tag{46}$$

This is the required result. In (37) $\lambda = \omega_0 r/c \gg 1$.

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