Domain Walls in Kaluza-Klein spacetime

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Abstract

Three families of exact solutions of Einstein field equations are found. Each family contains three parameters. Two of these families represent thick domain walls in a five dimensional Kaluza-Klein spacetime. The dynamical behaviour of our models is briefly discussed. The spacetime in all the cases is found to be reflection symmetric with respect to the wall.

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1 Introduction

The phase transitions in the early universe, due to spontaneous breaking of a discrete symmetry, could have produced the topological defects such as domain walls, strings and monopoles [1]. Hill, Schramm and Fry [2] have suggested that light domain walls of large thickness may have been produced during the late time phase transitions such as those occuring after the decoupling of matter and radiation. Recently the study of the thick domain walls and spacetimes associated with them have received considerable attention due to their application in structure formation in the Universe.
Vilenkin [3] first showed that the gravitational field fo an infinite thin domain wall with planar symmetry cannot be described by a static metric. Subsequently, Widrow [4] noted that nor could a thick domain wall be described by a regular static metric. These considerations suggest that non-static metrics are suitable for description of the field of a thick domain wall. Many authors have discussed non-static solutions of the Einstein scalar field equations for thick domain walls [4-6]. But these solutions have peculiar behaviour. In these solutions the energy scalar is independent of time whereas the metric tensor depends on both space and time. Letelier and Wang [7] have obtained exact solutions to the Einstein field equations that represent the collision of plane thin walls.
Later on Wang [8] has derived a two parameter family of solutions of the Einstein field equations representing gravitational collapse of a thick domain wall. Thick domain walls are characterized by the energy momentum tensor

$$T_{ik} = \rho(g_{ik} + w_i w_k) + p w_i w_k, \;\; w_i w^i = -1 \tag{1.1}$$

where ρ is the energy density of the wall, p is the pressure in the direction normal to the plane of the wall and w_i is a unit spacelike vector in the same direction. In Wang's solution, the energy scalar and the metric tensor are dependent on space as well as time coordinates.
There are two approaches for the study of thick domain walls. In the first approach one studies the field equations as well as the equations of domain wall treated as the self-interacting scalar field. In the second approach one assumes the energy momentum tensor in the form (1.1) and then the field equations are solved. The second approach seems to be easier. In the present paper we adopt the second approach but apply it to a more general five di-

mensional Kaluza-Klein spacetime. The advances in supergravity in 11-D and superstring in 10-D indicate that the multidimensionality of space is apparently a fairly adequate reflection of the dynamics of interactions over the distance where all forces unify. The cosmological implications of higher dimensions were first discussed by Chodos and Detweiler [9]. They have Kasner-type vaccum solutions in a five dimensional spacetime. Their solutions possess the property of dimensional reduction. It would be worthwhile to study the time-dependent thick domain walls in a five dimensional Kaluza-Klein spacetime. Banerjee and Das [10] have considered thick domain walls in higher dimensions and obtained some exact solutions. The purpose of the present work is to report some other new exact solutions of the Einstein equations representing gravitation field of thick domain walls in a five-dimensional spacetime.
In Sec.2 we set up the field equations and solve them and we conclude in Sec.3 with a discussion.

2 Field equations and their solutions

In this section, we engage ourselves to the problem of construction of general relativistic models of plane symmetric thick domain walls in a five-dimensional spacetime. Here the spacetime admits one additional killing vector. The general five-dimensional plane symmetric metric can be expressed in the form

$$ds^2 = A^2(dt^2 - dx^2) - B^2(dy^2 + dz^2) - E^2 d\psi^2 \tag{2.1}$$

where A, B and E are functions of x and t, and ψ is the fifth coordinate corresponding to the extra dimension.

We introduce the pentad

$$\theta^0 = Adt, \theta^1 = Adx, \theta^2 = Bdy, \theta^3 = Bdz, \theta^4 = Ed\psi. \tag{2.2}$$

Here and in what follows all the components would refer to base frame. The surviving Ricci components R_{ab} are listed below for ready reference:

$$A^2 R_{00} \quad = \quad \frac{\ddot{A}}{A} + \frac{2\ddot{B}}{B} + \frac{\ddot{E}}{E} - \frac{\dot{A}}{A}(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B} + \frac{\dot{E}}{E}) -$$

$$[\frac{A''}{A}+\frac{A'}{A}(\frac{2B'}{B}+\frac{E'}{E}-\frac{A'}{A})] \tag{2.3}$$

$$A^2R_{01}=\frac{2\dot{B}'}{B}+\frac{\dot{E}'}{E}-\frac{\dot{A}}{A}(\frac{2B'}{B}+\frac{E'}{E})-\frac{A'}{A}(\frac{2\dot{B}}{B}+\frac{\dot{E}}{E}) \tag{2.4}$$

$$\begin{aligned} A^2R_{(11)} \quad = \quad & \frac{A''}{A}+\frac{2B''}{B}+\frac{E''}{E}-\frac{A'}{A}(\frac{A'}{A}+\frac{2B'}{B}+\frac{E'}{E}) \\ & -[\frac{\ddot{A}}{A}+\frac{\dot{A}}{A}(2\frac{\dot{B}}{B}+\frac{\dot{E}}{E}-\frac{\dot{A}}{A})] \end{aligned} \tag{2.5}$$

$$A^2R_{22}=A^2R_{33}=\frac{B''}{B}+\frac{B'}{B}(\frac{B'}{B}+\frac{E'}{E})-[\frac{\ddot{B}}{B}+\frac{\dot{B}}{B}(\frac{\dot{B}}{B}+\frac{\dot{E}}{E})] \tag{2.6}$$

$$A^2R_{44}=\frac{E''}{E}+2\frac{E'B'}{EB}-(\frac{\ddot{E}}{E}+\frac{2\dot{B}\dot{E}}{BE}) \tag{2.7}$$

where a prime and a dot indicate derivatives with respect to x and t respectively.
We now have to solve the Einstein field equations

$$R_{ab}=-8\pi[T_{ab}-\frac{1}{3}Tg_{ab}] \tag{2.8}$$

where the energy stress components in the comoving coordinates for the thick domain wall are given by

$$T_0^0=T_2^2=T_3^3=\rho, T_1^1=-p, T_1^0=0, T_4^4=0 \tag{2.9}$$

Here ρ is the energy density of the wall which is also equal to the tension along y and z directions in the plane of the wall, p is the pressure along x-direction. The stress component T_4^4 corresponding to the extra dimension is assumed to be zero. In view of (2.9), equations (2.8) lead ot the following relations:

$$R_{01}=0 \tag{2.10}$$

$$R_{22}=-R_{00} \tag{2.11}$$

$$3R_{22} + R_{11} - R_{44} = 0 \tag{2.12}$$

$$8\pi p = 3R_{22} \tag{2.13}$$

$$8\pi\rho = -(R_{11} + 2R_{22}). \tag{2.14}$$

The general solution of the above system of equations is quite difficult to obtain. So we make the following separability assumptions for the metric potentials:

$$A = cosh^a(mx)e^{\alpha kt}, B = cosh^b(mx)e^{\beta kt}, E = cosh^d(mx)e^{\delta kt}. \tag{2.15}$$

Here $a, b, d, \alpha, \beta, \delta, m$ and k are real constants. With these assumptions, equation (2.10) leads to

$$2\beta(b-a) - \alpha(d+2b) + \delta(d-a) = 0. \tag{2.16}$$

$R_{(22)} = -R_{(00)}$ gives

$$(a-b)m^2[(2b+d-1)sech^2(mx)-(2b+d)]+k^2[\delta^2-\delta(\alpha+\beta)-2\alpha\beta] = 0. \tag{2.17}$$

Eqn. (2.12) on simplification leads to

$$m^2[8b^2-2ab-ad+bd]+m^2sech^2(mx)[a+5b-8b^2+2ab+ad-bd] = k^2(2\beta+\delta)(\alpha+3\beta-\delta). \tag{2.18}$$

Eqns.. (2-13) and (2-14) determine the physical parameters p and ρ. They are given by

$$\frac{8\pi pA^2}{3} = bm^2[(2b+d)+(1-2b-d)sech^2(mx)] - \beta(2\beta+\delta)k^2 \tag{2.19}$$

and

$$
\begin{aligned}
-8\pi\rho A^2 &= m^2(6b^2+d^2-2ab+bd-ad) \\
&+ m^2 sech^2(mx)[-6b^2-d^2+2ab+ad-2bd+a+d+4b] \\
&- (\alpha+2\beta)(2\beta+\delta)k^2. \qquad (2.20)
\end{aligned}
$$

From eqn. (2.17) it is clear that $(a-b)(2b+d-1)=0$, which will ipmply either $a=b$ or $2b+d=1$, and correspondingly we will have the following two cases.

Case I: $a=b$.

In this case, eqns (2.16) - (2.18) give

$$
a=b=1,\ \frac{m^2}{k^2}=\beta^2,\ \alpha=\frac{d(d-1)}{(d+2)}\beta,\ \delta=\beta d \qquad (2.21)
$$

where d, β, and k are arbitrary parameters. The pressure p and the energy density ρ are given by

$$
8\pi p=-3m^2(d+1)sech^4(mx)e^{-2\alpha kt} \qquad (2.22)
$$

and

$$
8\pi\rho=m^2(d^2-2)sech^4(mx)e^{-2\alpha kt}. \qquad (2.23)
$$

Therefore

$$
\rho=-\frac{(d^2-2)}{3(d+1)}p. \qquad (2.24)
$$

Thus we have a three parameter family of solutions describing thick domain walls. When $\beta=-1/2$, it reduces to the two parameter family of solutions discussed by Banerjee and Das [10] with slight change of notations.

Case II: $2b+d=1$.

Eqns (2.16) - (2.18) give

$$
\begin{aligned}
a=b(3b-2),\ d=1-2b,& \\
\alpha=6b(1-b)\beta+(1-3b^2)\delta,& \\
-3b(1-b)m^2=k^2(\delta^2-2\alpha\beta-\alpha\delta-\beta\delta),& \\
-3b(1+b)m^2=k^2(\delta-\alpha-3\beta)(2\beta+\delta).& \qquad (2.25)
\end{aligned}
$$

The last two equations lead to

$$(b-1)(1-4b^2)\frac{\beta^2}{\delta^2}+b(1+2b-4b^2)\frac{\beta}{\delta}-b^3=0 \tag{2.26}$$

which has the following two roots

$$\frac{\beta}{\delta}=\frac{b}{1-2b},\ \frac{b^2}{(1-b)(2b+1)}. \tag{2.27}$$

Therefore we have to consider two separate cases.

Case II (i)

In this case we obtain

$$\begin{aligned}\beta &= \frac{b\delta}{1-2b},\ \alpha=\frac{(1-2b+3b^2)\delta}{(1-2b)},\ d=1-2b\\ a &= b(3b-2),\ \frac{m^2}{k^2}=\frac{(1-b+4b^2)}{(1+b)(1-2b)}\delta^2\end{aligned} \tag{2.28}$$

and

$$\frac{8\pi p}{3}=\frac{2k^2b^2\delta^2(3b-4b^2-2)}{(1-2b)^2(1+b)}sech^{2a}(mx)e^{-2k\alpha t} \tag{2.29}$$

and

$$8\pi\rho=\frac{b\delta^2k^2(1+3b^2)(8b^2-12b+7)}{(1-2b)^2(1+b)}sech^{2a}(mx)e^{-2k\alpha t} \tag{2.30}$$

Here also the ratio ρ/p is a constant.

Case II (ii)

For this case we get

$$\begin{aligned}d &= 1-2b,\ a=b(3b-2),\ \beta=-\frac{b^2\delta}{(b-1)(2b+1)}\\ \alpha &= \frac{(1-b)(1+3b)}{(1+2b)},\ \frac{m^2}{k^2}=\frac{b(2b^2-b-2)}{(b-1)^2(2b+1)^2}\delta^2.\end{aligned} \tag{2.31}$$

The pressure and density are given by

$$8\pi p = \frac{3b^2k^2\delta^2(2b^2-2b-3)}{(b-1)^2(2b+1)^2} sech^{2a}(mx)e^{-2\alpha kt} \tag{2.32}$$

and

$$8\pi\rho = \frac{k^2\delta^2(-6b^5+6b^4+4b^3-b^2+4b+1)}{(b-1)^2(2b+1)^2} sech^{2a}(mx)e^{-2\alpha kt} \tag{2.33}$$

and they again bear constant ratio.
In case II, we thus obtain two three parameter families of solutions for domain walls, the parameters being b, δ and k.

3 Discussion

It is clear that the spacetimes of solutions of Case I and Case II are reflection symmetric with respect to the wall. For a thick domain wall it is desirable that pressure and density decrease on both sides of the wall away from the symmetry plane and fall off to zero as $x \rightarrow \pm\infty$.
For the domain wall solution of Case I the physical requirements $\rho > 0, p > 0$ and $\rho - p \geq 0$ would be satisfied provided we choose the parameter d such that $d < -(3+\sqrt{5})/2$. When $d = -(3+\sqrt{5})/2$ then it is $p = \rho$. Clearly ρ, p fall off to zero on either side of the wall.

For the solutions of Case II(i) and Case II(ii), the proper fall off behaviour would require $a > 0$. In Case II(i), this requirement would conflict with $\rho > 0, m^2/k^2 \geq 0$. Thus this family is not physically viable. However it is interesting to note that when $b = 0$, ρ and p vanish, resulting into an empty spacetime given by the metric,

$$ds^2 = e^{2nt}(dt^2 - dx^2) - dy^2 - dz^2 - e^{2nt}cosh^2(nx)d\psi^2 \tag{3.1}$$

where we have set $n = k\delta$.
It can be easily checked that when $b < \frac{1}{4}(1-\sqrt{17})$ (i.e. $2b^2 - b - 2 \leq 0$), then the solutions of Case II (ii) would satisfy the requirements $a > 0, \rho > 0$ and $m^2/k^2 > 0$. It would have the proper fall off behaviour as well as $\rho - p \geq 0$.
We shall now discuss the dynamical behaviour of our models under different restrictions imposed on various parameters occuring in the solutions. The general expression for the three space volume is given by

$$|g_3|^{1/2} = cosh^{a+2b}(mx)e^{kt(\alpha+2\beta)} \tag{3.2}$$

Thus for temporal behaviour would be

$$|g_3|^{1/2} \sim exp[kt(\alpha + 2\beta)] \tag{3.3}$$

Here it should be noted that when $\beta = -1/2$ in C ase I, we recover the Banerjee and Das solution [10]. So, for case I, we take β to be negative. If $d < -(3 + \sqrt{5})/2$, then we have $\alpha + 2\beta < 0$. Further if $k > 0$, 3-space collapses while the extra dimension inflates. In this process we get a singularity because as $t \to \infty$, ρ and p diverge. On the other hand if $k < 0$, the effective 3-space inflates while the extra dimension collapses in course of time.

On the similar lines we can discuss the dynamical behaviour of the domain wall solutions of Case II (ii). For the sake of brevity we shall not go into these details here.

The repulsive and attractive character of thick domain walls can be discussed by either studying the timelike geodesics in the spacetime or analysing the acceleration of an observer who is at rest relative to the wall [11]. Let us consider an observer with the four velocity $v_i = cosh^a(mx)e^{\alpha kt}\delta^t_i$. Then we obtain the acceleration vector A^i as

$$A^i = v^i_{;k}v^k = amtanh(mx)cosh^{-2a}(mx)e^{-2\alpha kt}\delta^i_x. \tag{3.4}$$

In case I, $a = 1$, and if $m > 0$, then A^x is positive. This implies that in order to keep the observer comoving with the wall it has to accelerate away from the symmetry plane or in other words it is attracted towards the wall. Similarly if $m < 0$, then the wall exhibites a repulsive nature to the observer. Similar conclusions can be drawn for the domain wall solutions of case II (ii). If we assume

$$T^0_0 = T^2_2 = T^3_3 = T^4_4 = \rho, T^1_1 = -p, T^0_1 = 0 \tag{3.5}$$

instead of (2.9), we get a domain wall solution in which $\rho = p$. But this solution is the same as that given by Banerjee and Das [10].

The function e^{x^2} is also reflection symmetric about the yz-plane. If we take e^{mx^2} in place of $cosh(mx)$ in the separability assumption (2.15), there cannot occur any domain wall solutions. But it does give a five-dimensional empty spacetime described by the metric

$$ds^2 = e^{2nt+n^2x^2}(dt^2 - dx^2) - e^{\frac{2nt}{\sqrt{6}}}(dy^2 + dz^2) - e^{\frac{4nt}{\sqrt{6}}} d\psi^2 \qquad (3.6)$$

where n is an arbitrary constant, and is the cause for the spacetime curvature. It is an inhomogeneos vacuum spacetime.

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